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ABSTRACT OF HABILITATION THESIS

Connections of Certain Inequalities Related to Convex Functions and to Inner Product Spaces

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In this habilitation thesis we have described the significant results achieved by the author after obtaining his PhD degree in Mathematics from Simion Stoilow Institute of Mathematics of the Romanian Academy, in 2012. Inequalities Theory represents an old topic of many mathematical areas which still remains an attractive research domain with many applications. The study of convex functions occupied and occupies a central role in *Inequalities Theory*, because the convex functions develop a series of inequalities.

The research results presented here are concerned with the improvement of classical inequalities resulting from convex functions and highlighting their applications.

A function $f : I \to \mathbb{R}$, where I is an interval, is called *convex* if we have

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b),$$

for all $a,b \in I, t \in [0,1]$.

Related to probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, known as *Jensen's inequality*, underlies many important inequalities.

Another important result related to convex function is the *Hermite–Hadamard inequality,* due to Hermite [107] and Hadamard [99], which asserts that for every continuous convex function $f : [a,b] \to \mathbb{R}$ the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

Related to the Hermite–Hadamard inequality, many mathematicians have worked with great interest to generalise, refine and extend it for different classes of functions such as: quasi-convex functions, log-convex, r-convex functions, etc and apply it for special means (logarithmic mean, Stolarsky mean, etc).

The habilitation thesis is focused on the study of important inequalities from Inequalities Theory and on their impact in some applications.

The thesis consists of four chapters. It also includes a list of notations and a bibliography with 211 references.

In the first part of this thesis we have presented the scientific and professional achievements and the evolution and development plans for career development.

The first chapter studies the inequalities developed from convex functions. This chapter contains several original results, many of them published in ISI journals. These studies are linked to several inequalities such as the Hermite-Hadamard inequality, the Fejér inequality, Hammer-Bullen's inequality and Young's inequality. In the last part of this chapter we present several Grüss-type inequalities in discrete form and in integral form. Here we show a refinement of Grüss's inequality via Cauchy–Schwarz's inequality for discrete random variables in finite case. In the end, we have analyzed the bounds of several statistical indicators and we have given a generalized form of Grüss type inequality and we have obtained other integral inequalities.

The second chapter studies the inequalities for functionals and inequalities for invertible positive operators. Here there are researched the Jensen functional under superquadraticity conditions and the Jensen functional related to a strongly convex function. We have shown several inequalities on generalized entropies. Generalized entropies have been studied by many researchers. Rényi [191] and Tsallis [201] entropies are well known as one-parameter generalizations of Shannon's entropy, being intensively studied not only in the field of classical statistical physics [202–204], but also in the field of quantum physics [198].

We have also studied the inequalities for invertible positive operators that have applications in operator equations, network theory and in quantum information theory.

The third chapter explores the inequalities in an inner product space (pre-Hilbert space). We remark the study of the Cauchy - Schwarz inequality in an inner product space and some reverse inequalities for the Cauchy-Schwarz inequality in an inner product space. We also make considerations about several inequalities and we mention a characterization of an inner product space.

In the second part of this habilitation thesis we have presented the evolution and development plans for career development.

The last chapter examines several future directions for research. We have identified three future directions for research, namely: future directions for research related to Hermite-Hadamard's inequality and Hammer-Bullen's inequality; future directions for research related to Young's inequality and Hardy's inequality and future directions for research related to inequalities in an inner product space.

Their study is initiated so as to improve some results on classical inequalities.

Original results of this habilitation thesis have been published in journals such as: Aequat. Math., Int. J. Number Theory, J. Inequal. Appl., Math. Inequal., J. Math. Inequal., Gen. Math., Appl. Math. Inf. Sci. etc.

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