Universitatea *Transilvania* din Braşov

REZUMATUL TEZEI DE ABILITARE

Conexiunile anumitor inegalități legate de funcțiile convexe și de spațiile vectoriale înzestrate cu produs scalar

Nicușor MINCULETE

În această teză de abilitate am descris rezultatele semnificative obținute de autor după ce a obținut titlul de doctor în matematică la Institutul de Matematică Simion Stoilow al Academiei Române în anul 2012. Teoria inegalităților reprezintă un subiect vechi al multor domenii matematice, care rămâne un domeniu de cercetare atractiv cu multe aplicații. Studiul funcțiilor convexe a ocupat și ocupă un rol central în teoria inegalităților, deoarece funcțiile convexe dezvoltă o serie de inegalități.

Rezultatele cercetărilor prezentate aici se referă la îmbunătățirea inegalităților clasice care rezultă din funcțiile convexe și evidențierea aplicațiilor acestora.

O funcție $f : I \to \mathbb{R}$, în care I este un interval, se numește convexă dacă avem

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b),$$

pentru orice $a,b \in I, t \in [0,1]$.

Legat de teoria probabilității, o funcție convexă aplicată la valoarea așteptată a unei variabile aleatoare este întotdeauna mai mică sau egală cu valoarea așteptată a funcției convexe a variabilei aleatoare. Acest rezultat, cunoscut sub numele de inegalitatea lui Jensen, stă la baza multor inegalități importante.

Un alt rezultat important legat de funcția convexă este inegalitatea Hermite-Hadamard, datorată lui Hermite [107] și Hadamard [99], care afirmă că pentru orice funcție convexă continuă $f : [a,b] \to \mathbb{R}$ avem următoarea inegalitate:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

Legat de inegalitatea Hermite-Hadamard, mulți matematicieni au lucrat cu mare interes la generalizarea, rafinarea si extinderea acesteia pentru diferite clase de funcții cum ar fi: funcțiile cvasi-convexe, funcțiile log-convexe, funcțiile r-convexe etc. și aplicarea lor pentru medii speciale (media logaritmică, media Stolarsky, etc).

Teza de abilitare se axează pe studierea inegalităților importante din teoria inegalităților și a impactului acestora în unele aplicații.

Teza constă din patru capitole. De asemenea, include o listă de notații și o bibliografie cu 211 de referințe.

În prima parte a acestei lucrări am prezentat realizările științifice și profesionale și planurile de evoluție și dezvoltare pentru dezvoltarea carierei.

Primul capitol studiază inegalitățile rezultate din funcțiile convexe. Acest capitol conține mai multe rezultate originale, multe dinte ele publicate în reviste ISI. Aceste studii sunt legate de câteva inegalități, precum: inegalitatea Hermite-Hadamard, inegalitatea Fejér, inegalitatea lui Hammer-Bullen și inegalitatea lui Young.

În ultima parte a acestui capitol prezentăm mai multe inegalități de tip Grüss în formă discretă și în formă integrală. Aici vom arăta o rafinare a inegalității lui Grüss prin inegalitatea Cauchy-Schwarz pentru variabile aleatoare discrete în cazul finit. În final, am analizat marginile mai multor indicatori statistici și am dat o formă generalizată a inegalității de tip Grüss și am obținut alte inegalități integrale.

În al doilea capitol studiem inegalitățile pentru funcționale și inegalități pentru operatorii inversabili pozitivi. Aici este cercetată funcționala Jensen în condiții de superpătricitate și funcționala Jensen legată de o funcție puternic convexă. Am arătat mai multe inegalități privind entropiile generalizate. Entropiile generalizate au fost studiate de mulți cercetători. Entropiile Rényi [191] și Tsallis [201] sunt bine cunoscute ca generalizări cu un parametru ale entropiei lui Shannon, fiind studiate intensiv nu numai în domeniul clasic al fizicii statistice [202-204], ci și în domeniul fizicii cuantice[198].

De asemenea, am studiat inegalitățile pentru operatorii inversabili pozitivi care au aplicații în: ecuațiile operatorilor, teoria rețelelor și teoria cuantică a informațiilor.

Al treilea capitol explorează inegalitățile într-un spațiu vectorial înzestrat cu produs scalar (prehilbertian). Remarcăm studiul inegalității Cauchy-Schwarz într-un spațiu vectorial înzestrat cu produs scalar și unele inegalități inverse pentru inegalitatea Cauchy-Schwarz într-un spațiu prehilbertian. De asemenea, facem câteva considerații cu privire la mai multe inegalități și menționăm o caracterizare a unui spațiului vectorial înzestrat cu produs scalar.

În a doua parte a acestei teze de abilitate am prezentat planurile de evoluție și dezvoltare pentru dezvoltarea carierei.

Ultimul capitol analizează mai multe direcții viitoare de cercetare. Am identificat trei direcții viitoare de cercetare, și anume: viitoare direcții de cercetare legate de inegalitatea lui Hermite-Hadamard și inegalitatea lui Hammer-Bullen; viitoarele direcții de cercetare referitoare la inegalitatea lui Young și inegalitatea lui Hardy și direcțiile viitoare de cercetare referitoare la inegalitățile dintr-un spațiu vectorial înzestrat cu produs scalar.

Studiul lor este inițiat pentru a îmbunătăți unele rezultate privind inegalitățile clasice.

Rezultatele originale ale acestei teze de abilitate au fost publicate în reviste precum: Aequat. Math., Int. J. Number Theory, J. Inequal. Appl., Math. Inequal., J. Math. Inequal., Gen. Math., Appl. Math. Inf. Sci. etc.

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