Universitatea *Transilvania* din Braşov

HABILITATION THESIS

Connections of Certain Inequalities Related to Convex Functions and to Inner Product Spaces

Domain: Mathematics

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CONTENT

(B-ii) The evolution and development plans for career development

List of notations

$\mathbb{R}$: the set of real numbers
$\mathbb{C}$: the set of complex numbers
[a, b]: interval
N *: the set of positive integers
$\mathbb{R}$: the set of real numbers
$\mathbb{R}_+$: the set of nonnegative real numbers
$\mathbb{R}^*$: the set of nonzero real numbers
exp(x): the exponential function
log(x): the logarithmic function with the base e
$\mathbb{R}^n$: Euclidean n-space
$M_n(R)$, $M_n(C)$: spaces of n × n-dimensional matrices
det A : determinant of A
$\frac{\partial f}{\partial x_k}$: partial derivative
A(s, t), G(s, t), H(s, t) : arithmetic, geometric and harmonic means
I(s, t) : identric mean
L(s, t) : logarithmic mean
Mp(s, t): Hölder (power) mean
$i=\overline{1,n}$: $i=1,2,...,n$
R([a, b]) : the space of Riemann-integrable functions on the interval [a, b]
C^0([a, b]) : the space of real-valued continuous functions on the interval [a, b]
$L_2(a,b)$: the space of integrable functions f on the interval [a, b], with $\int_a^b f^2(x)dx<\infty$
$B(H)$: algebra of bounded linear operators on a real Hilbert space H.
$\langle x,y\rangle$: inner product
$\|x\|$: norm of x
$A\#_{\alpha,p}B$: quasi-arithmetic power means for operators
$A\nabla_p B$: weighted arithmetic mean for operators
$A!_p B$: weighted harmonic mean for operators
$A\#_p B$: weighted geometric mean for operators
$A\#B$: geometric mean for operators
$H_q(p_1,p_2,...,p_n)$: the Tsallis entropy
$H(p_1,p_2,...,p_n)$: the Shannon entropy
$R_q(p_1,p_2,...,p_n)$: the Rényi entropy
$D_1^{\psi}(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)$: the quasilinear relative entropy
$R_q(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)$: the Rényi relative entropy
$D_q(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)$: the Tsallis relative entropy
$I_q^{\psi}(p_1,p_2,...,p_n)$: the Tsallis quasilinear entropy (q-quasilinear entropy)

$D_q^{\psi}\left(p_1, p_2, ..., p_n \| r_1, r_2, ..., r_n\right)$: the Tsallis quasilinear relative entropy
□: end of a proof

Abstract

In this habilitation thesis we have described the significant results achieved by the author after obtaining his PhD degree in Mathematics from Simion Stoilow Institute of Mathematics of the Romanian Academy, in 2012. Inequalities Theory represents an old topic of many mathematical areas which still remains an attractive research domain with many applications. The study of convex functions occupied and occupies a central role in *Inequalities Theory*, because the convex functions develop a series of inequalities.

The research results presented here are concerned with the improvement of classical inequalities resulting from convex functions and highlighting their applications.

A function $f : I \to \mathbb{R}$, where I is an interval, is called *convex* if we have

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b),$$

for all $a,b \in I, t \in [0,1]$.

Related to probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, known as *Jensen's inequality*, underlies many important inequalities.

Another important result related to convex function is the *Hermite–Hadamard inequality,* due to Hermite [107] and Hadamard [99], which asserts that for every continuous convex function $f : [a,b] \to \mathbb{R}$ the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Related to the Hermite–Hadamard inequality, many mathematicians have worked with great interest to generalise, refine and extend it for different classes of functions such as: quasi-convex functions, log-convex, r-convex functions, etc and apply it for special means (logarithmic mean, Stolarsky mean, etc).

The habilitation thesis is focused on the study of important inequalities from Inequalities Theory and on their impact in some applications.

The thesis consists of four chapters. It also includes a list of notations and a bibliography with 211 references.

In the first part of this thesis we have presented the scientific and professional achievements and the evolution and development plans for career development.

The first chapter studies the inequalities developed from convex functions. This chapter contains several original results, many of them published in ISI journals. These studies are linked to several inequalities such as the Hermite-Hadamard inequality, the Fejér inequality, Hammer-Bullen's inequality and Young's inequality. In the last part of this chapter we present several Grüss-type inequalities in discrete form and in integral form. Here we show a refinement of Grüss's inequality via Cauchy–Schwarz's inequality for discrete random variables in finite case. In the end, we have analyzed the bounds of several statistical

indicators and we have given a generalized form of Grüss type inequality and we have obtained other integral inequalities.

The second chapter studies the inequalities for functionals and inequalities for invertible positive operators. Here there are researched the Jensen functional under superquadraticity conditions and the Jensen functional related to a strongly convex function. We have shown several inequalities on generalized entropies. Generalized entropies have been studied by many researchers. Rényi [191] and Tsallis [201] entropies are well known as one-parameter generalizations of Shannon's entropy, being intensively studied not only in the field of classical statistical physics [202–204], but also in the field of quantum physics [198].

We have also studied the inequalities for invertible positive operators that have applications in operator equations, network theory and in quantum information theory.

The third chapter explores the inequalities in an inner product space (pre-Hilbert space). We remark the study of the Cauchy - Schwarz inequality in an inner product space and some reverse inequalities for the Cauchy-Schwarz inequality in an inner product space. We also make considerations about several inequalities and we mention a characterization of an inner product space.

In the second part of this habilitation thesis we have presented the evolution and development plans for career development.

The last chapter examines several future directions for research. We have identified three future directions for research, namely: future directions for research related to Hermite-Hadamard's inequality and Hammer-Bullen's inequality; future directions for research related to Young's inequality and Hardy's inequality and future directions for research related to inequalities in an inner product space.

Their study is initiated so as to improve some results on classical inequalities.

Original results of this habilitation thesis have been published in journals such as: Aequat. Math., Int. J. Number Theory, J. Inequal. Appl., Math. Inequal., J. Math. Inequal., Gen. Math., Appl. Math. Inf. Sci. etc.

Rezumat

În această teză de abilitate am descris rezultatele semnificative obținute de autor după ce a obținut titlul de doctor în matematică la Institutul de Matematică Simion Stoilow al Academiei Române în anul 2012. Teoria inegalităților reprezintă un subiect vechi al multor domenii matematice, care rămâne un domeniu de cercetare atractiv cu multe aplicații. Studiul funcțiilor convexe a ocupat și ocupă un rol central în teoria inegalităților, deoarece funcțiile convexe dezvoltă o serie de inegalități.

Rezultatele cercetărilor prezentate aici se referă la îmbunătățirea inegalităților clasice care rezultă din funcțiile convexe și evidențierea aplicațiilor acestora.

O funcție $f : I \to \mathbb{R}$, în care I este un interval, se numește convexă dacă avem

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b),$$

pentru orice $a,b \in I, t \in [0,1]$.

Legat de teoria probabilității, o funcție convexă aplicată la valoarea așteptată a unei variabile aleatoare este întotdeauna mai mică sau egală cu valoarea așteptată a funcției convexe a variabilei aleatoare. Acest rezultat, cunoscut sub numele de inegalitatea lui Jensen, stă la baza multor inegalități importante.

Un alt rezultat important legat de funcția convexă este inegalitatea Hermite-Hadamard, datorată lui Hermite [107] și Hadamard [99], care afirmă că pentru orice funcție convexă continuă $f : [a,b] \to \mathbb{R}$ avem următoarea inegalitate:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

Legat de inegalitatea Hermite-Hadamard, mulți matematicieni au lucrat cu mare interes la generalizarea, rafinarea si extinderea acesteia pentru diferite clase de funcții cum ar fi: funcțiile cvasi-convexe, funcțiile log-convexe, funcțiile r-convexe etc. și aplicarea lor pentru medii speciale (media logaritmică, media Stolarsky, etc).

Teza de abilitare se axează pe studierea inegalităților importante din teoria inegalităților și a impactului acestora în unele aplicații.

Teza constă din patru capitole. De asemenea, include o listă de notații și o bibliografie cu 211 de referințe.

În prima parte a acestei lucrări am prezentat realizările științifice și profesionale și planurile de evoluție și dezvoltare pentru dezvoltarea carierei.

Primul capitol studiază inegalitățile rezultate din funcțiile convexe. Acest capitol conține mai multe rezultate originale, multe dinte ele publicate în reviste ISI. Aceste studii sunt legate de câteva inegalități, precum: inegalitatea Hermite-Hadamard, inegalitatea Fejér, inegalitatea lui Hammer-Bullen și inegalitatea lui Young.

În ultima parte a acestui capitol prezentăm mai multe inegalități de tip Grüss în formă discretă și în formă integrală. Aici vom arăta o rafinare a inegalității lui Grüss prin inegalitatea Cauchy-Schwarz pentru variabile aleatoare discrete în cazul finit. În final, am analizat marginile mai multor indicatori statistici și am dat o formă generalizată a inegalității de tip Grüss și am obținut alte inegalități integrale.

În al doilea capitol studiem inegalitățile pentru funcționale și inegalități pentru operatorii inversabili pozitivi. Aici este cercetată funcționala Jensen în condiții de superpătricitate și funcționala Jensen legată de o funcție puternic convexă. Am arătat mai multe inegalități privind entropiile generalizate. Entropiile generalizate au fost studiate de mulți cercetători. Entropiile Rényi [191] și Tsallis [201] sunt bine cunoscute ca generalizări cu un parametru ale entropiei lui Shannon, fiind studiate intensiv nu numai în domeniul clasic al fizicii statistice [202-204], ci și în domeniul fizicii cuantice[198].

De asemenea, am studiat inegalitățile pentru operatorii inversabili pozitivi care au aplicații în: ecuațiile operatorilor, teoria rețelelor și teoria cuantică a informațiilor.

Al treilea capitol explorează inegalitățile într-un spațiu vectorial înzestrat cu produs scalar (prehilbertian). Remarcăm studiul inegalității Cauchy-Schwarz într-un spațiu vectorial înzestrat cu produs scalar și unele inegalități inverse pentru inegalitatea Cauchy-Schwarz într-un spațiu prehilbertian. De asemenea, facem câteva considerații cu privire la mai multe inegalități și menționăm o caracterizare a unui spațiului vectorial înzestrat cu produs scalar.

În a doua parte a acestei teze de abilitate am prezentat planurile de evoluție și dezvoltare pentru dezvoltarea carierei.

Ultimul capitol analizează mai multe direcții viitoare de cercetare. Am identificat trei direcții viitoare de cercetare, și anume: viitoare direcții de cercetare legate de inegalitatea lui Hermite-Hadamard și inegalitatea lui Hammer-Bullen; viitoarele direcții de cercetare referitoare la inegalitatea lui Young și inegalitatea lui Hardy și direcțiile viitoare de cercetare referitoare la inegalitățile dintr-un spațiu vectorial înzestrat cu produs scalar.

Studiul lor este inițiat pentru a îmbunătăți unele rezultate privind inegalitățile clasice.

Rezultatele originale ale acestei teze de abilitate au fost publicate în reviste precum: Aequat. Math., Int. J. Number Theory, J. Inequal. Appl., Math. Inequal., J. Math. Inequal., Gen. Math., Appl. Math. Inf. Sci. etc.

(B) Scientific and professional achievements and the evolution and development plans for career development

(B-i) Scientific and professional achievements

Chapter 1

Inequalities developed from convex functions

The study of optimization problems is distinguished by a number of properties characterized by convex functions. These functions play an important role in many areas of mathematics. The convex functions develop a series of inequalities.

A function $f : I \to \mathbb{R}$, where I is an interval, is called *convex* if the line segment between any two points on the graph of the function lies above or on the graph. Equivalently, a function is convex if the set of points on or above the graph of the function is a convex set. In fact, we have

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b),$$

for all $a,b \in I, t \in [0,1]$.

As applications of convex function we have the following: every *norm* is a convex function, by the triangle inequality and positive homogeneity; the function $-\log\det(X)$ on the domain of positive-definite matrices is convex; another example is *Euler's gamma function,* $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, x>0$ (in fact Euler's gamma function is a log-convex function, i.e., we have $f(ta+(1-t)b) \le f^t(a)f^{(1-t)}(b)$, for all $a,b \in I, t \in [0,1]$); if a function $f : I \to \mathbb{R}$ is log-convex, then it is also convex; related to probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, known as *Jensen's inequality*, which underlies many important inequalities, is given as: for a real convex function *f*, numbers $x_1, x_2, ..., x_n$ in its domain, and positive weights $w_1, w_2, ..., w_n$, we have:

$$f\left(\frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}\right) \le \frac{\sum_{i=1}^n w_i f(x_i)}{\sum_{i=1}^n w_i}. \tag{1.1}$$

When, we have $w_1 = w_2 = ... = w_n$, then, we deduce the classical variant of Jensen's inequality:

(1.2) $$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{\sum_{i=1}^{n} f(x_i)}{n}.$$

1.1 About the Hermite-Hadamard inequality

As a particular case, in Jensen's inequality, for $n = 2$ in inequality (1.2), we have:

(1.1.1) $$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

An important result related to inequality (1.1.1) is the *Hermite–Hadamard inequality,* due to Hermite [107] and Hadamard [99], which asserts that for every continuous convex function $f:[a,b] \to \mathbb{R}$ the following inequalities hold:

(1.1.2) $$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Hardy, Littlewood and Pólya presented in the book [106] the following result, which characterise the convex functions, given by:

Theorem 1.1.1. *A necessary and sufficient condition that a continuous function f be convex in (a, b) is that*

(1.1.3) $$f(x) \leq \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt.$$

for $a \leq x - h < x + h \leq b$. *It can be shown that this result is equivalent to the first inequality in* (1.1.2) *when f is continuous on* [*a*, *b*].

Related to the Hermite–Hadamard inequality, many mathematicians have worked with great interest to generalise, refine, counterpart and extend it for different classes of functions such as: quasi-convex functions, log-convex, r-convex functions, etc and apply it for special means (logarithmic mean, Stolarsky mean, etc).

In the monograph [51], Dragomir and Pearce presented many characterizations of the Hermite-Hadamard inequality.

Ioan Rașa [165] made the following remark in connection with the above refinement on Hermite-Hadamard inequality: if $f:[a,b] \to \mathbb{R}$ is a convex function, then

(1.1.4) $$\frac{1}{2}\left(f\left(\frac{a+b}{2}-c\right)+f\left(\frac{a+b}{2}+c\right)\right) \leq \frac{1}{b-a}\int_a^b f(t)dt,$$

for every $c \in \left[0,\frac{b-a}{2}\right]$, and $c = \frac{b-a}{4}$ is maximal with this property.

A series of proofs and improvements of the Hermite-Hadamard inequality were given over time (see [32, 54, 55, 68]). In [22], Bessenyei applied Hermite-Hadamard inequality on simplices and Bessenyei and Páles established in [21] several inequalities of Hermite-Hadamard type for generalized convex functions. An extension of the Hermite-Hadamard inequality through subarmonic function was also given by Mihăilescu and Niculescu in [139]. The Hermite-Hadamard inequality is the starting point to Choquet's theory [166].

Before stating the results, we recall some useful facts from literature. Dragomir, Cerone and Sofo present in [56, 57] the following estimates of the precision in the Hermite-Hadamard inequality:

Proposition 1.1.2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m\le f''\le M$. Then*

$$m\frac{(b-a)^2}{24}\le\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\le M\frac{(b-a)^2}{24} \tag{1.1.5}$$

and

$$m\frac{(b-a)^2}{12}\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\le M\frac{(b-a)^2}{12}. \tag{1.1.6}$$

These inequalities follow from the Hermite-Hadamard inequality, for the convex functions $f(x)-m\frac{x^2}{2}$ and $f(x)-M\frac{x^2}{2}$.

Theorem 1.1.3 (Minculete-Mitroi [145]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m\le f''\le M$. Then*

$$m\frac{\lambda(1-\lambda)}{2}(b-a)^2\le\lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b)\le M\frac{\lambda(1-\lambda)}{2}(b-a)^2 \tag{1.1.7}$$

for all $\lambda\in[0,1]$.

Remark 1.1.4. By integrating each term of the inequality (1.1.7) on [0, 1] with respect to the variable λ, we recover the inequality (1.1.6).

Corollary 1.1.5 (Minculete-Mitroi [145]). *Preserving the notation of Theorem* 1.1.3*, the following inequalities hold:*

(1.1.8)

$$m\frac{(1-2\lambda)^2}{8}(b-a)^2\le\frac{1}{2}\left(f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)\right)-f\left(\frac{a+b}{2}\right)\le M\frac{(1-2\lambda)}{8}(b-a)^2$$

for all $\lambda\in[0,1]$.

Remark 1.1.6. Notice that by integrating all terms of the inequality (1.1.8) on [0, 1] with respect to the variable λ, we recover the inequality (1.1.5).

The following result incorporates the classic statement of the Hermite-Hadamard inequality.

Corollary 1.1.7 (Minculete-Mitroi [145]). *Suppose $f:[a,b]\to\mathbb{R}$ is differentiable and convex. Then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\ge\frac{x-a}{b-a}\left(\frac{f(a)+f(x)}{2}-\frac{1}{x-a}\int_a^x f(t)g(t)dt\right)\ge 0 \tag{1.1.9}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\ge\frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x f(t)dt-f\left(\frac{a+x}{2}\right)\right)\ge 0, \tag{1.1.10}$$

for all $x,y\in(a,b)$.

1.2 Fejér type inequalities for convex functions

Fejér [71], studying trigonometric polynomials, obtained some inequalities, which generalise the Hermite-Hadamard inequality, and thus established the following well-known weighted generalization:

Theorem 1.2.1. *If* $f:[a,b]\to\mathbb{R}$ *is continuous and convex and if* $g:[a,b]\to\mathbb{R}_+$ *is integrable and symmetric with respect to the line* $x=(a+b)/2$, *that is,* $g((a+b)/2+t)=g((a+b)/2-t)$. *Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt\le\int_a^b f(t)g(t)dt\le\frac{f(a)+f(b)}{2}\int_a^b g(t)dt. \tag{1.2.1}$$

Motivated by the above results, in the paper [145] we have shown other inequalities of Fejér type:

Theorem 1.2.2 (Minculete-Mitroi [145]). *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function such that there exist real constants m and M so that* $m\le f''\le M$. *Assume* $g:[a,b]\to\mathbb{R}_+$ *is integrable and symmetric about* $\frac{a+b}{2}$. *Then the following inequalities hold:*

$$\frac{m}{2}\int_a^b (t-a)(b-t)g(t)dt\le\frac{f(a)+f(b)}{2}\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\le\frac{M}{2}\int_a^b (t-a)(b-t)g(t)dt \tag{1.2.2}$$

and

$$\frac{m}{8}\int_a^b (2t-a-b)^2 g(t)dt\le\int_a^b f(t)g(t)dt-f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt\le\frac{M}{8}\int_a^b (2t-a-b)^2 g(t)dt. \tag{1.2.3}$$

Remark 1.2.3. For the particular case $g(x)=1$, if we apply Theorem 1.2.2 on the intervals $\left[a,\frac{a+b}{2}\right],\left[\frac{a+b}{2},b\right]$, we get:

$$m\frac{(b-a)^2}{48}\le\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)-\frac{1}{b-a}\int_a^b f(t)dt\le M\frac{(b-a)^2}{48} \tag{1.2.4}$$

which represents an improvement of *the Hammer-Bullen inequality* [166], given by:

$$\frac{2}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right). \tag{1.2.5}$$

The following theorem gives new Fejér-type inequalities.

Theorem 1.2.4 (Minculete-Mitroi [145]). *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable, convex function with* $f''\ge 0$ *and* $g:[a,b]\to\mathbb{R}_+$ *be continuous. Then the following statements hold.*

i) If g is monotonically decreasing, then

$$\frac{f(a)+f(b)}{2}\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\ge\frac{f(a)+f(x)}{2}\int_a^x g(t)dt-\int_a^x f(t)g(t)dt\ge 0; \tag{1.2.6}$$

ii) If g is monotonically increasing, then

$$\int_a^b f(t)g(t)dt-f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt\ge\int_a^x f(t)g(t)dt-f\left(\frac{a+x}{2}\right)\int_a^x g(t)dx\ge 0, \tag{1.2.7}$$

for all $x,y\in(a,b)$.

We end this section with weighted statement of a known result concerning convex functions. In the light of Proposition 1.2.1, the following statement appears as a trivial generalization of a result due to Vasić and Lacković [205], and Lupaș [128] (cf. Pećarić *et. al* [176]) and we omit its proof.

Proposition 1.2.5. *Let p and q be two positive numbers and $a_1 \le a \le b \le b_1$. Let $g:[a,b]\to\mathbb{R}_+$ be integrable and symmetric about $A=\frac{pa+qb}{p+q}$. Then the inequalities*

(1.2.8) $$f\left(\frac{pa+qb}{p+q}\right)\int_{A-y}^{A+y} g(t)dt \le \int_{A-y}^{A+y} f(t)g(t)dt \le \frac{pf(a)+qf(b)}{p+q}\int_{A-y}^{A+y} g(t)dt$$

hold for $y>0$ and all continuous convex function $f:[a_1,b_1]\to\mathbb{R}$ if and only if $y \le \frac{b-a}{p+q}min\{p,q\}$.

This inequality is due to *Brenner and Alzer* [25].

From inequality (1.2.6) applied to the convex function t^p, with $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$, we have (see [145])

(1.2.9) $$(b-a)\left\{[A_p(a,b)]^p-[S_p(a,b)]^p\right\}\ge(x-a)\left\{[A_p(a,x)]^p-[S_p(a,x)]^p\right\},$$

where $x\in[a,b]$. Here $A_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}$ is *the power mean* and $S_p(a,b)=\left(\frac{a^p-b^p}{p(a-b)}\right)^{1/(p)-1}$, $p\ne 0,1$, is *the Stolarsky mean*. Also the limit case $p\to-1$ (or we may equivalently say the case of the convex function $\frac{1}{t}$) gives us

(1.2.10) $$(b-a)\left[\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right]\ge(x-a)\left[\frac{1}{H(a,x)}-\frac{1}{L(a,x)}\right],$$

where $H(a,b)=\frac{2ab}{a+b}$ is *the harmonic mean* and $L(a,b)=\frac{b-a}{log\, b-log\, a}$ is *the logarithmic mean*.

Some of the previous results where mentioned in the following papers, thus: in [168], Niezgoda, established some generalizations of Fejér inequality for convex sequences, in {169} he gave several inequalities for convex sequences and nondecreasing convex functions and in [122], Kunt *et al.* found new inequalities of Hermite-Hadamard-Fejér type for harmonically convex functions via fractional integrals.

1.3 Two reverse inequalities of Hammer-Bullen's inequality

For certain constraints of *f*, in [51], Dragomir and Pearce found an improvement of Hammer-Bullen 's inequality given by the following:

Theorem 1.3.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m\le f''\le M$. Then the following inequalities hold:*

(1.3.1) $$m\frac{(b-a)^2}{24}\le\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(t)dt\le M\frac{(b-a)^2}{24}.$$

This result was mentioned in Remark 1.2.2.

Next, we provide two reverse inequalities of Hammer-Bullen 's inequality.

Lemma 1.3.2 (Minculete-Dicu-Rațiu [146]). *Whenever* $f:[a,b]\to\mathbb{R}$ *is a twice differentiable function, we have the following equality:*

$$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)q(x)f''(x)dx, \tag{1.3.2}$$

where

$$q(x)=\begin{cases} a-x, & x\in\left[a,\frac{a+b}{2}\right) \\ b-x, & x\in\left[\frac{a+b}{2},b\right] \end{cases}.$$

Remark 1.3.3. a) Clearly for $x\in[a,b]$, one has $\left(x-\frac{a+b}{2}\right)q(x)\geq 0$. By some elementary computations one obtains:

$$\int_a^b\left(x-\frac{a+b}{2}\right)q(x)=\frac{(b-a)^3}{24}.$$

Therefore, for every $x\in[a,b]$, we can write

$$m\left(x-\frac{a+b}{2}\right)q(x)\leq\left(x-\frac{a+b}{2}\right)q(x)f''(x)\leq M\left(x-\frac{a+b}{2}\right)q(x).$$

Integrating from a to b, multiplying by $1/(b-a)$ and using relation (1.3.2), we obtain the inequalities from (1.3.1). b) Inequalities (1.3.1) can also be obtained by applying the Hammer-Bullen inequality for the convex functions $f(x)-m\frac{x^2}{2}$ and $f(x)-M\frac{x^2}{2}$.

In the following, we give a reverse inequality of Hammer-Bullen 's inequality.

Theorem 1.3.4 (Minculete-Dicu-Rațiu [146]). *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable and convex function. Then the following inequality holds*

$$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(t)dt\leq\frac{(b-a)(f'(b)-f'(a))}{16}. \tag{1.3.3}$$

Applying the inequality of Grüss (see [98]), we obtain the following:

Theorem 1.3.5 (Minculete-Dicu-Rațiu [146]). *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function and assume there exist real constants m and M such that:* $m\leq f''(x)\leq M$ *for all* $x\in[a,b]$. *Then*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(t)dt-\frac{(b-a)(f'(b)-f'(a))}{16}\right|\leq\frac{(M-m)(b-a)^2}{64}. \tag{1.3.4}$$

If we consider $f(x)=x^p$, $p>1$. Obviously f is a convex function. According to Theorem 1.3.4 one has:

$$A(a^p,b^p)+A^p(a,b)-2S_p^p(a,b)\leq\frac{p(b-a)(b^{p-1}-a^{a-1})}{16},$$

where $A(a,b)=\dfrac{a+b}{2}$ is the arithmetic mean and $S_p(a,b)=\left[\dfrac{a^p-b^p}{p(a-b)}\right]^{1/(p-1)}$, $p\neq 0,1$, is the Stolarsky mean.

For $f(x)=-\log x$, $x>0$, we have that f is a convex function. Applying Theorem 1.3.4 for *f*, we find the inequality

$$A(a,b)G(a,b)e^{\frac{(b-a)^2}{16ab}}\geq I^2(a,b),$$

where $G(a,b)=\sqrt{ab}$ is *the geometric mean* and $I(a,b)=\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$ is *the identric mean*.

In [Minculete-Florea-Furuichi, 147], our purpose was to establish several inequalities related to Hermite-Hadamard inequality. We also proved a generalization of the Hammer-Bullen inequality.

Let $H_a[f],H_b[f]:[a,b]\to\mathbb{R}$ be two functions defined by:

$$H_a[f](x)=(x-a)\frac{f(a)+f(x)}{2}-\int_a^x f(t)dt$$

and

$$H_b[f](x)=(b-x)\frac{f(b)+f(x)}{2}-\int_x^b f(t)dt.$$

Lemma 1.3.6. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that $f''(x)\geq 0$, for all $x\in[a,b]$. Then we have that the functions $H_a[f]$ and $H_b[f]$ are nonnegative and convex.*

Since the functions $H_a[f]$ and $H_b[f]$are convex, then applying the Hermite-Hadamard inequality, we obtain the following:

Theorem 1.3.7 (Minculete-Florea-Furuichi [147]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that $f''(x)\geq 0$, for all $x\in[a,b]$. Then, we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\geq\frac{2}{(b-a)^2}\int_a^b H_a[f](x)dx\geq \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}-\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt\geq 0 \tag{1.3.5}$$

and

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\geq\frac{2}{(b-a)^2}\int_a^b H_b[f](x)dx\geq \frac{f(b)+f\left(\frac{a+b}{2}\right)}{2}-\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(t)dt\geq 0. \tag{1.3.6}$$

Remark 1.3.8. By adding relations (1.3.5) and (1.3.6) we deduce the following inequality:

(1.3.7) $$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \geq \frac{1}{(b-a)^2}\int_a^b H_a[f](x)+H_b[f](x)dx \geq$$
$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt\right) \geq 0.$$

Theorem 1.3.9 (Minculete-Florea-Furuichi [147]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that $f''(x)\geq 0$, for all $x\in[a,b]$. Then, we have*

(1.3.8)
$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \geq \frac{2}{(b-a)^2}\int_a^b H_a[f](x)dx+\frac{4}{(b-a)^3}\int_a^b H_s[H_a[f](x)]dx \geq 0$$

and

(1.3.9)
$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \geq \frac{2}{(b-a)^2}\int_a^b H_b[f](x)dx+\frac{4}{(b-a)^3}\int_a^b H_s[H_b[f](x)]dx \geq 0,$$

where $s\in\{a,b\}$.

To generalize the above results, we can extend the functions $H_a[f], H_b[f]:[a,b]\to\mathbb{R}$ to the functions $H_a[f,g], H_b[f,g]:[a,b]\to\mathbb{R}$ which are defined by:

$$H_a[f,g](x)=\left(\int_a^x g(t)dt\right)\frac{f(a)+f(x)}{2}-\int_a^x f(t)g(t)dt$$

and

$$H_b[f](x)=\left(\int_x^b g(t)dt\right)\frac{f(b)+f(x)}{2}-\int_x^b f(t)g(t)dt.$$

If we take the following functions: $\overline{H}_a[f,g], \overline{H}_b[f,g]:[a,b]\to\mathbb{R}$ defined by:

$$\overline{H}_a[f,g](x)=\int_a^x f(t)g(t)dt-f\left(\frac{a+x}{2}\right)\left(\int_a^x g(t)dt\right)$$

and

$$\overline{H}_b[f](x)=\int_x^b f(t)g(t)dt-f\left(\frac{b+x}{2}\right)\left(\int_x^b g(t)dt\right),$$

then, we deduce the following:

Theorem 1.3.10 (Minculete-Florea-Furuichi [147]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function and $g:[a,b]\to\mathbb{R}_+$ is a differentiable function symmetric about $\frac{a+b}{2}$. Then the following inequalities hold:*

(1.3.10) $$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt \leq \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\int_a^b g(t)dt \leq \int_a^b f(t)g(t)dt \leq$$
$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)\int_a^b g(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)dt.$$

We also established an estimation of Féjer inequalities for different kinds of functions. In this context, we show an alternative proof and a generalization of Theorem 1.2.4 in [142], considering the integrability Riemann-Stieltjes.

Future directions for research related to Hammer-Bullen's inequality will be analyzed in the paper [Minculete-Niezgoda-Mitroi, 142].

1.4 Young type inequalities

The *Young integral inequality* is the source of many basic inequalities. Young [208] proved the following:

Theorem 1.4.1. *Suppose that* $f:[0,\infty)\to[0,\infty)$ *is an increasing continuous function such that* $f(0)=0$ *and* $\lim\limits_{x\to\infty} f(x)=\infty$. *Then*

(1.4.1) $$ab \le \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = Y(f;a,b).$$

There has been much work on different proofs and generalisations of (1.4.1) (Bullen [27] and Mitrinović *et al.* [155]).

It is easy to see that in relation (1.4.1), *ab* is a lower bound for the Young functional *Y* (*f*;*a,b*).

In 1974, Merkle [138] showed that there cannot be an upper bound to *Y* (*f*;*a,b*) which is independent of *f*. He proves the following theorem which provides a reverse inequality.

Suppose the conditions of Theorem 1.4.1 hold. Then

(1.4.2) $$Y(f;a,b) \le max\{af(a), bf^{-1}(b)\}.$$

Lemma 1.4.2. *If f satisfies the assumptions of Theorem 1.4.1, then*

(1.4.3) $$af(a) = \int_0^a f(x)dx + \int_0^{f(a)} f^{-1}(x)dx = Y(f;a,f(a)).$$

We remark the relation:

(1.4.4) $$Y(f;a,b) + Y\left(f;f^{-1}(b),f(a)\right) = af(a) + bf^{-1}(b).$$

Witkowski, in his paper [206], showed another reverse Young's integral inequality, thus: under the assumptions of Theorem 1.4.1, the inequality

(1.4.5) $$ab \le Y(f;a,b) \le af(a) + f^{-1}(b)(b - f(a)).$$

holds with equality if and only if $b = f(a)$. In [140], Minguzzi generalizes this inequality.

Using conveniently inequality (1.4.5), for $f^{-1}(b), f(a)$, we find the following inequality:

(1.4.6) $$f(a)f^{-1}(b) \le Y\left(f;f^{-1}(b),f(a)\right) \le bf^{-1}(b) + a(f(a) - b).$$

Combining relations (1.4.4) and (1.4.6), we obtain again inequality (1.4.5).

Again, Witkowski [206] gave another result related to Young's integral inequality, thus, under the assumptions of Theorem 1.4.1, the inequality

(1.4.7) $$ab \ge min\left\{1, \frac{b}{f(a)}\right\}\int_0^a f(x)dx + min\left\{1, \frac{a}{f^{-1}(b)}\right\}\int_0^b f^{-1}(x)dx\,.$$

holds with equality if and only if $b = f(a)$.

Cerone, in [33], proved that the upper bound obtained by Witkowski given in (1.4.5) is always better than that of Merkle (1.4.2).

For $f(x)=x^{p-1}$, $p>1$, in Theorem 1.4.8, we deduce the inequality:

(1.4.8) $$ab \geq min\left\{1,\frac{ab}{a^p}\right\}\frac{a^p}{p}+min\left\{1,\frac{ab}{b^q}\right\}\frac{b^q}{q}.$$

For $f(x)=x^{p-1}$, $p>1$, in Theorem 1.4.1, we deduce *the Young inequality*:

(1.4.9) $$ab \leq \frac{a^p}{p}+\frac{b^q}{q},$$

for all $a,b \geq 0$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Minguzzi, in [140], proved a reverse Young's inequality in the following way:

(1.4.10) $$0 \leq \frac{a^p}{p}+\frac{b^q}{q}-ab \leq \left(b-a^{p-1}\right)\left(b^{q-1}-a\right),$$

for all $a,b \geq 0$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

This inequality is equivalent to the following inequality, for $p=\frac{1}{u}, a=x^u, b=x^{\frac{u}{u-1}}$:

If $a,b>0$ and $p\in(0,1)$, we change p and $\frac{1}{p}$, a by a^p and b by b^{1-p}, then the Young inequality becomes:

(1.4.11) $$a^p b^{1-p} \leq pa+(1-p)b,$$

But, this is true, when $a,b>0$ and $p\in[0,1]$.

Especially, when we talk about Young's inequality, we will refer to the last form.

Next, we present some refinements and some reverse inequalities of Young's inequality, which we have used in our research.

One of reverse inequalities for Young inequality was given by Tominaga in [200], using the Specht ratio, in the following way

(1.4.12) $$pa+(1-p)b \leq S\left(\frac{a}{b}\right)a^p b^{1-p},$$

for positive real numbers a, b and $p\in[0,1]$, where the Specht ratio [78, 93] was defined by

$$S(h)\equiv\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}, h\neq 1,$$

for a positive real number h.

Note that $\lim\limits_{h\to 1} S(h)=1$ and $S(h) = S(1/h) > 1$ for $h\neq 1, h>0$. We call the inequality (1.4.12) a ratio-type reverse inequality for Young's inequality.

Tominaga also showed in [200] the following inequality:

(1.4.13) $$pa+(1-p)b \leq L(a,b)\log S\left(\frac{a}{b}\right)+a^p b^{1-p},$$

for positive real numbers a,b and $p\in[0,1]$, where the logarithmic mean [26] $L(x,y)$ is defined by

$$L(x,y) \equiv \frac{x-y}{\log x - \log y}, (x \neq y),\ L(x,x) = x.$$

We call the inequality (1.4.13) a difference-type reverse inequality for Young's inequality. Based on the scalar inequalities (1.4.12) and (1.4.13), Tominaga showed two reverse inequalities for invertible positive operators.

In [Furuichi-Minculete, 76], we presented two inequalities which give two different reverse inequalities for Young's inequality, namely:

$$(1.4.14) \qquad 0 \le \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \le a^{\lambda}b^{1-\lambda} \exp\left\{\frac{\lambda(1-\lambda)(a-b)^2}{m^2}\right\} - a^{\lambda}b^{1-\lambda},$$

and

$$(1.4.15) \qquad 0 \le \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \le \lambda(1-\lambda)\left\{\log\frac{a}{b}\right\}^2 M,$$

where $a,b>0$, $m = \min\{a,b\}, M = \max\{a,b\}$, for all $\lambda \in [0,1]$.

The above results are the particular cases of the following theorem from [Furuichi-Minculete, 76]:

Theorem 1.4.3. *Let $f:[a,b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constant M so that $0 \le f'' \le M$, for $x \in [a,b]$. Then the following inequalities hold:*

$$(1.1.16) \qquad 0 \le \lambda f(a) + (1-\lambda)f(b) - f(\lambda a + (1-\lambda)b) \le M\lambda(1-\lambda)(b-a)^2$$

for all $\lambda \in [0,1]$.

For $n=2$, Cartwright-Field's inequality (see e. g. [30]) may be written as follows:

$$(1.4.17) \qquad \frac{\lambda(1-\lambda)}{2M}(b-a)^2 \le \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \le \frac{\lambda(1-\lambda)}{2m}(b-a)^2,$$

where $a,b>0$, $m = \min\{a,b\}, M = \max\{a,b\}$, for all $\lambda \in [0,1]$. This inequality is an improvement of Young's inequality and, at the same time, gives a reverse inequality for Young inequality.

Remark 1.4.3. The first inequality of (1.4.17) clearly gives an improvement of the first inequality in (1.4.15) and (1.4.16). For $0 < a, b < 1$, we find the right hand side of the second inequality of (1.4.17) gives tighter upper bound than that of (1.4.16), from the inequality $\frac{x-y}{\log x - \log y} < \frac{x+y}{2}$, for $x, y > 0$. For $a, b > 1$, we find the right hand side of the second inequality of (1.4.15) gives tighter upper bound than that of (1.4.17), from the inequality $\frac{x-y}{\log x - \log y} < \frac{x+y}{2}$, for $x, y > 0$. In addition, we find the right hand side of the second inequality of (1.4.17) gives tighter upper bound than that of (1.4.15) for $a,b>0$, from $e^x > 1+x$.

Next, we focus on two immediate particular cases of Theorem 1.3 (Minculete-Mitroi, [145]) that help us to give improvements of the well known arithmetic-geometric mean inequality (also known as Young's inequality).

1) We apply relation 1.4.17 to the function $f:[a,b] \to \mathbb{R}$ ($a > 0$) defined by $f(x) = -\log x$, which leads to

$$(1.4.18) \qquad \exp\left(\frac{p(1-p)(a-b)^2}{2b^2}\right) \le \frac{pa+(1-p)b}{a^p b^{1-p}} \le \exp\left(\frac{p(1-p)(a-b)^2}{2a^2}\right),$$

Since $exp\left(\frac{p(1-p)(a-b)^2}{2b^2}\right) \geq 1$, we obtain a refinement of Young's inequality, where $p \in [0,1]$.

We also obtained a reverse inequality for Young's inequality.
2) Next, we apply relation 1.4.17 to the function $f:[log\,b, log\,a] \to \mathbb{R}$, defined by $f(x) = exp(x)$, and we arrive at the following inequality:

(1.4.19) $$\frac{p(1-p)}{2} b\,log^2\left(\frac{a}{b}\right) \leq pa + (1-p)b - a^p b^{1-p} \leq \frac{p(1-p)}{2} a\,log^2\left(\frac{a}{b}\right),$$

where $0 < b \leq a$ and $p \in [0,1]$.

Young's inequality was refined by Kittaneh and Manasrah, in [116], thus

(1.4.20) $$pa + (1-p)b \geq a^p b^{1-p} + r\left(\sqrt{a} - \sqrt{b}\right)^2,$$

where $p \in [0,1]$ and $r = min\{p, 1-p\}$. They use this inequality for the study of matrix norm inequalities.

In [78], Furuichi improves inequality (1.4.11) thus

(1.4.21) $$pa + (1-p)b \geq S\left(\left(\frac{a}{b}\right)^r\right) a^p b^{1-p},$$

where $p \in [0,1]$ and $r = min\{p, 1-p\}$ and the function S was given above.

Kober proved in [119] a general result related to an improvement of the inequality between arithmetic and geometric means, which for $n = 2$ implies the inequality:

(1.4.22) $$a^p b^{1-p} + r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq pa + (1-p)b \leq a^p b^{1-p} + (1-r)\left(\sqrt{a} - \sqrt{b}\right)^2,$$

where $p \in [0,1]$ and $r = min\{p, 1-p\}$. This inequality was rediscovered by Kittaneh and Manasrah, in [116].

A generalization of inequality (1.4.11) can be found in a paper of Aldaz [11].

In [Minculete, 151], we present other improvement of Young's inequality and a reverse inequality as follows

(1.4.23) $$a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} \leq pa + (1-p)b \leq a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)},$$

for the positive real numbers a, b and $p \in [0,1]$ and $r = min\{p, 1-p\}$.

This inequality can be presented with Kantorovich constant:

(1.4.24) $$K^r(h,2) a^p b^{1-p} \leq pa + (1-p)b \leq K^{1-r}(h,2) a^p b^{1-p},$$

where a, $b>0$, $p \in [0,1]$, $r = min\{p, 1-p\}$, $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{b}{a}$. Notice that the first inequality in (1.4.24) was obtained by Zou *et al.* in [211] while the second was obtained by Liao *et al.* [124].

Finally, we gave, in [Minculete, 152], another improvement of Young's inequality and a reverse inequality, given as:

(1.4.25)
$$r\left(\sqrt{a} - \sqrt{b}\right)^2 + A(p)log^2\left(\frac{a}{b}\right) \leq pa + (1-p)b - a^p b^{1-p} \leq (1-r)\left(\sqrt{a} - \sqrt{b}\right)^2 + B(p)log^2\left(\frac{a}{b}\right),$$

where $a, b \geq 1$, $p \in (0,1)$, $r = min\{p, 1-p\}$,

$$A(p)=\frac{p(1-p)}{2}-\frac{r}{4} \text{ and } B(p)=\frac{p(1-p)}{2}-\frac{1-r}{4}.$$

Remark 1.4.4. a). Since $A(p)=\frac{p(1-p)}{2}-\frac{r}{4}\geq 0$ and $B(p)=\frac{p(1-p)}{2}-\frac{1-r}{4}\leq 0$, we obtain a refinement of the Kittaneh-Manasrah inequality and a refinement of Young's inequality.
b) Inequalities (1.4.18) and (1.4.19) give two improvements of Young's inequality.
c) Inequality (1.4.19) can be found in [Minculete-Mitroi, 145] and in many other paper of Dragomir (see e. g. [53]).
d) For $p\to 1-p$ in (1.4.19) we obtain

(1.4.26) $$\frac{p(1-p)}{2}b\,log^2\left(\frac{a}{b}\right)\leq (1-p)a+pb-a^{1-p}b^p\leq \frac{p(1-p)}{2}a\,log^2\left(\frac{a}{b}\right),$$

where $0<b\leq a$ and $p\in[0,1]$. By the sum of relations (1.4.19) and (1.4.25), we deduce

(1.4.27) $$\frac{p(1-p)}{2}b\,log^2\left(\frac{a}{b}\right)\leq \frac{a+b}{2}-\frac{a^p b^{1-p}+a^{1-p}b^p}{2}\leq \frac{p(1-p)}{2}a\,log^2\left(\frac{a}{b}\right).$$

The Heinz mean [83] is defined as

$$H_p(a,b)=\frac{a^p b^{1-p}+a^{1-p}b^p}{2},$$

where $0\leq a,b$ and $p\in[0,1]$. It is easy to see that

$$\sqrt{ab}\leq H_p(a,b)\leq \frac{a+b}{2}.$$

But, using relation (1.4.27), we have

$$\frac{p(1-p)}{2}b\,log^2\left(\frac{a}{b}\right)\leq \frac{a+b}{2}-\frac{a^p b^{1-p}+a^{1-p}b^p}{2}\leq \frac{p(1-p)}{2}a\,log^2\left(\frac{a}{b}\right),$$

so, we deduce

$$\frac{p(1-p)}{2}b\,log^2\left(\frac{a}{b}\right)\leq A(a,b)-H_p(a,b)\leq \frac{p(1-p)}{2}a\,log^2\left(\frac{a}{b}\right).$$

From inequality (1.4.25), we deduce another inequality for the Heinz mean, thus:
(1.4.28)

$$r\left(\sqrt{a}-\sqrt{b}\right)^2+A(p)log^2\left(\frac{a}{b}\right)\leq A(a,b)-H_p(a,b)\leq (1-r)\left(\sqrt{a}-\sqrt{b}\right)^2+B(p)log^2\left(\frac{a}{b}\right),$$

where $a,b\geq 1$, $p\in(0,1)$, $r=min\{p,1-p\}$,

$$A(p)=\frac{p(1-p)}{2}-\frac{r}{4}=A(1-p) \text{ and } B(p)=\frac{p(1-p)}{2}-\frac{1-r}{4}=B(1-p).$$

Next, we make a little synthesis of some recent results about Young's inequality.

In the recent paper [209], Zhao and Wu provided two refining terms of Young's inequality, thus:
Let $a,b\geq 0$ and $p\in[0,1]$.
(i) If $p\in\left[0,\frac{1}{2}\right]$, then

(1.4.40) $$(1-p)a+pb\geq a^{1-p}b^p+p\left(\sqrt{a}-\sqrt{b}\right)^2+r_0\left(\sqrt{a}-\sqrt[4]{ab}\right)^2,$$

(ii) If $p\in\left[\frac{1}{2},1\right]$, then

(1.4.41) $$(1-p)a+pb\geq a^{1-p}b^{p}+(1-p)\left(\sqrt{a}-\sqrt{b}\right)^{2}+r_0\left(\sqrt{b}-\sqrt[4]{ab}\right)^{2},$$

where $r=min\{p,1-p\}$ and $r_0=min\{2r,1-2r\}$.

In the same paper, we find the reverse versions of above inequalities:
Let $a,b\geq 0$ and $p\in[0,1]$.

(i) If $p\in\left[0,\frac{1}{2}\right]$, then

(1.4.42) $$(1-p)a+pb\leq a^{1-p}b^{p}+(1-p)\left(\sqrt{a}-\sqrt{b}\right)^{2}-r_0\left(\sqrt{b}-\sqrt[4]{ab}\right)^{2},$$

(ii) If $p\in\left[\frac{1}{2},1\right]$, then

(1.4.43) $$(1-p)a+pb\leq a^{1-p}b^{p}+p\left(\sqrt{a}-\sqrt{b}\right)^{2}-r_0\left(\sqrt{a}-\sqrt[4]{ab}\right)^{2},$$

where $r=min\{p,1-p\}$ and $r_0=min\{2r,1-2r\}$.

Quite recently, in [194], Sababheh and Moslehian gave a full description of all other refinements of the reverse Young's inequality, thus:
Let $a,b\geq 0$ and $p\in[0,1]$.

(i) If $p\in\left[0,\frac{1}{2}\right]$, then

(1.4.44) $$(1-p)a+pb\leq a^{1-p}b^{p}+(1-p)\left(\sqrt{a}-\sqrt{b}\right)^{2}-S_n\left(2p,\sqrt{ab},b\right),$$

(ii) If $p\in\left[\frac{1}{2},1\right]$, then

(1.4.45) $$(1-p)a+pb\leq a^{1-p}b^{p}+p\left(\sqrt{a}-\sqrt{b}\right)^{2}-S_n\left(2(1-p),\sqrt{ab},a\right),$$

where $[x]$ is the greatest integer less than or equal to x and

$$S_n(p,a,b)=\sum_{k=1}^{n}s_k(p)\left(\sqrt[2^k]{b^{2^{k-1}-j_k(p)}a^{j_k(p)}}-\sqrt[2^k]{b^{2^{k-1}-j_k(p)-1}a^{j_k(p)+1}}\right),\ j_k(p)=\left[2^{k-1}p\right],$$
$$r_k(p)=\left[2^k p\right],\ s_k(p)=(-1)^{r_k(p)}2^{k-1}p+(-1)^{r_k(p)+1}\left[\frac{r_k(p)+1}{2}\right].$$

Furuichi, Ghaemi and Gharakhanlu gave in [83] a reverse Young's inequality for
$p\in\mathbb{R}$, namely: Let $a,b\geq 0$, $n\in\mathbb{N}$ such that $n\geq 2$ and $\frac{1}{2}\neq p\in\mathbb{R}$. Then,

(i) If $p\notin\left[\frac{1}{2},\frac{2^{n-1}+1}{2^n}\right]$, then

(1.4.46) $$(1-p)a+pb\leq a^{1-p}b^{p}+(1-p)\left(\sqrt{a}-\sqrt{b}\right)^{2}+(2p-1)\sqrt{ab}\sum_{k=2}^{n}2^{k-2}\left(\sqrt[2^k]{\frac{b}{a}}-1\right)^{2},$$

(i) If $p\notin\left[\frac{2^{n-1}-1}{2^n},\frac{1}{2}\right]$, then

(1.4.47) $$(1-p)a+pb\leq a^{1-p}b^{p}+p\left(\sqrt{a}-\sqrt{b}\right)^{2}+(1-2p)\sqrt{ab}\sum_{k=2}^{n}2^{k-2}\left(\sqrt[2^k]{\frac{a}{b}}-1\right)^{2}.$$

1.5 Grüss-type inequalities in discrete form and in integral form

In this section we prove an inequality which will helps us find a new refinement of the discrete version of Grüss inequality. We have also continued the research in this field and we show some inequalities that have been obtained ([Minculete-Rațiu-Pečarić,143], [Minculete-Ciurdariu, 149]).

The discrete version of Grüss inequality [32, 110, 113] has the following form:

$$\left|\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i\right| \leq \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2),$$

where x_i, y_i are real numbers so that $\gamma_1 \leq x_i \leq \Gamma_1$ and $\gamma_2 \leq y_i \leq \Gamma_2$ for all $i = \overline{1,n}$.

In 1935, Grüss (see [98]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions:

Let f and g be two bounded functions defined on $[a,b]$ *with* $\gamma_1 \leq f(x) \leq \Gamma_1$ *and* $\gamma_2 \leq g(x) \leq \Gamma_2$, *where* $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ *are four constants. Then, we have:*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2)$$

and the inequality is sharp, in the sense that the constant $1/4$ *can't be replaced by a smaller one.*

After the number of papers published there can be noticed a great interest for this inequality. It is well known that an important resource for studying inequalities is [4, 155, 193]. In [181], Peng and Miao established a form of inequality of Gruss type for functions whose first and second derivatives are absolutely continuous and the third derivative is bound. Also, in [59], Dragomir presented several integral inequalities of Gruss type, and in [60], he showed some Gruss type inequalities in inner product spaces and applications for the integral. Another improvement of Gruss inequality was obtained by Mercer in [136]. Moreover, in [125], a Gruss type inequality was used in order to obtain some sharp Ostrowski-Gruss type inequalities by Liu.

Kechriniotis and Delibasis showed in [113] several refinements of Gruss inequality in inner product spaces using Kurepa's results for Gramians. New generalizations of the inequality of Gruss were presented in [47] using Riemann-Liouville fractional integrals. Cerone and Dragomir studied in [32] some refinements of Gruss' inequality.

As applicable, we obtain some properties of bounds of the variance, the standard deviation, the coefficient of variation and of the covariance related to several statistical indicators for discrete random variables in finite case ([Minculete-Rațiu-Pečarić,143], [Minculete-Ciurdariu, 149]).

1.5.1 A refinement of Grüss's inequality via Cauchy–Schwarz's inequality for discrete random variables in finite case

The *variance* of a random variable $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1 \le i \le n}$ with probabilities $P(X = x_i) = p_i = \frac{1}{n}$ for any $i = \overline{1,n}$ is its second central moment, the expected value of the squared deviation from mean $\mu_X = E[X] = \frac{1}{n}\sum_{i=1}^{n} x_i$:

$$Var(X) = E\left[(X - \mu_X)^2\right] = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_X)^2 .$$

The expression for the variance can be thus expanded:

$$Var(X) = E\left[X^2\right] - E^2[X].$$

We note by **RV** the set of random variables $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1 \le i \le n}$ with probabilities $P(X = x_i) = p_i = \frac{1}{n}$ for any $i = \overline{1,n}$.

The *covariance* is a measure of how much two random variables change together at the same time and is defined as

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])],$$

and is equivalent to the form

$$Cov(X,Y) = E[XY] - E[X]E[Y].$$

Using the inequality of Cauchy-Schwarz for discrete random variables we find the inequality given by

$$|Cov(X,Y)|^2 \le Var(X)Var(Y)$$

or in the form

$$|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)} .$$

Next, we show a refinement of this inequality.

Lemma 1.5.1. *If X and Y are discrete random variables in finite case, then there is the following equality*

(1.5.1) $$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X,Y),$$

where a and b are real numbers.

Corollary 1.5.2. *If X and Y are discrete random variables in finite case, then there are the following equalities:*

(1.5.2) $$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

and

(1.5.3) $$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X,Y) .$$

Remark 1.5.3. From relations (1.5.1) and (1.5.2), we find the parallelogram law in terms of variance, namely:

(1.5.4) $$Var(X + Y) + Var(X - Y) = 2Var(X) + 2Var(Y) .$$

Lemma 1.5.4. *If X , Y, Z and T are discrete random variables in finite case, then there is the following equality*

(1.5.5)

$$Cov(aX + bY, cZ + dT) = acCov(X,Z) + adCov(X,T) + bcCov(Y,Z) + bdCov(Y,T),$$

where a, b, c and d are real numbers.

Theorem 1.5.5 (Minculete-Rațiu-Pečarić,[143]). *If X, Y and Z are discrete random variables in finite case, with* $X \neq kZ$*, then we have the inequality*

$$(1.5.6)\quad 0 \le \frac{[Cov(X,Y)Cov(X,Z) - Cov(Y,Z)Var(X)]^2}{Var(X)Var(Z) - [Cov(X,Z)]^2} \le Var(X)Var(Y) - [Cov(X,Y)]^2.$$

Proof. For the discrete random variables *X, Y* and *Z* given in finite case, with $Var(X) \neq 0$, we take the following random variable: $W = \frac{Cov(X,Y) + \lambda Cov(X,Z)}{Var(X)} X - Y - \lambda Z$. We calculate the variance of random variable *W*, thus: $Var(W) = Var\left(\left(\frac{Cov(X,Y)}{Var(X)} X - Y\right) - \lambda\left(\frac{Cov(X,Z)}{Var(X)} X - Z\right)\right)$ and applying relation (1.5.1), we have

$$Var(W) = Var\left(\frac{Cov(X,Y)}{Var(X)} X - Y\right) + \lambda^2 Var\left(\frac{Cov(X,Z)}{Var(X)} X - Z\right) -$$

$$- 2\lambda Cov\left(\frac{Cov(X,Y)}{Var(X)} X - Y, \frac{Cov(X,Z)}{Var(X)} X - Z\right) =$$

$$= Var(Y) - \frac{[Cov(X,Y)]^2}{Var(X)} + \lambda^2\left(Var(Z) - \frac{[Cov(X,Z)]^2}{Var(X)}\right) -$$

$$- 2\lambda Cov\left(\frac{Cov(X,Y)}{Var(X)} X - Y, \frac{Cov(X,Z)}{Var(X)} X - Z\right).$$

Using Lemma 1.5.4, we deduce the following inequality

$$Cov\left(\frac{Cov(X,Y)}{Var(X)} X - Y, \frac{Cov(X,Z)}{Var(X)} X - Z\right) = \frac{Cov(X,Y)Cov(X,Z)}{Var(X)Var(X)} Cov(X,X) -$$

$$- \frac{Cov(X,Y)Cov(X,Z)}{Var(X)} - \frac{Cov(X,Z)Cov(X,Y)}{Var(X)} + Cov(Y,Z) =$$

$$= Cov(Y,Z) - \frac{Cov(X,Y)Cov(X,Z)}{Var(X)}.$$

Returning to calculate the variance for random variable *W*, we have

$$Var(W) = Var(Y) - \frac{[Cov(X,Y)]^2}{Var(X)} + \lambda^2\left(Var(Z) - \frac{[Cov(X,Z)]^2}{Var(X)}\right) -$$

$$- 2\lambda\left(Cov(Y,Z) - \frac{Cov(X,Y)Cov(X,Z)}{Var(X)}\right).$$

Therefore, we deduce the equality

$$Var(X)Var(W) = Var(X)Var(Y) - [Cov(X,Y)]^2 + \lambda^2\left(Var(X)Var(Z) - [Cov(X,Z)]^2\right) -$$

$$- 2\lambda\left(Var(X)Cov(Y,Z) - Cov(X,Y)Cov(X,Z)\right).$$

Since $Var(X)Var(W) \ge 0$, it follows that

$$\lambda^2\left(Var(X)Var(Z) - [Cov(X,Z)]^2\right) - 2\lambda\left(Var(X)Cov(Y,Z) - Cov(X,Y)Cov(X,Z)\right) +$$

$$+ Var(X)Var(Y) - [Cov(X,Y)]^2 \ge 0$$

for every $\lambda \in R$.

This implies that

(1.5.7) $$\begin{aligned}&\left(Var(X)Var(Z)-[Cov(X,Z)]^2\right)\left(Var(X)Var(Y)-[Cov(X,Y)]^2\right)\\&\geq\left(Var(X)Cov(Y,Z)-Cov(X,Y)Cov(X,Z)\right)^2\end{aligned}.$$

Taking into account that $Var(X)Var(Z)-[Cov(X,Z)]^2\neq 0$, because $X\neq kZ$ and dividing by $Var(X)Var(Z)-[Cov(X,Z)]^2$, we obtain the inequality of the statement.

□

Remark 1.5.6. *Let X, Y and Z be discrete random variables in finite case, with $Var(Y)\neq 0$ and $Var(Z)\neq 0$, if we take the following random variable: $W=X-\dfrac{Cov(X,Y)}{Var(Y)}Y-\lambda Z$, then we have the inequality*

(1.5.8) $$0\leq\frac{[Cov(X,Y)Cov(Y,Z)-Cov(X,Z)Var(Y)]^2}{Var(Y)Var(Z)}\leq Var(X)Var(Y)-|Cov(X,Y)|^2.$$

Let $x_1,x_2,\dots,x_n$ be real numbers, assume $\gamma_1\leq x_i\leq\Gamma_1$ for all $i=\overline{1,n}$ and the average $\mu_X=\frac{1}{n}\sum_{i=1}^n x_i$.

In 1935, Popoviciu (see e.g. [20, 84]) proved the following inequality

(1.5.9) $$Var(X)=\frac{1}{n}\sum_{i=1}^n(x_i-\mu_X)^2\leq\frac{1}{4}(\Gamma_1-\gamma_1)^2.$$

The discrete version of Grüss inequality has the following form:

(1.5.10) $$\left|\frac{1}{n}\sum_{i=1}^n x_iy_i-\frac{1}{n}\sum_{i=1}^n x_i\frac{1}{n}\sum_{i=1}^n y_i\right|\leq\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2),$$

where x_i,y_i are real numbers so that $\gamma_1\leq x_i\leq\Gamma_1$ and $\gamma_2\leq y_i\leq\Gamma_2$ for all $i=\overline{1,n}$.

From the relation $Cov(X,Y)=E[XY]-E[X]E[Y]=\frac{1}{n}\sum_{i=1}^n x_iy_i-\frac{1}{n}\sum_{i=1}^n x_i\frac{1}{n}\sum_{i=1}^n y_i$ and using the inequality of Cauchy-Schwarz for discrete random variables given by $|Cov(X,Y)|\leq\sqrt{Var(X)Var(Y)}$, we obtain a proof for Grüss's inequality.

Bhatia and Davis show in [20] that the following inequality

(1.5.11) $$Var(X)=\frac{1}{n}\sum_{i=1}^n(x_i-\mu_X)^2\leq(\Gamma_1-\mu_X)(\mu_X-\gamma_1).$$

The inequality of Bhatia and Davis represents an improvement of Popoviciu's inequality, because $(\Gamma_1-\gamma_1)^2\geq 4(\Gamma_1-\mu_X)(\mu_X-\gamma_1)$. Therefore, we will first have an improvement of Grüss's inequality given by the following relation:

(1.5.12)

$$\left|\frac{1}{n}\sum_{i=1}^n x_iy_i-\frac{1}{n}\sum_{i=1}^n x_i\frac{1}{n}\sum_{i=1}^n y_i\right|\leq\sqrt{(\Gamma_1-\mu_X)(\mu_X-\gamma_1)(\Gamma_2-\mu_Y)(\mu_Y-\gamma_2)}\leq\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).$$

If X, Y and Z are discrete random variables in finite case, with $X\neq kZ$, then we have from inequality (1.5.8) the following relation:

(1.5.13) $$[Cov(X,Y)]^2+\frac{[Cov(X,Y)Cov(X,Z)-Cov(Y,Z)Var(X)]^2}{Var(X)Var(Z)-[Cov(X,Z)]^2}\leq Var(X)Var(Y).$$

Let $x_1,x_2,\dots,x_n$, $y_1,y_2,\dots,y_n$, $z_1,z_2,\dots,z_n$, be real numbers, assume $x_i\neq kz_i$ for all $i=\overline{1,n}$ and for any real number k. Then applying inequality (1.5.13) we deduce a second refinement of Grüss's inequality given by

(1.5.14)

$$\left[\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}y_i\right]^2+S\le\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}y_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)^2\right),$$

where $S=\frac{[A-B]^2}{C}$ with

$$A=\left(\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}y_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}x_iz_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}z_i\right),$$

$$B=\left(\frac{1}{n}\sum_{i=1}^{n}y_iz_i-\frac{1}{n}\sum_{i=1}^{n}y_i\frac{1}{n}\sum_{i=1}^{n}z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)$$

and

$$C=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}z_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}z_i\right)^2\right)-\left(\frac{1}{n}\sum_{i=1}^{n}x_iz_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}z_i\right)^2.$$

Remark 1.5.7. In [113], Kechriniotis and Delibasis demonstrated other refinements of the discrete version of Grüss inequality.

Corollary 1.5.8 (Minculete-Raţiu-Pečarić,[143]). *If X and Y are discrete random variables in finite case, then there is the following inequality*

(1.5.15) $$\sqrt{Var(X+Y)}\le\sqrt{Var(X)}+\sqrt{Var(Y)}$$

Remark 1.5.9. This inequality in terms of sums becomes

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i-\mu_X-\mu_Y)^2}\le\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\mu_Y)^2}$$

Dividing by $\sqrt{\frac{1}{n}}$ and making the following substitutions: $x_i-\mu_X=a_i$ and $y_i-\mu_Y=b_i$, we obtain the inequality

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2}\le\sqrt{\sum_{i=1}^{n}a_i^2}+\sqrt{\sum_{i=1}^{n}b_i^2}$$

which is in fact *the Minkowski inequality*, in the case $\sum_{i=1}^{n}a_i=0$ and $\sum_{i=1}^{n}b_i=0$.

Corollary 1.5.10. *If X and Y are discrete random variables in finite case, then there is the following inequality*

(1.5.16) $$\sqrt{Var(X-Y)}\ge\left|\sqrt{Var(X)}-\sqrt{Var(Y)}\right|$$

Proof. From relation (1.5.3), we have

$$Var(X-Y)=Var(X)+Var(Y)-2Cov(X,Y)$$
$$=\left(\sqrt{Var(X)}-\sqrt{Var(Y)}\right)^2+2\left(\sqrt{Var(X)Var(Y)}-Cov(X,Y)\right).$$

Applying the inequality of Cauchy-Schwarz for discrete random variables, we obtain

$$Var(X-Y)\ge\left(\sqrt{Var(X)}-\sqrt{Var(Y)}\right)^2$$

which implies the inequality of the statement.

□

In [126, 127], the *Lukaszyk–Karmowski metric* is a function defining a distance between two random variables or two random vectors. In case the random variables X and Y are characterized by discrete probability distribution, the Lukaszyk–Karmowski metric D is defined as:

$$D(X,Y)=\sum_i\sum_j |x_i-y_j|P(X=x_i)P(Y=y_i).$$

Next, we will give another metric for the set **RV**. We can look the set **RV** as a vector space. The natural way is by introducing and using the standard inner product on **RV**. The inner product of any two random variables X and Y is defined by

$$\langle X,Y\rangle = Cov(X,Y).$$

The inner product of X with itself is always non-negative. This product allows us to define the "length" of a random variable X through square root:

$$\|X\|=\sqrt{\langle X,X\rangle}=\sqrt{Cov(X,X)}=\sqrt{Var(X)}.$$

This length function satisfies the required properties of a seminorm and is called the *Euclidean seminorm* on **RV**. A seminorm allowed assigning zero length to some non-zero vectors. The set **RV** with this seminorm is called *seminormed vector space*. Finally, one can use the norm to define a metric on **RV** by

$$d(X,Y)=\|X-Y\|=\sqrt{Var(X-Y)}.$$

This distance function is called the *Euclidean metric* on **RV**. Consequently, the set of random variables **RV** form a Hilbert space, and a seminormed vector space.

Some of the previous results were mentioned in the paper [133], where Masjed-Jamei and Omey explore the properties of the covariance leading to new classes of inequalities including the Ostrowski and Ostrowski-Grüss inequalities.

1.5.2 About the bounds of several statistical indicators

Statistical indicators play a very important role in the characterization of the various processes: economic, social and technological. In statistics, by the general notion of scattering (variance or dispersion) we refer to the individual values of measurable deviations from the central value.

Next, we will obtain some properties of bounds of the variance, the standard deviation, the coefficient of variation and of the covariance related to several statistical indicators for discrete random variables in finite case. The results are developments of the research presented in (Minculete-Raţiu-Pečarić,[143]).

The *weighted arithmetic mean* (mean value) of a random variable $X=\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1\le i\le n}$ with probabilities $P(X=x_i)=p_i$ for any $i=\overline{1,n}$ and $\sum_{i=1}^n p_i=1$ is given by $\overline{X}=E[X]=\sum_{i=1}^n p_i x_i$.

The *variance* of a random variable $X=\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1\le i\le n}$ with probabilities $P(X=x_i)=p_i$ for any $i=\overline{1,n}$ and $\sum_{i=1}^{n} p_i = 1$ is its second central moment, the expected value of the squared deviation from mean $\overline{X}$:

$$\sigma_{\overline{X}}^2 = Var(X) = E\left[\left(X-\overline{X}\right)^2\right] = \sum_{i=1}^{n} p_i\left(x_i-\overline{X}\right)^2 .$$

Standard deviation ($\sigma_{\overline{X}}$) has a similar role with average linear deviation, but keeping the dispersion characteristics; statistics used this indicator which is calculated as mean of individual deviations squared from their central tendency, and the interval ($\overline{X}-\sigma_{\overline{X}}$, $\overline{X}-\sigma_{\overline{X}}$) is the medium interval of variation, where we have $\sigma_{\overline{X}}=\sqrt{Var(X)}$. *Coefficient of variation* (*Cv*(*X*)) is a relative measure of scattering, which describes the ratio between the standard deviation and the arithmetic mean, and is given by the formula:

$$C_V(X)=\frac{\sigma_{\overline{X}}}{\overline{X}}=\frac{\sqrt{Var(X)}}{E[X]} .$$

Two variables have a strong statistical relationship with one another if they appear to move together. According to [69], correlation is a measure of a linear relationship between two variables, *X* and *Y*, and is measured by the *correlation coefficient*, given by:

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} .$$

It is easy to see that $-1\le \rho(X,Y)\le 1$.

There is the following inequality:

(1.5.2.1) $$p_1\left(x_1-\overline{X}\right)^2+p_2\left(x_2-\overline{X}\right)^2+...+p_n\left(x_n-\overline{X}\right)^2\le \frac{1}{4}(M-m)^2,$$

where $M=max\{x_1,x_2,...,x_n\}$ and $m=min\{x_1,x_2,...,x_n\}$. For $p_i=\frac{1}{n}$ with $i=\overline{1,n}$ and $\sum_{i=1}^{n} p_i=1$, we deduce Popoviciu's inequality:

$$\frac{1}{n}\left[\left(x_1-\overline{X}\right)^2+\left(x_2-\overline{X}\right)^2+...+\left(x_n-\overline{X}\right)^2\right]\le \frac{1}{4}(M-m)^2 .$$

This inequality suggests an uper bound for indicators for the variance, the standard deviation, the coefficient of variation and of the covariance, thus:

$$\sigma_{\overline{X}}^2\le \frac{1}{4}(M-m)^2, \sigma_{\overline{X}}\le \frac{1}{2}(M-m), C_V(X)\le \frac{M-m}{2\overline{X}}$$

and

(1.5.2.2) $$|Cov(X,Y)|\le \frac{1}{4}(M-m)(Q-q),$$

where $Q=max\{y_1,y_2,...,y_n\}$, and $q=min\{y_1,y_2,...,y_n\}$.

The discrete version of Grüss inequality in the weighted form has the following form:

$$\left|\sum_{i=1}^{n} p_i x_i y_i - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i y_i\right| \le \frac{1}{4}(M-m)(Q-q),$$

where x_i, y_i are real numbers so that $m \le x_i \le M$ and $q \le y_i \le Q$ for all $i = \overline{1,n}$.

The integral variant of inequality of Grüss [98], besides applications in mathematical analysis, has some statistical and actuarial applications. We known that, the discrete version of Grüss inequality has the following form:

$$\left|\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i\right| \le \frac{1}{4}(M-m)(Q-q),$$

where x_i, y_i are real numbers so that $m \le x_i \le M$ and $q \le y_i \le Q$ for all $i = \overline{1,n}$.

There are many articles which treated this inequality in integral variant (see e.g. [4], [59], [60], [110], [136]). We will focus attention on the discrete version of Grüss inequality, being motivaded by usefulness of this inequality, we study the inequality of Grüss in the context of elements of statistics, using the concepts of variance and covariance for the random variables.

Bhatia and Davis show in [20], for $p_i = \frac{1}{n}$, with $i = \overline{1,n}$, that:

$$\sigma_{\overline{X}}^2 \le \left(M - \overline{X}\right)\left(\overline{X} - m\right).$$

But, the inequality of Bhatia and Davis remains valid for any p_i with $\sum_{i=1}^{n} p_i = 1$.

Thus, we deduce upper bounds better than in relation (1.5.2.2), thus:

$$\sigma_{\overline{X}}^2 \le \left(M - \overline{X}\right)\left(\overline{X} - m\right),\ \sigma_{\overline{X}} \le \sqrt{\left(M - \overline{X}\right)\left(\overline{X} - m\right)},\ C_V(X) \le \frac{\sqrt{\left(M - \overline{X}\right)\left(\overline{X} - m\right)}}{\overline{X}}$$

and

(1.5.2.3) $$|Cov(X,Y)| \le \sqrt{\left(M - \overline{X}\right)\left(\overline{X} - m\right)\left(Q - \overline{Y}\right)\left(\overline{Y} - q\right)}.$$

It has been shown [136] by A. McD. Mercer that for a discrete random variables in finite case, we have:

(1.5.2.4) $$\sigma_{\overline{X}}^2 \le 2M\left(\overline{X} - \overline{X}_h\right),$$

where $\overline{X_h} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$ is *the harmonic mean* for discrete random variables in finite case.

From [Minculete, 144] by replacement with the correlation coefficient in inequality (1.5.5), we deduce the inequality:

$$\left[1 - \rho^2(X,Y)\right]\left[1 - \rho^2(X,Z)\right] \ge \left(\rho(Y,Z) - \rho(X,Y)\rho(X,Z)\right)^2.$$

Next, we will present several improvements of the above inequalities related to variance.

Proposition 1.5.2.1. *For a discrete random variable in finite case X there is the following inequality*

(1.5.2.5) $$2m\left(\overline{X} - \overline{X}_g\right) \le \sigma_{\overline{X}}^2 \le 2M\left(\overline{X} - \overline{X}_g\right),$$

*where the **geometric mean** $\left(\overline{X}_g\right)$ is that value which shows that if we replace each individual value, their product would not change and we have the formula:* $\overline{X}_g = x_1^{p_1} \cdot x_2^{p_2} \cdot \ldots \cdot x_n^{p_n}$ *with* $\sum_{i=1}^{n} p_i = 1$.

Proof. In the paper [30], Cartwright and Field proved the following inequality:

$$\frac{1}{2M}\sum_{i=1}^{n} p_i\left(x_i - \sum_{i=1}^{n} p_i x_i\right)^2 \le \sum_{i=1}^{n} p_i x_i - \prod_{i=1}^{n} x_i^{p_i} \le \frac{1}{2m}\sum_{i=1}^{n} p_i\left(x_i - \sum_{i=1}^{n} p_i x_i\right)^2,$$

where $p_i > 0, (\forall) i = \overline{1,n}$ and $\sum_{i=1}^{n} p_i = 1$. But

$$\sum_{i=1}^{n} p_i\left(x_i - \sum_{i=1}^{n} p_i x_i\right)^2 = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 = \sigma_{\overline{X}}^2,$$

which implies to the inequality of the statement.

□

Remark 1.5.2.2. a) For $p_i = \frac{1}{n}, (\forall) i = \overline{1,n}$, in this inequality, we obtain

$$\frac{1}{2M}\frac{\left(x_1 - \overline{X}\right)^2 + \left(x_2 - \overline{X}\right)^2 + \ldots + \left(x_n - \overline{X}\right)^2}{n} \le \overline{X} - \overline{X}_g \le$$
$$\le \frac{1}{2m}\frac{\left(x_1 - \overline{X}\right)^2 + \left(x_2 - \overline{X}\right)^2 + \ldots + \left(x_n - \overline{X}\right)^2}{n},$$

where $\overline{X}_g = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$.

b) It is easy to see that inequality (1.5.2.5) is a refinement of inequality (1.5.2.4), because the geometric mean is higher than the harmonic mean. Inequality (1.5.2.5) provides another bound for the variance, but it is very difficult to compare the terms $2M\left(\overline{X} - \overline{X}_g\right)$ and $\left(M - \overline{X}\right)\left(\overline{X} - m\right)$ to see which is better.

Combining the above inequalities and taking into account inequality (1.5.2.5), we found other bounds for the standard deviation, the coefficient of variation and of the covariance, thus:

(1.5.2.6) $$\sqrt{2m\left(\overline{X} - \overline{X}_g\right)} \le \sigma_{\overline{X}} \le \sqrt{2M\left(\overline{X} - \overline{X}_g\right)},$$

(1.5.2.7) $$\frac{\sqrt{2m\left(\overline{X} - \overline{X}_g\right)}}{\overline{X}} \le C_V(X) \le \frac{\sqrt{2M\left(\overline{X} - \overline{X}_g\right)}}{\overline{X}}$$

and

(1.5.2.8) $$\left|Cov(X,Y)\right| \le 2\sqrt{MQ\left(\overline{X} - \overline{X}_g\right)\left(\overline{Y} - \overline{Y}_g\right)}.$$

Now, we want to find an upper bound, better than the Bhatia and Davis, for the above indicators.

Theorem 1.5.2.3. *For a discrete random variable in finite case X there is the following inequality*

(1.5.2.9) $$\left(M - \overline{X}\right)\left(\overline{X} - m\right) - \sigma_{\overline{X}}^2 = \sum_{i=1}^{n} p_i (M - x_i)(x_i - m).$$

Proof. We evaluate the sum $\sum_{i=1}^{n} p_i (M - x_i)(x_i - m)$ and we deduce the following:

$$\sum_{i=1}^{n} p_i(M - x_i)(x_i - m) = (M + m)\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i x_i^2 - Mm,$$

so we have

$$\sum_{i=1}^{n} p_i(M - x_i)(x_i - m) = (M + m)\overline{X} - \sum_{i=1}^{n} p_i x_i^2 - Mm = (M + m)\overline{X} - \sigma_{\overline{X}}^2 - \overline{X}^2 - Mm,$$

which is equivalent to the equality

$$\sum_{i=1}^{n} p_i(M - x_i)(x_i - m) = \left(M - \overline{X}\right)\left(\overline{X} - m\right) - \sigma_{\overline{X}}^2.$$

□

1.5.3 A generalized form of Grüss type inequality and other integral inequalities

In 1935, Grüss (see [98]) proved the following integral inequality:
Let f and g be two bounded functions defined on $[a,b]$ *with* $\gamma_1 \le f(x) \le \Gamma_1$ *and* $\gamma_2 \le g(x) \le \Gamma_2$, *where* $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ *are four constants. Then, we have:*

(1.5.3.1) $$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \, \frac{1}{b-a}\int_a^b g(x)dx \right| \le \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2)$$

and the inequality is sharp, in the sense that the constant $1/4$ *can't be replaced by a smaller one.*

In the following research, on refining the Grüss inequality, we used the same work methods as the ones used in the discrete version. The following results were extracted from our paper [Minculete-Ciurdariu, 149].

Florea and Niculescu in [70] treated the problem of estimating the deviation of the values of a function from its mean value.

The estimation of the deviation of a function from its mean value is characterized in terms of random variables.
We denote by $R([a, b])$ the space of Riemann-integrable functions on the interval $[a, b]$, and by $C^0([a, b])$ the space of real-valued continuous functions on the interval $[a, b]$.

The integral arithmetic mean for a Riemann-integrable function $f : [a,b] \to \mathbb{R}$ is the number

$$M_1[f] = \frac{1}{b-a}\int_a^b f(x)dx.$$

If f and h are two integrable funtions on $[a,b]$ and $\int_a^b h(x)dx > 0$, then a generalization for the integral arithmetic mean is the number

$$M_h[f] = \frac{\int_a^b f(x)h(x)dx}{\int_a^b h(x)dx}$$

called the *h-integral arithmetic mean* for a Riemann-integrable function *f*.

We find the following property of the h-integral arithmetic mean for a Riemann-integrable function f:

$$M_h[f \pm k] = M_h[f] \pm k$$

where k is a real constant.

If the function f is a Riemann-integrable function, we denote by

$$var(f) = M_1\left[(f - M_1(f))^2\right]$$

the *variance* of f.

The expression for the variance of f can be expanded in this way:

$$var(f) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right)^2 dx.$$

In the same way, we defined the *h-variance* of a Riemann-integrable function f by

$$var_h(f) = M_h\left[(f - M_h(f))^2\right].$$

The expression for the h-variance can be thus expanded:

$$var_h(f) = \frac{1}{\int_a^b h(x)dx}\int_a^b \left(f(x) - \frac{\int_a^b f(t)h(t)dt}{\int_a^b h(t)dt} \right)^2 h(x)dx.$$

It is easy to see another form of the h-variance, given by the following:

$$var_h(f) = M_h\left[f^2\right] - M_h^2[f].$$

and we have

$$var_h(f \pm k) = var_h(f),$$

where k is a constant.

In [9], Aldaz showed a refinement of the AM-GM inequality and used in the proof that

$$\frac{1 - \int_a^b f^{1/2}(x)dx}{\left(\int_a^b f(x)dx\right)^{1/2}}$$

is a measure of the dispersion of $f^{1/2}$ about its mean value, which is, in fact, comparable to the variance,

$$var\left(\frac{f^{1/2}(x)}{\left\|f^{1/2}(x)\right\|_2}\right), \text{ where } \|f(x)\|_2 = \sqrt{\int_a^b f^2(x)dx}.$$

The *covariance* is a measure of how much two Riemann-integrable functions change together at the same time and is defined as

$$cov(f,g) = M_1[(f - M_1[f])(g - M_1[g])],$$

and is equivalent to the form

$$cov(f,g) = M_1[fg] - M_1[f]M_1[g] = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \, \frac{1}{b-a}\int_a^b g(x)dx.$$

In fact, the covariance is *the Chebyshev functional* attached to functions *f* and *g*. In [113] it is written as *T*(*f* , *g*). The properties of the Chebyshev functional have been studied by Elezović, Marangunić and Pečarić in their paper, [66]. For other generalizations of Grüss inequality, see [156, 175].

The *h*-covariance is a measure of how much two random variables change together and is defined as

$$cov_h(f,g)=M_h[(f-M_h[f])(g-M_h[g])],$$

and is equivalent to the form

$$cov_h(f,g)=M_h[fg]-M_h[f]M_h[g]=\frac{\int_a^b f(x)g(x)h(x)dx}{\int_a^b h(x)}-\frac{\int_a^b f(x)h(x)dx}{\int_a^b h(x)}\frac{\int_a^b g(x)h(x)dx}{\int_a^b h(x)}.$$

In [174], Pečarić used the generalization of the Chebyshev functional notion attached to functions *f* and *g* to the Chebyshev *h*-functional attached to functions *f* and *g* defined by $T(f,g;h)$. Here, Pečarić showed some generalizations of the inequality of Grüss by the Chebyshev *h*-functional. It is easy to see that, in terms of the covariance, this can be written as $T(f,g;h)=cov_h(f,g)$.

In terms of covariance, the inequality of Gruss becomes

(1.5.3.2) $$|cov(f,g)|\le\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2)$$

And, in terms of Chebyshev functional, the inequality of Gruss becomes

$$|T(f,g)|\le\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).$$

If there is additional information about the mean values of the two functions in the inequality of Grüss then Zitikis argued in his paper, [210], that the inequality can be sharpened and he also gave a probabilistic interpretation for it.

Lemma 1.5.3.1 ([Minculete-Ciurdariu, 149]). *Let f be a Riemann-integrable function defined on* [*a*,*b*] *with* $\gamma_1\le f(x)\le\Gamma_1$, *where* γ_1,Γ_1 *are two constants. Then we have:*

(1.5.3.3) $$var_h(f)\le\frac{1}{4}(\Gamma_1-\gamma_1)^2,$$

where $h:[a,b]\to[0,\infty)$ *is a Riemann- integrable function with* $\int_a^b h(x)dx>0$.

Lemma 1.5.3.2 ([Minculete-Ciurdariu, 149]). *Let f be a Riemann-integrable function defined on* [*a*,*b*] *with* $\gamma_1\le f(x)\le\Gamma_1$, *where* γ_1,Γ_1 *are two constants and a Riemann-integrable function* $h:[a,b]\to[0,\infty)$ *with* $\int_a^b h(x)dx>0$. *Then we have the following relations:*

(1.5.3.4) $$\mathrm{var}_h(f)\le\left(\Gamma_1-\frac{\int_a^b f(x)h(x)dx}{\int_a^b h(x)dx}\right)\left(\frac{\int_a^b f(x)h(x)dx}{\int_a^b h(x)dx}-\gamma_1\right).$$

We can prove an inequality for integrable functions similar to the inequality of Cauchy-Schwarz for random variables given by the following.

Theorem 1.5.3.3 ([Minculete-Ciurdariu, 149]). *If f , g, h* $\in R([a, b])$, *then we have the inequality*

(1.5.3.5)
$$|cov_h(f,g)|^2 \le var_h(f)var_h(g).$$

Proposition 1.5.3.4 ([Minculete-Ciurdariu, 149]). *Let f and g be two Riemann-integrable functions defined on* $[a,b]$ *with* $\gamma_1 \le f(x) \le \Gamma_1$ *and* $\gamma_2 \le g(x) \le \Gamma_2$, *where* $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ *are four constants and we have a Riemann-integrable function* $h:[a,b] \to [0,\infty)$ *with* $\int_a^b h(x)dx > 0$. *Then we have*

(1.5.3.6)
$$|cov_h(f,g)| = |T(f,g)| \le \sqrt{(\Gamma_1 - M_h[f])(M_h[f]-\gamma_1)(\Gamma_2 - M_h[g])(M_h[g]-\gamma_2)}$$
$$\le \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2).$$

Theorem 1.5.3.5 ([Minculete-Ciurdariu, 149]). *If f, g, q* $\in$ *R*([*a,b*]), *with* $f \neq kq$ *and* $var_h(f) \neq 0$, *then we have the inequality*

(1.5.3.7)
$$0 \le \frac{[cov_h(f,g)cov_h(f,q) - cov_h(g,q)var_h(f)]^2}{var_h(f)var_h(q) - [cov_h(f,q)]^2} \le var_h(f)var_h(g) - [cov_h(f,g)]^2.$$

Lemma 1.5.3.6 ([Minculete-Ciurdariu, 149]). *Let f and g be two Riemann-integrable functions defined on* [*a,b*]. *Then we have*

(1.5.3.8)
$$M_h^2[fg] \le M_h[f^2]M_h[f^2].$$

Applying the inequality between the arithmetic mean and the geometric mean and Lemma 1.5.3.6, we deduce the following relation:

Theorem 1.5.3.7 ([Minculete-Ciurdariu, 149]). *Let f and g be two Riemann-integrable functions defined on* [*a,b*]. *Then we have*

(1.5.3.9)
$$0 \le var_h(f)var_h(g) - [cov_h(f,g)] \le M_h[f^2]M_h[f^2] - M_h^2[fg].$$

Next, we show a refinement of Grüss' inequality for normalized isotonic linear functional. There are many directions in which the inequality of Gruss [98] has been generalized. Using the notion of *normalized isotonic linear functional* which appears in the paper [52], we will give a generalization of the inequality of Gruss which is related to a theorem of Andrica and Badea, [13].

Let *E* be a nonempty set, *L* a linear class of real-valued functions and $g : E \to \mathbb{R}$ having the properties:

(L1) *f* , *g* $\in L$ imply $(af + \beta g) \in L$ for all $a,\beta \in \mathbb{R}$,

(L2) **1** $\in L$, *i.e.* if $f_0(t) = 1$, $(\forall) t \in E$, then $f_0 \in L$.

An *isotonic linear functional* (in [13] is called *positive definite functional*) $A : L \to \mathbb{R}$ is a functional satisfying:

(A1) $A(af + \beta g) = aA(f) + \beta A(g)$, for all *f* , *g* $\in L$ and $a,\beta \in \mathbb{R}$.

(A2) If $f \in L$ and $f \ge 0$, then $A(f) \ge 0$.

(A3) The mapping *A* is said to be *normalized* if $A(\mathbf{1}) = 1$.

Theorem 1.5.3.7 ([Minculete-Ciurdariu, 149]). *Let* $f \in L$ *be such that* $f^2 \in L$ *and assume that there exist real numbers* γ_1 *and* Γ_1 *so that* $\gamma_1 \le f \le \Gamma_1$. *Then for any normalized isotonic linear functional* $A : L \to \mathbb{R}$ *one has the inequality*

(1.5.3.10)
$$A(f^2) - [A(f)]^2 \le (\Gamma_1 - A(f)(A(f) - \gamma_1)).$$

From the inequality of Cauchy-Schwarz for a normalized isotonic linear functional [52], we have for f , g, f^2, $g^2 \in L$ where f , g :$E \to \mathbb{R}$ and $A : L \to \mathbb{R}$ is any normalized isotonic linear functional:

(1.5.3.11) $$[A(fg)]^2 \le A(f^2)A(g^2).$$

Related to a counterpart of the Cauchy-Schwarz inequality, we have the following:

Theorem 1.5.3.8. *Let* f , g, $fg \in L$ *such that* f^2, $g^2 \in L$ *and* $\gamma_1 \le f(x) \le \Gamma_1$ *and* $\gamma_2 \le g(x) \le \Gamma_2$, *where* $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ *are given real numbers. Then for any normalized linear isotonic functional* $A : L \to \mathbb{R}$ *one has the inequality*

(1.5.3.12) $$A(fg) - A(f)A(g) \le \sqrt{(\Gamma_1 - A(f))(A(f) - \gamma_1)(\Gamma_2 - A(g))(A(g) - \gamma_2)}.$$

Finally, we find several applications. Taking into account the *integral arithmetic mean* and *h-integral arithmetic mean* for a Riemann-integrable function $f : [a, b] \to \mathbb{R}$ we can rewrite the following inequalities:

a) In the case when $p \ge 0$ the integral form of the inequality from Theorem 2.4 (see [17]) was given by Theorem 2.5. Under the conditions of Theorem 2.5, the inequality becomes

(1.5.3.13) $$M_1\left[\frac{f^{m+1}}{g^p}\right] \ge \frac{M_1^{m+1}[f]}{M_1^p[g]}.$$

(b) In [164], Mortici gave a new refinement of Radon's inequality. Using the integral form of the reverse of inequality from Theorem 2.5 (see [17]) we obtain, for $p \in (-1, 0)$, $m \in (-1, 0)$, $m \le p$, and f , $g : [a, b] \to \mathbb{R}_+$ are two integrable functions on $[a, b]$ with $g(x) > 0$, $(\forall) x \in [a, b]$, a continuous function on $[a, b]$, the inequality

(1.5.3.14) $$M_1\left[\frac{f^{m+1}}{g^p}\right] \le \frac{M_1^{m+1}[f]}{M_1^p[g]}.$$

In our paper [Rațiu-Minculete, 189], we have shown several refinements and counterparts of Radon's inequality. We establish that the inequality of Radon is a particular case of Jensen's inequality. Starting from several refinements and counterparts of Jensen's inequality by Dragomir and Ionescu, we obtain a counterpart of Radon's inequality. In this way, using a result of Simić, we find another counterpart of Radon's inequality. We obtain several applications using Mortici's inequality to improve Hölder's inequality and Liapunov's inequality.

To determine the best bounds for some inequalities, we used Matlab program for different cases.

Chapter 2

Inequalities for functionals and inequalities for invertible positive operators

In functional analysis and in the calculus of variations, a *functional* is a function from a vector space into its underlying field of scalars. Among the most studied functionals in the theory of inequalities we remark the Jensen functional and Chebychev functional. Next, we study the Jensen functional under superquadraticity conditions and the Jensen functional related to a strongly convex function.

Related to operators, an operator means a bounded linear operator on a complex Hilbert space *H* without specified. We study several properties which imply the establishment of inequalities between different types of operators.

2.1 Inequalities for functionals

If f is a real valued function defined on an interval *I*, $x_1, x_2, ..., x_n \in I$, and $p_1, p_2, ..., p_n \in (0,1)$ such that $\sum_{i=1}^{n} p_i = 1$, then the *Jensen functional* is defined by

$$J(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

and the *Chebychev functional* is defined by

$$T(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j\right) f(x_i).$$

Under the conditions from Definition 2.1.8, we have defined the *generalized Jensen functional* by

$$J_k(f, \mathbf{p}_1, ..., \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, ..., \mathbf{x}_k) := \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{ij_1} ... p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}\right)$$

and the *generalized Chebychev functional* by:

$$T_k(f, \mathbf{p}_1, ..., \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, ..., \mathbf{x}_k) := \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{ij_1} ... p_{kj_k} \sum_{i=1}^{k} q_i \left(x_{ij_i} - \sum_{j=1}^{n_i} p_{ij} x_{ij}\right) f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right).$$

In [179], Pečarić and Beesack discuss about the monotonicity property of discrete Jensen's functional. Dragomir (see [58]) investigated boundedness of normalized Jensen's functional, that is functional $J(f, \mathbf{p}, \mathbf{x})$ satisfying $\sum_{i=1}^{n} p_i = 1$. He obtained the following lower and upper bound for normalized functional:

$$0 \le \min_{1 \le i \le n} \left\{\frac{p_i}{q_i}\right\} J(\Phi, \mathbf{q}, \mathbf{x}) \le J(\Phi, \mathbf{q}, \mathbf{x}) \le \max_{1 \le i \le n} \left\{\frac{p_i}{q_i}\right\} J(\Phi, \mathbf{q}, \mathbf{x}),$$

where $\Phi : K \subset X \to X$ is a convex function on convex subset K of linear space X, $\mathbf{x} = (x_1, x_2, ..., x_n) \in K^n$ and $\mathbf{p} = (p_1, p_2, ..., p_n)$, $\mathbf{q} = (q_1, q_2, ..., q_n)$ are positive real n-tuples with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$.

The Jensen's inequality can be regarded in a more general manner, including positive linear functionals acting on a linear class of real valued functions.

2.1.1 The Jensen functional under superquadraticity conditions and the Jensen functional related to a strongly convex function

In this section, in the first part, we give a recipe which describes upper and lower bounds for the Jensen functional under superquadraticity conditions. Some results involve the Chebychev functional. We give a more general definition of these functionals and establish analogous results. These results were shown in our paper [Mitroi-Symeonidis-Minculete, 158].

For the reader's convenience, let us briefly state known facts regarding the principal tools, superquadraticity and the Jensen functional. See Abramovich and Dragomir [1] for details and proofs.

Definition 2.1.1.1 ([2]). *A function f defined on an interval $I = [0, a]$ or $[0, \infty)$, is **superquadratic** if for each x in I there exists a real number C(x) such that*

(2.1.1.1) $$f(y) - f(x) \geq f(|y - x|) + C(x)(y - x)$$

for all $y \in I$.

We say that f is a *subquadratic function* if $-f$ is superquadratic. The set of superquadratic functions is closed under addition and positive scalar multiplication.

Example ([3]). The function $f(x) = x^p$, $p \geq 2$ is superquadratic with $C(x) = f'(x) = px^{p-1}$. Similarly, $g(x) = -(1 + x^{1/p})^p$, $p > 0$ is superquadratic with $C(x) = 0$. Also $h(x) = x^2 \log x$ with $C(x) = h'(x) = x(2 \log x + 1)$ is a superquadratic function (but not monotone and not convex). Some elementary functions are not superquadratic, such as $f(x) = x$ and $f(x) = \exp x$.

Lemma 2.1.1.2 ([2]). *Let f be a superquadratic function with C(x) defined as above.*

(i) *Then* $f(0) \leq 0$.

(ii) *If* $f(0) = f'(0) = 0$, *then* $C(x) = f'(x)$, *whenever f is differentiable at* $x > 0$.

(iii) *If* $f \geq 0$, *then f is convex and* $f(0) = f'(0) = 0$.

Definition 2.1.1.3 ([1]). Let f be a real valued function defined on an interval I, let $x_1, x_2, ..., x_n \in I$, and let $p_1, p_2, ..., p_n \in (0,1)$ be such that $\sum_{i=1}^{n} p_i = 1$. The *Jensen functional* is defined by

(2.1.1.3) $$J(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

and the *Chebychev functional* is defined by

(2.1.1.4) $$T(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j\right) f(x_i).$$

Proposition 2.1.1.4 ([1]). *Let* $x_i \geq 0$, $i = \overline{1,n}$, *and* $p_i > 0$, $i = \overline{1,n}$, *with* $\sum_{i=1}^{n} p_i = 1$. *If f is superquadratic, then*

$$J(f, \mathbf{p}, \mathbf{x}) \geq \sum_{i=1}^{n} p_i f\left(\left|x_i - \sum_{j=1}^{n} p_j x_j\right|\right). \tag{2.1.1.5}$$

Theorem 2.1.1.5 ([Mitroi-Symeonidis-Minculete, 158]). *Let f be a superquadratic function defined on an interval* $I = [0,a]$ *or* $[0,\infty)$, $x_1,..., x_n \in I$ *and* $p_1,..., p_n \in (0,1)$ *such that* $\sum_{i=1}^{n} p_i = 1$ *and a real number* $\lambda \in [0,1]$. *Then we have*

$$\sum_{i=1}^{n} p_i f\left((1-\lambda)\sum_{i=1}^{n} p_i x_i + \lambda x_i\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f\left(\lambda\left|x_i - \sum_{i=1}^{n} p_i x_i\right|\right). \tag{2.1.1.6}$$

Proof. Let *f* be a superquadratic function with *C(x)* defined as above and a real number $\lambda \in [0,1]$. Then replacing *y* by $(1-\lambda)x + \lambda y$, where $\lambda \in [0,1]$, we deduce the inequality

$$f((1-\lambda)x + \lambda y) - f(x) \geq f(\lambda|y - x|) + \lambda C(x)(y - x). \tag{2.1.1.7}$$

Now, in inequality (2.1.1.7) we make the following substitutions: $x = \sum_{i=1}^{n} p_i x_i$ and $y = x_i$. Therefore, we have

$$f\left((1-\lambda)\sum_{i=1}^{n} p_i x_i + \lambda x_i\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq f\left(\lambda\left|x_i - \sum_{i=1}^{n} p_i x_i\right|\right) + \lambda C\left(\sum_{i=1}^{n} p_i x_i\right)\left(x_i - \sum_{i=1}^{n} p_i x_i\right).$$

Multiplying by $p_i > 0$ this inequality and summing from $i = \overline{1,n}$, we deduce the statement.

□

Remark 2.1.1.6. For $\lambda = 1$, we obtain inequality from Proposition 2.1.1.4.

Corollary 2.1.1.7 ([Mitroi-Symeonidis-Minculete, 158]). *Let* $f \geq 0$ *be a superquadratic function defined on an interval* $I = [0,a]$ *or* $[0,\infty)$, $x_1,..., x_n \in I$ *and* $p_1,..., p_n \in (0,1)$ *such that* $\sum_{i=1}^{n} p_i = 1$. *Then we have*

$$J(f, p, x) \geq 2\sum_{i=1}^{n} p_i f\left(\frac{1}{2}\left|x_i - \sum_{i=1}^{n} p_i x_i\right|\right). \tag{2.1.1.8}$$

Proof. For $\lambda = \frac{1}{2}$ in Theorem 2.1.1.5, we have the inequality

$$\sum_{i=1}^{n} p_i f\left(\frac{\sum_{i=1}^{n} p_i x_i + x_i}{2}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f\left(\frac{1}{2}\left|x_i - \sum_{i=1}^{n} p_i x_i\right|\right). \tag{2.1.1.9}$$

From Lemma 2.1.1.2, we know that *f* is convex . Therefore, applying Jensen's inequality, we have

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i + x_i}{2}\right) \le \frac{1}{2}\left[f\left(\sum_{i=1}^{n} p_i x_i\right) + f(x_i)\right].$$

Using this inequality and inequality (2.1.1.9), we obtain inequality
$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \ge 2\sum_{i=1}^{n} p_i f\left(\frac{1}{2}\left|x_i - \sum_{i=1}^{n} p_i x_i\right|\right)$, which implies the inequality (2.1.1.8).

□

Motivated by the above results, we introduce, in a natural way, other functionals.

Definition 2.1.1.8. *Assume that we have a real valued function f defined on an interval I, the real numbers* p_{ij}, $i=\overline{1,k}$ *and* $j=\overline{1,n_i}$ *are such that* $p_{ij}>0$, $\sum_{j=1}^{n_i} p_{ij} = 1$ *for all* $i=\overline{1,k}$ *(we put* $\mathbf{p}_i = (p_{i1}, p_{i2},\ldots, p_{in_i})$*),* $\mathbf{x}_i = (x_{i1}, x_{i2},\ldots, x_{in_i}) \in I^{n_i}$ *for all* $i=\overline{1,k}$ *and* $\mathbf{q} = (q_1, q_2,\ldots,q_k)$*,* $q_i > 0$ *are such that* $\sum_{i=1}^{k} q_i = 1$*. We define the* ***generalized Jensen functional*** *by*

(2.1.1.10)

$$J_k(f, \mathbf{p}_1,\ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1,\ldots, \mathbf{x}_k) := \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{ij_1}\ldots p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}\right)$$

and the ***generalized Chebychev functional*** *by:*

(2.1.1.11)

$$T_k(f, \mathbf{p}_1,\ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1,\ldots, \mathbf{x}_k) := \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{ij_1}\ldots p_{kj_k} \sum_{i=1}^{k} q_i \left(x_{ij_i} - \sum_{j=1}^{n_i} p_{ij} x_{ij}\right) f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right).$$

We also easily notice that for $k = 1$ this definition reduces to Definition 2.1.1.3. In [160], the following estimation is obtained: if f is a convex function then we have

(2.1.1.12)
$$\min_{\substack{1\le j_1\le n_1\\ \ldots\\ 1\le j_k\le n_k}} \left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\} J_k(f, \mathbf{r}_1,\ldots, \mathbf{r}_k, \mathbf{q}, \mathbf{x}_1,\ldots, \mathbf{x}_k) \le$$
$$J_k(f, \mathbf{p}_1,\ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1,\ldots, \mathbf{x}_k)$$
$$\le \max_{\substack{1\le j_1\le n_1\\ \ldots\\ 1\le j_k\le n_k}} \left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\} J_k(f, \mathbf{r}_1,\ldots, \mathbf{r}_k, \mathbf{q}, \mathbf{x}_1,\ldots, \mathbf{x}_k)$$

In this section, we investigate upper and lower bounds that we have if the function f is superquadratic.

Now we extend the earlier results. The following lemma describes the behavior of the functional under the superquadraticity condition:

Lemma 2.1.9. *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 2.1.1.8. *If f is superquadratic then we have*

(2.1.1.13) J_k (f, $\mathbf{p}_1,\ldots,\mathbf{p}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$) $\geq \sum_{j_1,\ldots j_k=1}^{n_1,\ldots,n_k} p_{ij_1}\ldots p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right)$

where $\bar{x} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}$.

Using the same recipe as in the proof of Corollary 2.1.1.7, we get:

Corollary 2.1.1.10 ([Mitroi-Symeonidis-Minculete, 158]). *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 2.1.1.8. *Let* $f \geq 0$ *be a superquadratic function defined on an interval* $I = [0,a]$ *or* $[0,\infty)$, $x_1,\ldots,x_n \in I$ *and* $p_1,\ldots,p_n \in (0,1)$ *such that* $\sum_{i=1}^{n} p_i = 1$. *Then we have*

(2.1.1.14) J_k (f, $\mathbf{p}_1,\ldots,\mathbf{p}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$) $\geq \sum_{j_1,\ldots j_k=1}^{n_1,\ldots,n_k} p_{ij_1}\ldots p_{kj_k} f\left(\frac{1}{2}\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right)$.

The next result can be expressed as:

Theorem 2.1.11([Mitroi-Symeonidis-Minculete, 158]). *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 2.1.1.8 *and the positive real numbers* r_{ij}, $i = \overline{1,k}$ *and* $j = \overline{1,n_i}$ *be such that* $\sum_{j=1}^{n_i} r_{ij} = 1$ *for all* $i = \overline{1,k}$. *We put* $\boldsymbol{r}_i = (r_{i1},\ldots,r_{in_i})$ *for all* $i = \overline{1,k}$, $m = \min\limits_{\substack{1\leq j_1\leq n_1\\ \ldots\\ 1\leq j_k\leq n_k}} \left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\}$ and

$M = \max\limits_{\substack{1\leq j_1\leq n_1\\ \ldots\\ 1\leq j_k\leq n_k}} \left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\}$.

If f is a superquadratic function, then:

(2.1.1.15) J_k (f, $\mathbf{p}_1,\ldots,\mathbf{p}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$) $- mJ_k$ (f, $\mathbf{r}_1,\ldots,\mathbf{r}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$)

$$\geq mf\left(\left|\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} (r_{ij} - p_{ij}) x_{ij}\right|\right) + \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} \left(p_{ij_1}\ldots p_{kj_k} - m r_{ij_1}\ldots r_{kj_k}\right) f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right)$$

and

(2.1.1.16) MJ_k (f, $\mathbf{r}_1,\ldots,\mathbf{r}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$) $- J_k$ (f, $\mathbf{p}_1,\ldots,\mathbf{p}_k$, $\mathbf{q}$, $\mathbf{x}_1,\ldots,\mathbf{x}_k$)

$$\geq f\left(\left|\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} (r_{ij} - p_{ij}) x_{ij}\right|\right) + \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} \left(M r_{ij_1}\ldots r_{kj_k} - p_{ij_1}\ldots p_{kj_k}\right) f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right)$$

where $\bar{x} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}$.

Remark 2.1.1.12. Let $\mathbf{p}_1 = \cdots = \mathbf{p}_k = \mathbf{p}$ and $\mathbf{x}_1 = \cdots = \mathbf{x}_k = \mathbf{x}$. In this case we see that Lemma 2.1.9 recover Proposition 2.1.1.4.

More results can be found in paper [Mitroi-Symeonidis-Minculete, 158]. In [118], Kluza and Niezgoda quoted the above results for the introduction and study of Jeffreys–Csiszár and Jensen–Csiszár f-divergences. Some bounds of Crooks and Lin types for such divergences are provided. To this end, the concavity of the composition of monotone functions is discussed.

Next, we describe some results concerning upper and lower bounds for the Jensen functional related to the concept of a strongly convex function.

Definition 2.1.1.13. *A function f defined on an interval I is strongly convex with modulus* $c > 0$ [*or c-strongly convex*] *if*

(2.1.1.17) $$f((1-\lambda)x+\lambda y)\le(1-\lambda)f(x)+\lambda f(y)-c\lambda(1-\lambda)(y-x)^2,$$
for all $x,y\in I,\lambda\in[0,1]$.

We call *f* strongly convex if there exists a $c>0$ such that is strongly convex with modulus *c*. Strongly convex functions were introduced by Polyak [182]. A function *f* is called *strongly concave with modulus c* (or *approximately convex of order* 2 [170]) if − *f* is strongly convex with modulus *c*.

Obviously, every strongly convex function is convex. Affine functions are not strongly convex. The function $f(x) = cx^2 + bx + a$ is strongly convex with modulus *c* and the inequality (2.1.1.17) holds with equality sign.

According to Hiriart–Urruty and Lemaréchal [109], we have:

Proposition 2.1.1.14. *The function f is strongly convex with modulus c if and only if the function* $g(x) = f(x) - cx^2$ *is convex.*

In [137], the following result is proved:

Proposition 2.1.1.15. *Considering* $p_i\ge 0, i=\overline{1,n}$, *with* $\sum_{i=1}^n p_i=1$ *and* $\bar{x}=\sum_{i=1}^n p_ix_i$, *the function f strongly convex with modulus c, we have*

(2.1.1.18) $$J(f,\mathbf{p},\mathbf{x})=\sum_{i=1}^n p_if(x_i)-f\left(\sum_{i=1}^n p_ix_i\right)\ge c\sum_{i=1}^n p_i\left(x_i-\bar{x}\right)^2.$$

This is re-proved using the probabilistic approach in a paper of Rajba and Wasowicz [187, Corollary2.3]. Notice that the set of strongly convex functions is closed under addition and positive scalar multiplication.

In what follows we shall also be interested in a more general Jensen functional and its behaviour in the context of strong convexity.

Theorem 2.1.1.16 ([Mitroi-Symeonidis-Minculete, 159]). *Let f be a strongly convex function with modulus c defined on an interval I,* $x_1,\dots,x_n\in I$ *and* $p_1,\dots,p_n\in(0,1)$ *such that* $\sum_{i=1}^n p_i=1$. *Then*

(2.1.1.19)
$$\sum_{i=1}^n p_if\left((1-\lambda\mu)\bar{x}+\lambda\mu x_i\right)\le(1-\lambda)f(\bar{x})+\lambda\sum_{i=1}^n p_if\left((1-\mu)\bar{x}+\mu x_i\right)-c\lambda(1-\lambda)\mu^2\sum_{i=1}^n p_i\left(x_i-\bar{x}\right)^2,$$
for $\lambda,\mu\in[0,1]$.

Moreover, from (2.1.1.19) for $x_i\to\frac{\bar{x}+x_i}{2}$ we get a double inequality which refines the Merentes-Nikodem inequality (2.1.1.18):

Proposition 2.1.1.17 ([Mitroi-Symeonidis-Minculete, 159]). *Let f be a strongly convex function with modulus c defined on an interval I,* $x_1,\dots,x_n\in I$ *and* $p_1,\dots,p_n\in(0,1)$ *such that* $\sum_{i=1}^n p_i=1$. *Then*

(2.1.1.20) $$J(f,\mathbf{p},\mathbf{x})\ge 2\left[\sum_{i=1}^n p_if\left(\frac{\bar{x}+x_i}{2}\right)-f(\bar{x})\right]+\frac{c}{2}\sum_{i=1}^n p_i\left(x_i-\bar{x}\right)^2\ge c\sum_{i=1}^n p_i\left(x_i-\bar{x}\right)^2.$$

We state the following lemma about the behaviour of the generalized Jensen functional under the strong convexity condition:

Lemma 2.1.1.18 ([Mitroi-Symeonidis-Minculete, 159]). *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 2.1.1.8. *If f is strongly convex with modulus c, then we have*

(2.1.1.21) $$J_k\,(f,\ \mathbf{p}_1,\ldots,\ \mathbf{p}_k,\ \mathbf{q},\ \mathbf{x}_1,\ldots,\ \mathbf{x}_k)\ \geq c\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{ij_1}\ldots p_{kj_k}\left(\sum_{i=1}^{k} q_i x_{ij_i}-\overline{x}\right)^2,$$

where $\overline{x}=\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i} p_{ij}x_{ij}$.

For strongly convex functions we have the following bounds:

Theorem 2.1.1.19 ([Mitroi-Symeonidis-Minculete, 159]). *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 2.1.1.8 *and the positive real numbers* r_{ij}, $i=\overline{1,k}$ *and* $j=\overline{1,n_i}$ *be such that* $\sum_{j=1}^{n_i} r_{ij}=1$ *for all* $i=\overline{1,k}$. *We put* $\boldsymbol{r}_i=\left(r_{i1},\ldots,r_{in_i}\right)$ *for all* $i=\overline{1,k}$, $m=\min\limits_{\substack{1\leq j_1\leq n_1\\ \ldots\\ 1\leq j_k\leq n_k}}\left\{\dfrac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\}$ and

$$M=\max_{\substack{1\leq j_1\leq n_1\\ \ldots\\ 1\leq j_k\leq n_k}}\left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\}.$$

If f is a strongly convex function with modulus c, then we have:

(2.1.1.22) $$J_k\,(f,\ \mathbf{p}_1,\ldots,\ \mathbf{p}_k,\ \mathbf{q},\ \mathbf{x}_1,\ldots,\ \mathbf{x}_k)\ -\ mJ_k\,(f,\ \mathbf{r}_1,\ldots,\ \mathbf{r}_k,\ \mathbf{q},\ \mathbf{x}_1,\ldots,\ \mathbf{x}_k)$$
$$\geq c\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{ij_1}\ldots p_{kj_k}-mr_{ij_1}\ldots r_{kj_k}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i}-\overline{x}\right)^2+mc\left(\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i}\left(r_{ij}-p_{ij}\right)x_{ij}\right)^2$$

and

(2.1.1.23) $$MJ_k\,(f,\ \mathbf{r}_1,\ldots,\ \mathbf{r}_k,\ \mathbf{q},\ \mathbf{x}_1,\ldots,\ \mathbf{x}_k)\ -\ J_k\,(f,\ \mathbf{p}_1,\ldots,\ \mathbf{p}_k,\ \mathbf{q},\ \mathbf{x}_1,\ldots,\ \mathbf{x}_k)$$
$$\geq c\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(Mr_{ij_1}\ldots r_{kj_k}-p_{ij_1}\ldots p_{kj_k}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i}-\overline{x}\right)^2+c\left(\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i}\left(r_{ij}-p_{ij}\right)x_{ij}\right)^2$$

where $\overline{x}=\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i} p_{ij}x_{ij}$.

We show in [Mitroi-Symeonidis-Minculete, 159] some applications to function gamma of Euler.

The function gamma is defined via a convergent improper integral as

$$\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx\text{, for all } t\geq 0,$$

it is known as Euler integral of the second kind. The following infinite product definition for the gamma function is due to Weierstrass,

$$\Gamma(t)=\frac{e^{-\gamma t}}{t}\prod_{n=1}^{\infty}\left(1+\frac{t}{n}\right)^{-1}e^{\frac{t}{n}},$$

where $\gamma=0.577216..$ is the Euler-Mascheroni constant. This relation can be written as

(2.1.1.24) $$\log\Gamma(t)=-\gamma t-\log t-\sum_{n=1}^{\infty}\left(\frac{t}{n}-\log\left(1+\frac{t}{n}\right)\right),$$

where the base of the logarithm is e.

Proposition 2.1.1.20 ([Mitroi-Symeonidis-Minculete, 159])**.** *The function defined by* $f:[0,\infty)\to\mathbb{R}$, $f(t)=\log\Gamma(t^2+1)+\gamma t^2+t\arctan t$ *is strongly convex with modulus* 1 *on* $[0,\infty)$.

Proof. From relation (2.1.1.24), we get

$$\log\Gamma(t^2+1)=-\gamma(t^2+1)-\log(t^2+1)-\sum_{n=1}^{\infty}\left(\frac{t^2+1}{n}-\log\left(1+\frac{t^2+1}{n}\right)\right). \tag{2.1.1.25}$$

We consider the function

$$g(t)=\log\Gamma(t^2+1)+\gamma t^2+t\arctan t-t^2$$

defined on $[0,\infty)$. It easy to see that

$$g'(t)=-\frac{t}{t^2+1}+2t\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{t^2+n+1}\right)+\arctan t-2t$$

and

$$g''(t)=\frac{2t^2}{(t^2+1)^2}+4t^2\sum_{n=1}^{\infty}\frac{1}{(t^2+n+1)^2}+2\sum_{n=0}^{\infty}\left(\frac{1}{n}-\frac{1}{t^2+n+1}\right)-2.$$

The inequality

$$2\sum_{n=0}^{\infty}\left(\frac{1}{n}-\frac{1}{t^2+n+1}\right)-2\ge 2\sum_{n=0}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-2=0$$

yields $g''(t)\ge 0$, therefore g is convex, so f is strongly convex with modulus 1 on $[0,\infty)$.

□

It is straightforward that:

Corollary 2.1.1.21. *The function* $f:[0,\infty)\to\mathbb{R}$, $f(t)=\log\Gamma(t^2+1)+t\arctan t$ *is strongly convex with modulus* $1-\gamma$ *on* $[0,\infty)$.

Next, we give inequalities related to a strongly convex function.

An important inequality is given by F. C. Mitroi [157], as a particular case of the Dragomir inequality [58], for a convex function *f* on [*a*,*b*], we have the following inequality:

(2.1.1.26)

$$2\min\{\lambda,1-\lambda\}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\le \lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b)\le 2\max\{\lambda,1-\lambda\}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right),$$

for all $\lambda\in[0,1]$.

Lemma 2.1.1.22. *If f is a function integrable and convex on* $[a,b]$, *we have the following inequality:*

$$\left(1-\frac{|a+b-2x|}{b-a}\right)F\le\frac{bf(a)-af(b)}{b-a}+x\frac{f(b)-f(a)}{b-a}-f(x)\le\left(1+\frac{|a+b-2x|}{b-a}\right)F, \tag{2.1.1.27}$$

where $F=\dfrac{f(a)+f(b)}{2}-f\left(\dfrac{a+b}{2}\right)$, *for every* $x\in[a,b]$.

Proof. For $\lambda=\dfrac{b-x}{b-a}\in[0,1]$ when $x\in[a,b]$, we have $1-\lambda=\dfrac{x-a}{b-a}$,

$min\{\lambda,1-\lambda\}=\dfrac{1-|1-2\lambda|}{2}=1-\dfrac{|a+b-2x|}{b-a}$ and $max\{\lambda,1-\lambda\}=\dfrac{1+|1-2\lambda|}{2}=1+\dfrac{|a+b-2x|}{b-a}$.

If we replace these in inequality (2.1.1.26), we prove the inequality of the statement. □

Next, we obtain a reverse inequality of Jensen's inequality.

Proposition 2.1.1.23. *If f is a function integrable and convex on* $[a,b]$*, we have the following inequality:*

(2.1.1.28)

$$\frac{1}{n}\sum_{i=1}^{n}f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\le\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|a+b-2x_i|}{b-a}-\frac{\left|a+b-\frac{2}{n}\sum_{i=1}^{n}x_i\right|}{b-a}\right)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right),$$

for every $x_i\in[a,b]$.

Proof. If $x_i\in[a,b]$, for all $i=\overline{1,n}$, then using inequality (2.1.1.27), we have

$$\left(1-\frac{|a+b-2x_i|}{b-a}\right)F\le\frac{bf(a)-af(b)}{b-a}+x_i\frac{f(b)-f(a)}{b-a}-f(x_i)\le\left(1+\frac{|a+b-2x_i|}{b-a}\right)F$$

where $F=\dfrac{f(a)+f(b)}{2}-f\left(\dfrac{a+b}{2}\right)$. By summing from *1* to *n*, we find the following inequality:

(2.1.1.29)

$$\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\left(1-\frac{1}{n}\sum_{i=1}^{n}\frac{|a+b-2x_i|}{b-a}\right)\le$$
$$\frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{n(b-a)}\sum_{i=1}^{n}x_i-\frac{1}{n}\sum_{i=1}^{n}f(x_i)\le$$
$$\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\left(1+\frac{1}{n}\sum_{i=1}^{n}\frac{|a+b-2x_i|}{b-a}\right),$$

If $x_i\in[a,b]$, for all $i=\overline{1,n}$, then $\dfrac{1}{n}\sum_{i=1}^{n}x_i\in[a,b]$ and using inequality (2.1.1.27), we have

(2.1.1.30)

$$\left(1-\frac{\left|a+b-\frac{2}{n}\sum_{i=1}^{n}x_i\right|}{b-a}\right)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\le$$
$$\frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{n(b-a)}\sum_{i=1}^{n}x_i-f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$
$$\left(1+\frac{\left|a+b-\frac{2}{n}\sum_{i=1}^{n}x_i\right|}{b-a}\right)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right),$$

Therefore, combining inequalities (2.1.1.29) and (2.1.1.30), we obtain the inequality from statement.

□

From Proposition 2.1.1.14, we have that: if the function f is strongly convex with modulus c then the function $g(x) = f(x) - cx^2$ is convex.

We apply the above results for the function g, thus:

Corollary 2.1.1.24. *If f is a strongly convex function with modulus c, then we have:*

(2.1.1.27)

$$\left(1-\frac{|a+b-2x|}{b-a}\right)F_1 \le \frac{bf(a)-af(b)}{b-a}+x\frac{f(b)-f(a)}{b-a}-f(x)+c(x-a)(x-b) \le \left(1+\frac{|a+b-2x|}{b-a}\right)F_1$$

where $F_1 = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) - c\frac{(a-b)^2}{2}$, *for every* $x \in [a,b]$.

Proposition 2.1.1.25. *If f is a strongly convex function with modulus c, we have the following inequality:*

(2.1.1.28)

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - c\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) \le \left(\frac{1}{n}\sum_{i=1}^{n}\frac{|a+b-2x_i|}{b-a} - \frac{\left|a+b-\frac{2}{n}\sum_{i=1}^{n} x_i\right|}{b-a}\right)F_1,$$

for every $x_i \in [a,b]$, *where* $F_1 = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) - c\frac{(a-b)^2}{2}$.

2.1.2 Several inequalities on generalized entropies

Generalized entropies have been studied by many researchers (we refer the interested reader to [6]). Rényi [191] and Tsallis [201] entropies are well known as one-parameter generalizations of Shannon's entropy, being intensively studied not only in the field of classical statistical physics [202–204], but also in the field of quantum physics in relation to the entanglement [198].

The Tsallis entropy is a natural one-parameter extended form of the Shannon entropy, hence it can be applied to known models which describe systems of great interest in atomic physics [84]. However, to our best knowledge, the physical relevance of a parameter of the Tsallis entropy was highly debated and it has not been completely clarified yet, the parameter being considered as a measure of the non-extensivity of the system under consideration.

One of the authors of the present paper studied the Tsallis entropy and the Tsallis relative entropy from a mathematical point of view. Firstly, fundamental properties of the Tsallis relative entropywere were discussed in [81]. The uniqueness theorem for the Tsallis entropy and Tsallis relative entropy was studied in [85]. Following this result, an axiomatic characterization of a two-parameter extended relative entropy was given in [86].

In [74], information theoretical properties of the Tsallis entropy and some inequalities for conditional and joint Tsallis entropies were derived. In [87], matrix trace inequalities for the Tsallis entropy were studied. And, in [88], the maximum entropy principle for the Tsallis entropy and the minimization of the Fisher information in Tsallis statistics were studied.

Quite recently, we provided mathematical inequalities for some divergences in [89], considering that it is important to study the mathematical inequalities for the development of new entropies. We show several results from our paper [Furuichi-Minculete-Mitroi, 75], here we define a further generalized entropy based on Tsallis and Rényi entropies and study mathematical properties by the use of scalar inequalities to develop the theory of entropies.

We start from the weighted quasilinear mean for some continuous and strictly monotonic function $\psi : I \to \mathbb{R}$, defined by

(2.1.2.1) $$M_{\psi}(x_1, x_2, \ldots, x_n) \equiv \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi(x_j)\right),$$

where $\sum_{j=1}^{n} p_j = 1$, $p_j > 0$, $x_j \in I$, for $j = \overline{1,n}$, and $n \geq 1$.

If we take $\psi(x) = x$, then $M_{\psi}(x_1, x_2, \ldots, x_n)$ coincides with the weighted arithmetic mean $A(x_1, x_2, \ldots, x_n) \equiv \sum_{j=1}^{n} p_j x_j$. If we take $\psi(x) = log\,x$, then $M_{\psi}(x_1, x_2, \ldots, x_n)$ coincides with the weighted geometric mean $G(x_1, x_2, \ldots, x_n) \equiv \prod_{j=}^{n} x_j^{p_j}$. If $\psi(x) = x$ and $x_j = ln_q \frac{1}{p_j}$, then $M_{\psi}(x_1, x_2, \ldots, x_n)$ is equal to Tsallis entropy [201]:

(2.1.2.2) $$H_q(p_1, p_2, \ldots, p_n) \equiv -\sum_{j=1}^{n} p_j^q \, ln_q \, p_j = \sum_{j=1}^{n} p_j \, ln_q \frac{1}{p_j}, \ (q \geq 0, q \neq 1).$$

where $\{p_1, p_2, \ldots, p_n\}$is a probability distribution with $p_j > 0$ for all $j = \overline{1,n}$ and the q-logarithmic function for $x > 0$ is defined by $ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q}$, which uniformly converges to the usual logarithmic function *log(x)* in the limit $q \to 1$. Therefore, the Tsallis entropy converges to Shannon entropy in the limit $q \to 1$:

(2.1.2.3) $$\lim_{q \to 1} H_q(p_1, p_2, \ldots, p_n) = H(p_1, p_2, \ldots, p_n) \equiv -\sum_{j=1}^{n} p_j \, log \, p_j .$$

Thus, we find that Tsallis entropy is one of the generalizations of Shannon entropy. It is known that Renyi entropy [191] is also a generalization of Shannon entropy. Here, we review a quasilinear entropy [6] as another generalization of Shannon entropy. For a continuous and strictly monotonic function φ on (0, 1], the quasilinear entropy is given by

(2.1.2.4) $$I^{\phi}(p_1, p_2, \ldots, p_n) \equiv -log\,\phi^{-1}\left(\sum_{j=1}^{n} p_j \phi(p_j)\right).$$

If we take $\phi(x) = log\,x$ in (2.1.2.4), then we have $I^{log}(p_1, p_2, \ldots, p_n) = H_1(p_1, p_2, \ldots, p_n)$. We may redefine the quasilinear entropy by

(2.1.2.5) $$I^{\psi}(p_1, p_2, \ldots, p_n) \equiv log\,\psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right),$$

for a continuous and strictly monotonic function ψ on $(0, \infty)$. If we take $\psi(x) = log\,x$

in (2.1.2.5), we have $I_1^{log}(p_1, p_2,..., p_n) = H_1(p_1, p_2,..., p_n)$. The case $\psi(x) = x^{1-q}$ is also useful in practice, since we recapture Rényi entropy, namely $I_1^{x^{1-q}}(p_1, p_2,..., p_n) = R_q(p_1, p_2,..., p_n)$, where Rényi entropy [191] is defined by

(2.1.2.6) $$R_q(p_1, p_2,..., p_n) \equiv \frac{1}{1-q} log\left(\sum_{j=1}^{n} p_j^q\right).$$

From a viewpoint of application on source coding, the relation between the weighted quasilinear mean and Renyi entropy has been studied in Chapter 5 of [191] in the following way:

Theorem A ([191]) *For all real numbers* $q > 0$ *and integers* $D > 1$, *there exists a code* $(x_1, x_2, \ldots, x_n)$ *such that*

(2.1.2.7) $$\frac{R_q(p_1, p_2,..., p_n)}{log D} \le M_{D^{\frac{1-q}{q}x}}(x_1, x_2,..., x_n) < \frac{R_q(p_1, p_2,..., p_n)}{log D},$$

where the exponential function $D^{\frac{1-q}{q}x}$ *is defined on* $[1,\infty)$.

By simple calculations, we find that

$$\lim_{q\to 1} M_{D^{\frac{1-q}{q}x}}(x_1, x_2,..., x_n) = \sum_{j=1}^{n} p_j x_j,$$

and

$$\lim_{q\to 1} \frac{R_q(p_1, p_2,..., p_n)}{log D} = -\sum_{j=1}^{n} p_j \, log_D \, p_j.$$

Motivated by the above results and recent advances on the Tsallis entropy theory, we investigate the mathematical results for generalized entropies involving Tsallis entropies and quasilinear entropies, using some inequalities obtained by improvements of Young's inequality.

Definition 2.1.2.1. *For a continuous and strictly monotonic function* ψ *on* $(0,\infty)$ *and two probability distributions* $\{p_1, p_2, \ldots, p_n\}$ *and* $\{r_1, r_2, \ldots, r_n\}$ *with* $p_j > 0$, $r_j > 0$ *for all* $j = \overline{1,n}$, *the quasilinear relative entropy is defined by*

(2.1.2.7) $$D_1^{\psi}(p_1, p_2,..., p_n \| r_1, r_2,..., r_n) \equiv -log \, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right).$$

The quasilinear relative entropy coincides with the Shannon relative entropy if $\psi(x) = log\, x$, *i.e.*,

$$D_1^{log}(p_1, p_2,..., p_n \| r_1, r_2,..., r_n) \equiv -\sum_{j=1}^{n} p_j \, log \frac{r_j}{p_j} = D_1(p_1, p_2,..., p_n \| r_1, r_2,..., r_n).$$

We denote by $R_q(p_1, p_2,..., p_n \| r_1, r_2,..., r_n)$ the Rényi relative entropy [3] defined by

(2.1.2.6) $$R_q(p_1, p_2,..., p_n \| r_1, r_2,..., r_n) \equiv \frac{1}{q-1} log\left(\sum_{j=1}^{n} p_j^q r_j^{1-q}\right).$$

This is another particular case of the quasilinear relative entropy, namely for $\psi(x) = x^{1-q}$ we have

$$D_1^{x^{1-q}}\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right)\equiv -log\left(\sum_{j=1}^{n}p_j\left(\frac{r_j}{p_j}\right)^{1-q}\right)^{\frac{1}{1-q}}$$

$$=\frac{1}{q-1}log\left(\sum_{j=1}^{n}p_j^q r_j^{1-q}\right)=R_q\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right).$$

If we use the inequality (1.4.23), then we obtain

$$p_j^{\,q}r_j^{\,1-q}\left(\frac{p_j+r_j}{2\sqrt{p_jr_j}}\right)^{2r}\le qp_j+(1-q)r_j\le p_j^{\,q}r_j^{\,1-q}\left(\frac{p_j+r_j}{2\sqrt{p_jr_j}}\right)^{2(1-r)},$$

where $q\in(0,1)$ and $r=min\{q,1-q\}$. It follows that

$$\frac{1}{q-1}log\left(\sum_{j=1}^{n}(qp_j+(1-q)r_j)\left(\frac{p_j+r_j}{2\sqrt{p_jr_j}}\right)^{2(r-1)}\right)\le R_q\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right)$$

$$\le\frac{1}{q-1}log\left(\sum_{j=1}^{n}(qp_j+(1-q)r_j)\left(\frac{p_j+r_j}{2\sqrt{p_jr_j}}\right)^{-2r}\right).$$

We denote by

(2.1.2.7) $$D_q\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right)\equiv\sum_{j=1}^{n}p_j^q\left(ln_q\,p_j-ln_q\,r_j\right)=-\sum_{j=1}^{n}p_j\,ln_q\frac{r_j}{p_j},$$

the Tsallis relative entropy which converges to the usual relative entropy (divergence, Kullback-Leibler information) in the limit $q\to1$:

$$\lim_{q\to1}D_q\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right)=D_1\left(p_1,p_2,\dots,p_n\|r_1,r_2,\dots,r_n\right)=-\sum_{j=1}^{n}p_j\,log\frac{r_j}{p_j}.$$

On the other hand, the studies on refinements for Young's inequality have given a great progress in the papers [10, 11, 53, 76, 77, 78]. In the present paper, we give some inequalities on Tsallis entropies applying two types of inequalities obtained in [77, 157].

As an analogy with (2.1.2.5), we may define in our paper [75] the following entropy:

Definition 2.1.2.2. For a continuous and strictly monotonic function ψ on $(0,\infty)$ and $q>0$ with $q\neq1$, the Tsallis quasilinear entropy (q-quasilinear entropy) is defined by

(2.1.2.8) $$I_q^{\psi}\left(p_1,p_2,\dots,p_n\right)\equiv ln_q\,\psi^{-1}\left(\sum_{j=1}^{n}p_j\psi\left(\frac{1}{p_j}\right)\right),$$

where $\{p_1, p_2, \dots, p_n\}$ is a probability distribution with $p_j>0$ for all $j=\overline{1,n}$.

We notice that if ψ does not depend on q, then

$$\lim_{q\to1}I_q^{\psi}\left(p_1,p_2,\dots,p_n\right)=I^{\psi}\left(p_1,p_2,\dots,p_n\right),$$

For $x>0$ and $q>0$ with $q\neq1$, we define the q-exponential function as the inverse function of the q-logarithmic function by $exp_q(x)=[1+(1-q)x]^{\frac{1}{1-q}}$, if $1+(1-q)x>0$, otherwise it is undefined. If we take $\psi(x)=ln_q(x)$, then we have

$I_q^{ln_q}(p_1,p_2,...,p_n)=H_q(p_1,p_2,...,p_n)$. Furthermore, we have

$$I_q^{x^{1-q}}(p_1,p_2,...,p_n)=ln_q\left(\sum_{j=1}^n p_j p_j^{\,q-1}\right)^{\frac{1}{1-q}}=ln_q\left(\sum_{j=1}^n p_j^{\,q}\right)^{\frac{1}{1-q}}$$

$$=\frac{\left[\left(\sum_{j=1}^n p_j^{\,q}\right)^{\frac{1}{1-q}}\right]^{1-q}-1}{1-q}=\frac{\sum_{j=1}^n\left(p_j^{\,q}-p_j\right)}{1-q}=H_q(p_1,p_2,...,p_n).$$

Proposition 2.1.2.3 ([Furuichi-Minculete-Mitroi, 75]). *The Tsallis quasilinear entropy is nonnegative:*

$$I_q^{\psi}(p_1,p_2,...,p_n)\geq 0.$$

We note here that the q-exponential function gives us the following connection between Renyi entropy and Tsallis entropy [201]:

(2.1.2.9) $$exp\,R_q(p_1,p_2,...,p_n)=exp_q\,H_q(p_1,p_2,...,p_n).$$

We should note here $exp_q\,H_q(p_1,p_2,...,p_n)$ is always defined, since we have

(2.1.2.10) $$1+(1-q)H_q(p_1,p_2,...,p_n)=\sum_{j=1}^n p_j^q>0.$$

Definition 2.1.2.4. For a continuous and strictly monotonic function ψ on $(0,\infty)$ and two probability distributions $\{p_1, p_2, \ldots, p_n\}$ and $\{r_1, r_2, \ldots, r_n\}$ with $p_j > 0$, $r_j > 0$ for all $j=\overline{1,n}$, the Tsallis quasilinear relative entropy is defined by

(2.1.2.11) $$D_q^{\psi}(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)\equiv -ln_q\,\psi^{-1}\left(\sum_{j=1}^n p_j\psi\left(\frac{r_j}{p_j}\right)\right),$$

For $\psi(x)=ln_q(x)$, the Tsallis quasilinear relative entropy becomes Tsallis relative entropy,

$$D_q^{ln_q}(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)\equiv -\sum_{j=1}^n p_j\,ln_q\left(\frac{r_j}{p_j}\right)=D_q(p_1,p_2,...,p_n\|r_1,r_2,...,r_n),$$

and for $\psi(x)=x^{1-q}$, we have

$$D_q^{x^{1-q}}(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)=-ln_q\left(\sum_{j=1}^n p_j\left(\frac{r_j}{p_j}\right)^{1-q}\right)^{\frac{1}{1-q}}=-ln_q\left(\sum_{j=1}^n p_j^q r_j^{\,1-q}\right)^{\frac{1}{1-q}}$$

$$=-\frac{1}{1-q}\left[\left[\left(\sum_{j=1}^n p_j\left(\frac{r_j}{p_j}\right)^{1-q}\right)^{\frac{1}{1-q}}\right]^{1-q}-1\right]=\frac{\sum_{j=1}^n\left(p_j-p_j^q r_j^{\,1-q}\right)}{1-q}=D_q(p_1,p_2,...,p_n\|r_1,r_2,...,r_n).$$

Proposition 2.1.2.5 ([Furuichi-Minculete-Mitroi, 75]). *If ψ is a concave increasing function or a convex decreasing function, then we have nonnegativity of the Tsallis quasilinear relative entropy*:

$$D_q^{\psi}(p_1,p_2,...,p_n\|r_1,r_2,...,r_n)\geq 0.$$

Proof. We firstly assume that ψ is a concave increasing function. The concavity of ψ shows that we have

$$\psi\left(\sum_{j=1}^{n} p_j\left(\frac{r_j}{p_j}\right)\right) \geq \sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)$$

which is equivalent to

$$\psi(1) \geq \sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)$$

From the assumption, ψ^{-1} is also increasing so that we have $1 \geq \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right)$.

Therefore, we have $-ln_q\, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right) \geq 0$, since $\ln_q x$ is increasing and $\ln_q(1) = 0$. For the case that ψ is a convex decreasing function, we can similarly prove the nonnegativity of the Tsallis quasilinear relative entropy.

□

Remark 2.1.2.6. The following two functions satisfy the sufficient condition in the above proposition:
(i) $\psi(x) = ln_q(x)$ for $q > 0, q \neq 1$.
(ii) $\psi(x) = x^{1-q}$ for $q > 0, q \neq 1$.

It is notable that the following identity holds:

(2.1.2.12) $$exp\, R_q(p_1, p_2, ..., p_n \| r_1, r_2, ..., r_n) = exp_{2-q}\, D_q(p_1, p_2, ..., p_n \| r_1, r_2, ..., r_n).$$

Next, we give inequalities for the Tsallis quasilinear entropy and f-divergence. For this purpose, we review the results obtained in [157] as one of the generalizations of refined Young's inequality.

Proposition 2.1.2.7 ([157]). *For two probability vectors* $\mathbf{p} = \{p_1, p_2, ..., p_n\}$ *and* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *such that* $p_j > 0, r_j > 0$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} r_j = 1$ *and* $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ *such that* $x_j \geq 0$, *we have*

(2.1.2.13) $$0 \leq \min_{1\leq i\leq n}\left\{\frac{r_i}{p_i}\right\} T(f, \mathbf{x}, \mathbf{p}) \leq T(f, \mathbf{x}, \mathbf{r}) \leq \max_{1\leq i\leq n}\left\{\frac{r_i}{p_i}\right\} T(f, \mathbf{x}, \mathbf{p}),$$

where $T(f, \mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n} p_j f(x_j) - f\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j \psi(x_j)\right)\right)$

for a continuous increasing function $\psi : I \to I$ *and a function* $f : I \to J$ *such that*

$$f\left(\psi^{-1}\left((1-\lambda)\psi(a) + \lambda\psi(b)\right)\right) \leq (1-\lambda) f(a) + \lambda f(b)$$

for any $a, b \in I$ *and any* $\lambda \in [0,1]$.

We have the following inequalities on the Tsallis quasilinear entropy and Tsallis entropy:

Theorem 2.1.2.8 ([Furuichi-Minculete-Mitroi, 75]). *For* $q \geq 0$, *a continuous and strictly monotonic function* ψ *on* $(0,\infty)$ *and a probability distribution* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *with* $r_j > 0$ *for all* $j = \overline{1,n}$, *and* $\sum_{j=1}^{n} r_j = 1$, *we have*

(2.1.2.14) $$0 \leq n \min_{1\leq i\leq n}\{r_i\}\left\{ln_q\, \psi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}\psi\left(\frac{1}{r_j}\right)\right) - \frac{1}{n}\sum_{j=1}^{n} ln_q \frac{1}{r_j}\right\},$$

$$\le I_q^{\psi}(r_1,r_2,...,r_n)-H_q(r_1,r_2,...,r_n)$$
$$\le n\max_{1\le i\le n}\{r_i\}\left\{ln_q\,\psi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}\psi\left(\frac{1}{r_j}\right)\right)-\frac{1}{n}\sum_{j=1}^{n}ln_q\frac{1}{r_j}\right\}.$$

Proof. If we take the uniform distribution $\mathbf{p}=\left\{\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right\}\equiv\mathbf{u}$ in Proposition 2.1.2.7, then we have

(2.1.2.15) $$0\le n\min_{1\le i\le n}\{r_i\}T(f,\mathbf{x},\mathbf{u})\le T(f,\mathbf{x},\mathbf{r})\le n\max_{1\le i\le n}\{r_i\}T(f,\mathbf{x},\mathbf{u}),$$

(which coincides with Theorem 3.3 in [157]). In the inequalities (2.1.2.15), we put $f(x)=-ln_q(x)$ and $x_j=\frac{1}{r_j}$ for any $j=\overline{1,n}$, then we obtain the statement.

□

Corollary 2.1.2.9 ([Furuichi-Minculete-Mitroi, 75]). *For $q\ge 0$ and a probability distribution $\mathbf{r}=\{r_1,r_2,...,r_n\}$ with $r_j>0$, for all $j=\overline{1,n}$ and $\sum_{j=1}^{n}r_j=1$, we have*

(2.1.2.16) $$0\le n\min_{1\le i\le n}\{r_i\}\left\{ln_q\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\right)-\frac{1}{n}\sum_{j=1}^{n}ln_q\frac{1}{r_j}\right\},$$
$$\le ln_q\,n-H_q(r_1,r_2,...,r_n)$$
$$\le n\max_{1\le i\le n}\{r_i\}\left\{ln_q\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\right)-\frac{1}{n}\sum_{j=1}^{n}ln_q\frac{1}{r_j}\right\}.$$

Proof. Put $f(x)=-ln_q(x)$ in Theorem 2.1.2.8.

Remark 2.1.2.10. Corollary 2.1.2.9 improves the well-known inequalities $0\le H_q(r_1,r_2,...,r_n)\le ln_q\,n$. If we take the limit $q\to 1$, the inequalities (2.1.2.16) recover Proposition 1 in [58].

Corollary 2.1.2.11 ([Furuichi-Minculete-Mitroi, 75]). *For two vectors $\mathbf{a}=\{a_1,a_2,...,a_n\}$ and $\mathbf{b}=\{b_1,b_2,...,b_n\}$ for all $j=\overline{1,n}$, we have*

(2.1.2.17) $$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\left(\sum_{i=1}^{n}a_ib_i\right)^2=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)^2=\sum_{1\le i\le j\le n}(a_ib_j-a_jb_i)^2.$$

Theorem 2.1.2.12 ([Furuichi-Minculete-Mitroi, 75]). *Let $f:I\to\mathbb{R}$ be a twice differentiable function such that there exists real constant m and M so that $0\le m\le f''(x)\le M$ for any $x\in I$. Then we have*

(2.1.2.18) $$\frac{m}{2}\sum_{1\le i<j\le n}p_ip_j(x_j-x_i)^2\le\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\le\frac{M}{2}\sum_{1\le i<j\le n}p_ip_j(x_j-x_i)^2,$$

where $p_j>0$, $\sum_{j=1}^{n}p_j=1$ and $x_j\in I$ for all $j=\overline{1,n}$.

Corollary 2.1.2.13 ([Furuichi-Minculete-Mitroi, 75]). *For two vectors $\mathbf{a}=\{a_1,a_2,...,a_n\}$ and $\mathbf{b}=\{b_1,b_2,...,b_n\}$ for all $j=\overline{1,n}$, we have*

(2.1.2.19) $$\sum_{1\le i<j\le n} p_i p_j (x_j - x_i)^2 = \sum_{j=1}^{n} p_j \left(x_j - \sum_{i=1}^{n} p_i x_i \right)^2 .$$

Corollary 2.1.2.14. *Under the assumptions of Theorem* 2.1.2.12, *we have*

(2.1.2.20) $$\frac{m}{2}\sum_{j=1}^{n} p_j \left(x_j - \sum_{i=1}^{n} p_i x_i \right)^2 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \le \frac{M}{2}\sum_{j=1}^{n} p_j \left(x_j - \sum_{i=1}^{n} p_i x_i \right)^2 ,$$

where $p_j > 0$, $\sum_{j=1}^{n} p_j = 1$ *and* $x_j \in I$ *for all* $j = \overline{1,n}$.

We also have the following inequalities for Tsallis entropy:

Theorem 2.1.2.15 ([Furuichi-Minculete-Mitroi, 75])**.** *For two probability distribution* $\mathbf{p} = \{p_1, p_2, ..., p_n\}$ *and* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *such that* $p_j > 0, r_j > 0$, $j = \overline{1,n}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} r_j = 1$, *we have*

(2.1.2.21) $$ln_q\left(\sum_{j=1}^{n} \frac{p_j}{r_j} \right) - ln_q n + \frac{m_q}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 - \frac{M_q}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2$$
$$\le \sum_{j=1}^{n} p_j \, ln_q \frac{1}{r_j} - \sum_{j=1}^{n} p_j \, ln_q \frac{1}{p_j}$$
$$\le ln_q\left(\sum_{j=1}^{n} \frac{p_j}{r_j} \right) - ln_q n + \frac{M_q}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 - \frac{m_q}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 ,$$

where m_q *and* M_q *are positive numbers depending on the parameter* $q \ge 0$ *and satisfying* $m_q \le q r_j^{-q-1} \le M_q$ *and* $m_q \le q p_j^{-q-1} \le M_q$, *for all* $j = \overline{1,n}$.

Corollary 2.1.2.16 ([Furuichi-Minculete-Mitroi, 75])**.** *For two probability distribution* $\mathbf{p} = \{p_1, p_2, ..., p_n\}$ *and* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *such that* $p_j > 0, r_j > 0$, $j = \overline{1,n}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} r_j = 1$, *we have*

(2.1.2.22) $$log\left(\sum_{j=1}^{n} \frac{p_j}{r_j} \right) - log\, n + \frac{m_1}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 - \frac{M_1}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2$$
$$\le \sum_{j=1}^{n} p_j \, log \frac{1}{r_j} - \sum_{j=1}^{n} p_j \, log \frac{1}{p_j}$$
$$\le log\left(\sum_{j=1}^{n} \frac{p_j}{r_j} \right) - log\, n + \frac{M_1}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 - \frac{m_1}{2} \sum_{1\le i<j\le n} p_i p_j \left(\frac{1}{p_j} - \frac{1}{p_i} \right)^2 ,$$

where m_1 *and* M_1 *are positive numbers satisfying* $m_1 \le r_j^{-2} \le M_1$ *and* $m_1 \le p_j^{-2} \le M_1$, *for all* $j = \overline{1,n}$.

Proof. Take the limit $q \to 1$ in Theorem 2.1.2.15.

□

Remark 2.1.2.17. The second part of the inequalities (2.1.2.22) gives the reverse inequality for the so-called *information inequality*

(2.1.2.23) $$0 \le \sum_{j=1}^{n} p_j \log \frac{1}{r_j} - \sum_{j=1}^{n} p_j \log \frac{1}{p_j},$$

which is equivalent to the non-negativity of the relative entropy

$$D_1(p_1, p_2, ..., p_n \| r_1, r_2, ..., r_n) \ge 0.$$

Using the inequality (2.1.2.23), we derive the following result.

Proposition 2.1.2.18 ([Furuichi-Minculete-Mitroi, 75]). *For two probability distribution* $\mathbf{p} = \{p_1, p_2, ..., p_n\}$ *and* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *such that* $p_j > 0, r_j > 0,\ j = \overline{1,n}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} r_j = 1$, *we have*

(2.1.2.24) $$\sum_{j=1}^{n} (1 - p_j) \log \frac{1}{1 - r_j} \ge \sum_{j=1}^{n} (1 - p_j) \log \frac{1}{1 - p_j}.$$

Proof. In the inequality (2.1.2.23), we take the substitutions $p_j \to \frac{1 - p_j}{n - 1}$ and $r_j \to \frac{1 - r_j}{n - 1}$, which satisfy $\sum_{j=1}^{n} \frac{1 - p_j}{n - 1} = \sum_{j=1}^{n} \frac{1 - r_j}{n - 1} = 1$. Then we have the present proposition.

□

Above we consider $\mathbf{p} = \{p_1, p_2, ..., p_n\}$ *and* $\mathbf{r} = \{r_1, r_2, ..., r_n\}$ *such that* $p_j > 0, r_j > 0,\ j = \overline{1,n}$, to be probability distributions. Tsallis relative entropy (divergence) is given by

$$D_q(\mathbf{p} \| \mathbf{r}) = D_q(p_1, p_2, ..., p_n \| r_1, r_2, ..., r_n) = -\sum_{j=1}^{n} p_j \ln_q \frac{r_j}{p_j}.$$

It converges to the classic Kullback-Leibler information:

$$\lim_{q \to 1} D_q(\mathbf{p} \| \mathbf{r}) = D_1(\mathbf{p} \| \mathbf{r}) = -\sum_{j=1}^{n} p_j \log \frac{r_j}{p_j}.$$

The Jeffreys divergence is defined by

(2.1.2.25) $$J_1(\mathbf{p} \| \mathbf{r}) \equiv D_1(\mathbf{p} \| \mathbf{r}) + D_1(\mathbf{r} \| \mathbf{p}).$$

and the Jensen-Shannon divergence is defined as

(2.1.2.26) $$JS_1(\mathbf{p} \| \mathbf{r}) \equiv \frac{1}{2} D_1\left(\mathbf{p} \middle\| \frac{\mathbf{p} + \mathbf{r}}{2}\right) + \frac{1}{2} D_1\left(\mathbf{r} \middle\| \frac{\mathbf{p} + \mathbf{r}}{2}\right),$$

(see e.g. [161]).

Before stating the results we establish the notation. The two-parameter extended logarithmic function (see e.g. [161]) to the (r,q)-logarithmic function for $x > 0$ is defined by

$$\ln_{r,q}(x) \equiv \ln_q \exp \ln_r x = \frac{\exp \frac{1 - q}{1 - r}\left(x^{1-r} - 1\right) - 1}{1 - q},$$

which uniformly converges to the usual logarithmic function *log(x)* in the limit *q*→1 and *r*→1.

This is a decreasing function with respect to indices. Correspondingly, the inverse function of $\ln_{r,q}(x)$ is denoted by

$$\exp_{r,q}(x) \equiv \exp_q \log \exp_r x.$$

We start from the Tsallis (r,q)-quasilinear entropies and Tsallis (r,q)-quasilinear divergences as they were defined in [89].

Definition 2.1.2.19. For a continuous and strictly monotonic function ψ on (0,∞) and q,r > 0 with $q,r \neq 1$, the Tsallis quasilinear entropy ((r,q)-quasilinear entropy) is defined by

$$I_{r,q}^{\psi}(p_1, p_2, \dots, p_n) \equiv ln_{r,q}\, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right), \tag{2.1.2.27}$$

For $\psi(x) = ln_{r,q}(x)$ we have the following entropic functional:

$$H_{r,q}(\mathbf{p}) = \sum_{j=1}^{n} p_j\, ln_{r,q} \frac{1}{p_j}. \tag{2.1.2.28}$$

This also gives rise to another case of interest

$$I_{\frac{2r-1}{r},q}^{x^{1-r}}(\mathbf{p}) = ln_q\, exp\, ln_{\frac{2r-1}{2}}\left(\sum_{j=1}^{n} p_j^r\right)^{\frac{1}{1-r}} = ln_q\, exp\left(\frac{r}{1-r}\left(\left(\sum_{j=1}^{n} p_j^r\right)^{\frac{1}{r}} - 1\right)\right),$$

which in particular case coincides with *Arimoto's entropy*.

Definition 2.1.2.20. For a continuous and strictly monotonic function ψ on (0,∞), $q,r > 0$ with $q,r \neq 1$, and two probability distributions $\{p_1, p_2, \dots, p_n\}$ and $\{r_1, r_2, \dots, r_n\}$ with $p_j > 0$, $r_j > 0$ for all $j = \overline{1,n}$, the (r,q)-quasilinear divergence is defined by

$$D_{r,q}^{\psi}(\mathbf{p}\|\mathbf{r}) \equiv -ln_{r,q}\, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right). \tag{2.1.2.29}$$

For $\psi(x) = ln_{r,q}(x)$ we the following:

$$D_{r,q}(\mathbf{p}\|\mathbf{r}) = -\sum_{j=1}^{n} p_j\, ln_{r,q} \frac{r_j}{p_j}. \tag{2.1.2.30}$$

By analogy to the entropy computation, we find the following Arimoto type divergence:

$$D_{\frac{2r-1}{r},q}^{x^{1-r}}(\mathbf{p}\|\mathbf{r}) = -ln_q\, exp\left(-\frac{r}{1-r}\left(1 - \left(\sum_{j=1}^{n} p_j^r r_j^{1-r}\right)^{\frac{1}{r}}\right)\right).$$

Proposition 2.1.2.21 ([Mitroi-Minculete, 161]). *Let r be a real number. Assume $p > 0, q > 0$ satisfy $p = \frac{1}{q}$. If $1 < p < 2$ or if $1 < q < 2$, then we have*

$$D_{r,q}(\mathbf{p}\|\mathbf{r}) + H_{2-r,2-q}(\mathbf{p}) \geq 1 - \sum_{j=1}^{n} p_j\, exp\left(ln_r \frac{r_j}{p_j} + ln_r\, p_j\right). \tag{2.1.2.31}$$

Theorem 2.1.2.22 ([Mitroi-Minculete, 161]). *Assume that real numbers p, q satisfy $\frac{1}{1-p} + \frac{1}{1-q} = 1$. If $1 < p < 2$ or if $1 < q < 2$, then we have*

$$\begin{aligned} J_{r,p}(\mathbf{p}\|\mathbf{r}) + J_{r,q}(\mathbf{p}\|\mathbf{r}) \geq {} & 2 - \sum_{j=1}^{n}\left[p_j\, exp\left(2 ln_r \frac{r_j}{p_j}\right) + r_j\, exp\left(2 ln_r \frac{p_j}{r_j}\right)\right] \\ & - \alpha \sum_{j=1}^{n}\left[p_j E\left(\frac{r_j}{p_j}\right) + r_j\, E\left(\frac{p_j}{r_j}\right)\right], \end{aligned} \tag{2.1.2.32}$$

where $E(x)=\left[exp\left(\frac{1-p}{2}ln_r\, x\right)-exp\left(\frac{1-q}{2}ln_r\, x\right)\right]^2,\ \alpha=min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

As we have seen in all these examples, in many cases the use of the (r,q) – generalized logarithmic function nicely completes the picture obtained with the q-logarithm and can be useful in applied areas (signal and image processing, information theory).

2.2 Inequalities for invertible positive operators

In Theory of Operators we found various characterizations and the relationship between operator monotonicity and operator convexity given by Hansen and Pedersen [104],Chansangiam [34].

In [121], Kubo-Ando has studied the connections between operator monotone functions and operator means. The operator monotone function plays an important role in the Kubo-Ando theory of operator connections and operator means. Other information about applications of operator monotone functions to theory of operators mean can be found in [180]. Theory of operator mean plays a central role in operator inequalities, operator equations, network theory, and quantum information theory.

Let H be a real Hilbert space. Denote by $B(H)$ the algebra of bounded linear operators on H. We write $A>0$ to means that A is a strictly positive operator, or equivalently, $A\geq 0$ and A is invertible. We note that I is the identity operator.

In [19], we found the quasi-arithmetic power mean $\#_{\alpha,p}$ with exponent α and weight p given by

$$A\#_{\alpha,p}B=\left[(1-p)A^{\alpha}+pB^{\alpha}\right]^{1/\alpha},\quad A,B\geq 0.$$

Several special cases of the family of quasi-arithmetic power means are the following: for $\alpha=1$, we have the weighted arithmetic mean as follows

$$A\nabla_p B:=A\#_{1,p}B=(1-p)A+pB,\quad A,B\geq 0;$$

for $\alpha=-1$, we obtain the weighted harmonic mean given as

$$A!_p B:=A\#_{-1,p}B=\left[(1-p)A^{-1}+pB^{-1}\right]^{-1},\quad A,B\geq 0;$$

for $\alpha\to 0$, we have the weighted geometric mean given by

$$A\#_p B=\lim_{\alpha\downarrow 0}A\#_{\alpha,p}B=A^{1-p}B^p,\quad A,B\geq 0 \text{ and } A,\ B \text{ commutes.}$$

The geometric mean was defined by Pusz and Woronowicz in [186]:

$$A\#B=max\left\{T\geq 0:\left|\langle Tx,y\rangle\right|\leq\left\|A^{1/2}x\right\|\cdot\left\|B^{1/2}y\right\|,\forall x,y\in H\right\},\quad A,B\geq 0.$$

In fact, this definition is the formula given by Ando in [15]:

$$A\#B=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2},\quad A,B>0.$$

Another definition of the geometric mean (see e.g. [14], [16]) is given by

$$A\#B=sup\left\{X\mid 0\leq X\quad and\quad \begin{bmatrix}A & X\\ X & B\end{bmatrix}\geq 0\right\},\quad A,B\geq 0.$$

An important remark [14] is that the geometric mean $A\#B$ is the unique positive solution to the Riccati equation

$$XA^{-1}X = B.$$

The p-weighted geometric mean is defined [16] by

$$A\#_p B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2},$$

where $p \in (0,1)$ and $A, B > 0$.

Furuta-Yanagida proved, in [92], the following inequality

$$A!_p B \le A\#_p B \le A\nabla_p B.$$

From the known inequality

$$\{(1-p)+pt^{-1}\}^{-1} \le t^p \le (1-p)+pt,$$

which implies

$$\{(1-p)+pt^{-\alpha}\}^{-1/\alpha} \le t^p \le \{(1-p)+pt^{\alpha}\}^{1/\alpha},\ p \in (0,1), \alpha > 0,$$

we deduce an inequality for the quasi-arithmetic power mean $\#_{\alpha,p}$

$$A\#_{-\alpha,p} B \le A\#_p B \le A\#_{\alpha,p} B.$$

Theorem B ([200]) *For invertible positive operators A and B with* $0 < mI \le A, B \le MI$*, we have*

(*i*) *(Ratio-type reverse inequality)*

(2.2.1) $$(1-p)A + pB \le S(h)A\#_p B,$$

(*ii*) *(Difference-type reverse inequality)*

(2.2.2) $$(1-p)A + pB \le A\#_p B + L(1,h)S(h)B,$$

where $p \in [0,1]$.

Next, we show two reverse inequalities which are different from (2.2.1) and (2.2.2) given in our paper [Furuichi-Minculete, 76].

We first show the following remarkable scalar inequality:

Theorem. 2.2.1 ([Furuichi-Minculete, 76]). *Let* $f:[a,b] \to \mathbb{R}$ *be a twice differentiable function such that there exist real constant M so that* $0 \le f'' \le M$*, for* $x \in [a,b]$*. Then the following inequalities hold:*

$$0 \le pf(a) + (1-p)f(b) - f(pa + (1-p)b) \le Mp(1-p)(b-a)^2$$

for all $p \in [0,1]$.

If we take, in inequality from above Theorem, $f(x) = -log\, x$ and afterwards $f(x) = -log\, x$, then we obtain

$$0 \le pa + (1-p)b - a^p b^{1-p} \le a^p b^{1-p}\, exp\left\{\frac{p(1-p)(a-b)^2}{m^2}\right\} - a^p b^{1-p},$$

and

$$0 \le pa + (1-p)b - a^p b^{1-p} \le p(1-p)\left\{log\,\frac{a}{b}\right\}^2 M.$$

From here, we consider bounded linear operators acting on a complex Hilbert space H. If a bounded linear operator A satisfies $A = A^*$, then A is called a *selfadjoint operator*. If a self-adjoint operator A satisfies $\langle x \mid A \mid x\rangle \ge 0$ for any $\mid x\rangle \in H$, then A is called a *positive operator*. In addition, $A \ge B$ means $A - B \ge 0$.

Theorem 2.2.2 ([Furuichi-Minculete, 76]). *For* $p \in [0,1]$, *two invertible positive operators* A *and* B *satisfying the ordering* $0 < mI \le A, B \le MI \le I$ *with* $h = \frac{M}{m}$ *we have*

(*i*) *(Ratio-type reverse inequality)*

$$A \#_p B \le (1-p)A + pB \le exp\left(p(1-p)\left(1-\frac{1}{h}\right)^2\right) A \#_p B, \tag{2.2.3}$$

(*ii*) *(Difference-type reverse inequality)*

$$A \#_p B \le (1-p)A + pB \le A \#_p B + p(1-p)(log^2 h)B. \tag{2.2.4}$$

Remark 2.2.3. It is natural to consider that our inequalities are better than Tominaga's inequalities under the assumption $A \le B$. The inequality that underlies the proof of inequality (2.2.1) is one of reverse inequalities for Young inequality that was given by Tominaga [200] by

$$pa + (1-p)b \le S\left(\frac{a}{b}\right) a^p b^{1-p}.$$

Therefore, we compare this inequality with the inequality

$$0 \le pa + (1-p)b - a^p b^{1-p} \le a^p b^{1-p} exp\left\{\frac{p(1-p)(a-b)^2}{m^2}\right\} - a^p b^{1-p}$$

used in the proof of inequality (2.2.3), thus [Furuichi-Minculete, 76]:

(i) Take $h = \frac{1}{2}$ and $p = \frac{1}{20}$, then we have

$$exp\left\{p(1-p)\left(1-\frac{1}{h}\right)^2\right\} - S(h) \cong -0.0128295.$$

(ii) Take $h = \frac{1}{2}$ and $p = \frac{1}{10}$, then we have

$$exp\left\{p(1-p)\left(1-\frac{1}{h}\right)^2\right\} - S(h) \cong 0.0326986.$$

Thus, we can conclude that there is no ordering between (2.2.3) and (2.2.1).

In [201], Tsallis defined the one-parameter extended entropy for the analysis of a physical model in statistical physics. The properties of Tsallis relative entropy was studied in [81] and [82], by Furuichi, Yanagi and Kuriyama.

The relative operator entropy

$$S(A \mid B) := A^{1/2} log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

for two invertible positive operators A and B on a Hilbert space was introduced by Fujii and Kamei in [73]. The parametric extension of the relative operator entropy was introduced by Furuta in [91] as

$$S_p(A \mid B) := A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^p log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2},$$

for $p \in \mathbb{R}$ and two invertible positive operators A and B on a Hilbert space. Note that $S_0(A \mid B) = S(A \mid B)$.

In [207], Yanagi, Kuriyama and Furuichi introduced a parametric extension of relative operator entropy by the concept of Tsallis relative entropy for operators, thus

$$T_p(A\,|\,B) := \frac{A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2} - A}{p},\ p \in (0,1],$$

where A and B are two positive invertible operators on a Hilbert space.

The relation between relative operator entropy $S\,(A\,|\,B)$ and Tsallis relative operator entropy $T_p(A\,|\,B)$ was considered in [82], as follows:

(2.2.5) $$A - AB^{-1}A \le T_{-p}(A\,|\,B) \le S(A\,|\,B) \le T_p(A\,|\,B) \le B - A.$$

The following known property of the Tsallis relative operator entropy is given in [108]:

Proposition 2.2.4. *For any strictly positive operators* A *and B and* $p,q \in [-1,0) \cup (0,1]$ *with* $p \le q$*, we have*

(2.2.6) $$T_p(A\,|\,B) \le T_q(A\,|\,B).$$

This proposition can be proved by the monotone increasing $\frac{x^p - 1}{p}$ on $p \in \mathbb{R}$ for any $x > 0$, and implies the following inequalities (which include the inequalities (2.2.5)):

$$A - AB^{-1}A = T_{-1}(A\,|\,B) \le T_{-p}(A\,|\,B) \le \mathrm{S}(A\,|\,B) \le T_p(A\,|\,B) \le \mathrm{T}_1(A\,|\,B) = B - A,$$

for any strictly positive operators A and B and $p \in (0, 1]$.

The main result from our paper [Moradi-Furuichi-Minculete,163] is a set of bounds that are complementary to (2.2.5). Some of our inequalities improve well-known ones. Among other inequalities, it is shown that if A, B are invertible positive operators and $p \in (0,1]$, then

$$A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I\right)A^{\frac{1}{2}} \le T_p(A\,|\,B) \le \frac{1}{2}\left(A\#_p B - A\#_{p-1} B + B - A\right),$$

which is a considerable refinement of (2.2.5), where I is the identity operator. We also prove a reverse inequality involving Tsallis relative operator entropy $T_p(A\,|\,B)$.

Theorem 2.2.5 ([Moradi-Furuichi-Minculete,163])**.** *For any invertible positive operator A and B such that* $A \le B$*, and* $p \in (0,1]$ *we have*

(2.2.7)

$$A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I\right)A^{\frac{1}{2}} \le T_p(A\,|\,B) \le \frac{1}{2}\left(A\#_p B - A\#_{p-1} B + B - A\right),$$

Proof. Consider the function $f(t) = t^{p-1}$, $p \in (0,1]$. It is easy to check that $f\,(t)$ is convex on $[1,\infty)$. Bearing in mind the fact

$$\int_1^x t^{p-1}dt = \frac{x^p - 1}{p},$$

and utilizing the left-hand side of Hermite-Hadamard inequality, one can see that

$$\left(\frac{x+1}{2}\right)^{p-1}(x-1) \le \frac{x^p - 1}{p},$$

where $x \ge 1$ and $p \in (0,1]$. On the other hand, it follows from the right-hand side of Hermite-Hadamard inequality that

$$\frac{x^p - 1}{p} \le \left(\frac{x^{p-1}+1}{2}\right)(x-1),$$

for each $x \ge 1$ and $p \in (0,1]$.

Replacing x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in above inequalities, and multiplying $A^{\frac{1}{2}}$ on both sides, we get the desired result.

□

Proposition 2.2.6 ([Moradi-Furuichi-Minculete,163]). For $x \ge 1$ and $\frac{1}{2} \le p \le 1$, we have

(2.2.8) $$\frac{x-1}{\sqrt{x}} \le \left(\frac{x+1}{2}\right)^{p-1}(x-1).$$

Proof. In order to prove (2.2.7), we set the function $f_p(x) \equiv \left(\frac{x+1}{2}\right)^{p-1} - \frac{1}{\sqrt{x}}$, where $x \ge 1$ and $\frac{1}{2} \le p \le 1$. Since $f_p{}'(x) = \left(\frac{x+1}{2}\right)^{p-1} log\left(\frac{x+1}{2}\right) \ge 0$, for $x \ge 1$. Therefore, we have $f_p(x) \ge f_{1/2}(x) = \frac{\sqrt{2x} - \sqrt{x+1}}{\sqrt{x(x+1)}} \ge 0$, for $x \ge 1$. Consequently, we deduce the inequality of the statement.

□

Corollary 2.2.7 ([Moradi-Furuichi-Minculete,163]). *For any invertible positive operators A and B such that $A \ge B$, and $p \in (0,1]$, we have*

(2.2.9) $$A\#_p B - A\#_{p-1} B \le \frac{1}{2}\left(A\#_p B - A\#_{p-1} B + B - A\right) \le T_p(A \mid B)$$
$$\le A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I\right)A^{\frac{1}{2}} \le A\#_{p+1} B - A\#_p B \le 0.$$

In [81], we found several results about the Tsallis relative operator entropy. Furuta [91] showed two reverse inequalities involving Tsallis relative operator entropy $T_p(A \mid B)$ via generalized Kantorovich constant $K(p)$.

Dragomir, Cerone and Sofo in [56, 57] and Niculescu and Persson in [166] present the following estimates of the precision in the Hermite-Hadamard inequality:

Proposition 2.2.8. *Let $f:[a,b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \le f'' \le M$. Then*

(2.2.10) $$m\frac{(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \le M\frac{(b-a)^2}{24}$$

and

(2.2.11) $$m\frac{(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le M\frac{(b-a)^2}{12}.$$

In this context, since $\int_1^x t^{p-1}dt = \frac{x^p - 1}{p}$, we have

Lemma 2.2.9. *For the real numbers* $x \geq 1$ *and* $p \in [-1,0) \cup (0,1)$*, the following inequalities*

(2.2.12) $$0 < (p-1)(p-2)x^{p-3}\frac{(x-1)^3}{24} \leq \frac{x^p - 1}{p} - (x-1)\left(\frac{x+1}{2}\right)^{p-1} \leq (p-1)(p-2)\frac{(x-1)^3}{24}$$

and

(2.2.13) $$0 < (p-1)(p-2)x^{p-3}\frac{(x-1)^3}{12} \leq \frac{x^p - x^{p-1} + x - 1}{2} - \frac{x^p - 1}{p} \leq (p-1)(p-2)\frac{(x-1)^3}{12}$$

hold.

Proof. For $x = 1$, we obtain the equality in relations (2.2.12) and (2.2.13).
We consider $x > 1$ and the function $f : [1, x] \to \mathbb{R}$ defined by $f(t) = t^{p-1}$ with $p \in [-1,0) \cup (0,1)$. It follows that $f'(t) = (p-1)t^{p-2}$ with $f''(t) = (p-1)(p-2)t^{p-3} \geq 0$, so the function *f* is convex and $(p-1)(p-2)x^{p-3} = m \leq f''(t) \leq M = (p-1)(p-2)$,
Therefore, we apply the above theorem and we have

$$(p-1)(p-2)x^{p-3}\frac{(x-1)^2}{24} \leq \frac{x^p - 1}{p(x-1)} - \left(\frac{x+1}{2}\right)^{p-1} \leq (p-1)(p-2)\frac{(x-1)^2}{24},$$

which is equivalent to inequality (2.2.12).
Using the second inequality from the above theorem we have

$$(p-1)(p-2)x^{p-3}\frac{(x-1)^2}{12} \leq \frac{x^{p-1} + 1}{2} - \frac{x^p - 1}{p(x-1)} \leq (p-1)(p-2)\frac{(x-1)^2}{12}.$$

□

Theorem 2.2.10 ([Moradi-Furuichi-Minculete,163]). For any invertible positive operator A and B such that $A \leq B$, *and* $p \in [0,1]$, we have

(2.2.14)
$$\frac{(p-1)(p-2)}{24}\left(pT_p(A \mid B) - 3(p-1)T_{p-1}(A \mid B) + 3(p-2)T_{p-2}(A \mid B) - (p-3)T_{p-3}(A \mid B)\right) \leq$$
$$T_p(A \mid B) - (B-A)A^{-1/2}\left(\frac{A^{-1/2}BA^{-1/2} + I}{2}\right)^{p-1}A^{1/2} \leq$$
$$\frac{(p-1)(p-2)}{24}\left(A\#_3 B - 3A\#_2 B + 3B - A\right)$$

and

(2.2.15)
$$\frac{(p-1)(p-2)}{12}\left(pT_p(A \mid B) - 3(p-1)T_{p-1}(A \mid B) + 3(p-2)T_{p-2}(A \mid B) - (p-3)T_{p-3}(A \mid B)\right) \leq$$
$$\frac{1}{2}\left(A\#_p B - A\#_{p-1} B + B - A\right) - T_p(A \mid B) \leq$$
$$\frac{(p-1)(p-2)}{12}\left(A\#_3 B - 3A\#_2 B + 3B - A\right)$$

Proof. If *A* and *B* are positive invertible operators such that $A \leq B$, then replacing *x* with the positive operator $A^{-1/2}BA^{-1/2}$ and multiplying by $A^{1/2}$ relations (2.2.12) and (2.2.13) we obtain

$$\frac{(p-1)(p-2)}{24}\left(A\#_p B-3A\#_{p-1}B+3A\#_{p-2}B-A\#_{p-3}B\right)\leq$$
$$T_p(A\,|\,B)-(B-A)A^{-1/2}\left(\frac{A^{-1/2}BA^{-1/2}+I}{2}\right)^{p-1}A^{1/2}\leq$$
$$(p-1)(p-2)\frac{A\#_3 B-3A\#_2 B+3B-A}{24}$$

and

$$\frac{(p-1)(p-2)}{12}\left(A\#_p B-3A\#_{p-1}B+3A\#_{p-2}B-A\#_{p-3}B\right)\leq$$
$$\frac{1}{2}\left(A\#_p B-A\#_{p-1}B+B-A\right)-T_p(A\,|\,B)\leq$$
$$\frac{(p-1)(p-2)}{12}\left(A\#_p B-3A\#_{p-1}B+3A\#_{p-2}B-A\#_{p-3}B\right).$$

But, replacing $A\#_p B=pT_p(A\,|\,B)+A$ in the above inequalities implies the inequalities of the statement.

□

In this moment, we see that equality

$$\int_1^t (x-1)\left(\frac{x+1}{2}\right)^{p-2}dx=\frac{2}{p-1}(t-1)\left(\frac{t+1}{2}\right)^{p-1}-\frac{4}{p(p-1)}\left[\left(\frac{t+1}{2}\right)^{p}-1\right]$$

which can be written as

$$\int_1^t (x-1)\left(\frac{x+1}{2}\right)^{p-2}dx=\frac{4}{p}\left[\left(\frac{t+1}{2}\right)^{p}-1\right]-\frac{4}{p-1}\left[\left(\frac{t+1}{2}\right)^{p-1}-1\right].$$

Remark 2.2.11. Therefore, the inequality from Theorem 2.2.10, can be rewritten as:
For any strictly positive operators A and B such that $A\leq B$, *and* $p\in(0,1)$, *we have*

(2.2.16) $$4\left[T_p\left(A\,|\,\frac{A+B}{2}\right)-T_{p-1}\left(A\,|\,\frac{A+B}{2}\right)\right]\leq\frac{1}{p-1}\left[T_p(A\,|\,B)-T_1(A\,|\,B)\right]$$
$$\leq\frac{1}{2}\left[T_p(A\,|\,B)-T_{p-1}(A\,|\,B)\right]+\frac{1}{4}A\#_2(B-A)$$

or, multiplying by $p-1<0$, we obtain

(2.2.17) $$\frac{p-1}{2}\left[T_p(A\,|\,B)-T_{p-1}(A\,|\,B)\right]+\frac{p-1}{4}A\#_2(B-A)\leq T_p(A\,|\,B)-T_1(A\,|\,B)$$
$$\leq 4(p-1)\left[T_p\left(A\,|\,\frac{A+B}{2}\right)-T_{p-1}\left(A\,|\,\frac{A+B}{2}\right)\right]$$

The below inequality implies inequality (2.2.15).

$$\frac{4}{p}\left[\left(\frac{t+1}{2}\right)^{p}-1\right]-\frac{4}{p-1}\left[\left(\frac{t+1}{2}\right)^{p-1}-1\right]\leq\frac{t^p-1}{p(p-1)}-\frac{t-1}{p-1}\leq\frac{1}{2}\left(\frac{t^p-1}{p}-\frac{t^{p-1}-1}{p-1}\right)+\frac{1}{4}(t-1)^2.$$

We can prove that

$$\frac{1}{2}\left(\frac{t^p-1}{p}-\frac{t^{p-1}-1}{p-1}\right)\le\frac{t^p-1}{p(p-1)}-\frac{t-1}{p-1},$$

Because we put $h_p(t)=\frac{t^p-1}{p(p-1)}-\frac{t-1}{p-1}-\frac{1}{2}\left(\frac{t^p-1}{p}-\frac{t^{p-1}-1}{p-1}\right)$, for $t\ge1, 0<p<1$.

But $\frac{dh_p(t)}{dt}=\frac{t^{p-1}-1}{p-1}-\frac{1}{2}\left(t^{p-1}-t^{p-2}\right)$, $\frac{d^2h_p(t)}{dt^2}=\frac{(2-p)t^{p-3}}{2}\left(\frac{3-p}{2-p}t-1\right)>0$, for $t\ge1, 0<p<1$, thus we have $\frac{dh_p(t)}{dt}\ge\frac{dh_p(1)}{dt}=0$. It follows that $h_p(t)\ge h_p(1)=0$.

Therefore, we deduce the inequality

$$\frac{1}{2}\left(\frac{t^p-1}{p}-\frac{t^{p-1}-1}{p-1}\right)\le\frac{t^p-1}{p(p-1)}-\frac{t-1}{p-1}\le\frac{1}{2}\left(\frac{t^p-1}{p}-\frac{t^{p-1}-1}{p-1}\right)+\frac{1}{4}(t-1)^2,$$

which implies the inequality

(2.2.18)
$$\frac{1}{2}\left[T_p(A\,|\,B)-T_{p-1}(A\,|\,B)\right]\le\frac{1}{p-1}\left[T_p(A\,|\,B)-T_1(A\,|\,B)\right]$$
$$\le\frac{1}{2}\left[T_p(A\,|\,B)-T_{p-1}(A\,|\,B)\right]+\frac{1}{4}A\#_2(B-A).$$

In a recent study, Furuichi and Minculete showed that:

Theorem 2.2.12. *For any strictly positive operators A and B such that $A\le B$, and $p\in(0,1)$, we have*

(2.2.19)
$$\frac{A\nabla_p B-A\#_p B}{p(1-p)}\le\frac{A\nabla_q B-A\#_q B}{q(1-q)}.$$

Proof. We have the identity: $\int_1^x\int_1^t y^{p-2}dydt=\frac{x^p-1}{p(p-1)}-\frac{x-1}{p-1}$.

Therefore, we deduce the following inequality, for $x\ge1, 0<p<1$,

$$\int_1^x\int_1^t\left(y^{p-2}-y^{q-2}\right)dydt=\frac{x^p-1-p(x-1)}{p(p-1)}-\frac{x^q-1-p(x-1)}{q(q-1)},$$

But, for $y\ge1, 0<p,q<1$, we have $y^{p-2}\le y^{q-2}$, which implies

$$\int_1^x\int_1^t\left(y^{p-2}-y^{q-2}\right)dydt\le0,$$

so we obtain

$$\frac{x^p-1-p(x-1)}{p(p-1)}\le\frac{x^q-1-p(x-1)}{q(q-1)}.$$

It follows that

$$\frac{px+(1-p)-x^p}{p(1-p)}\le\frac{qx+(1-q)-x^q}{q(1-q)},$$

which, replacing *x* by $A^{-1/2}BA^{-1/2}$, and multiplying by $A^{1/2}$ to left and to right, implies the statement.

□

More interesting things happen when we apply these considerations to the operators.

For instance, from the inequality (1.4.24) it follows that:

Theorem 2.2.13 ([Moradi-Furuichi-Minculete,163]). *Let A, B be two invertible positive operators such that* $I < h'I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le hI$ *or* $0 < hI \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le h'I < I$, *then we have*

(2.2.20) $$K^r(h',2)A\#_p B \le A\nabla_p B \le K^{1-r}(h',2)A\#_p B.$$

where $p \in [0,1]$, $r = min\{p, 1-p\}$.

Ando's inequality [16] says that if *A*, *B* are positive operators and Φ is a positive linear mapping, then

(2.2.21) $$\Phi(A\#_p B) \le \Phi(A)\#_p \Phi(B),\ p \in [0,1]\ .$$

Concerning inequality (2.2.21), we have the following corollary:

Corollary 2.2.14 ([Moradi-Furuichi-Minculete,163]). *Let A, B be two invertible positive operators such that* $I < h'I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le hI$ *or* $0 < hI \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le h'I < I$. *Let* Φ *is a positive linear mapping on* $B(H)$, *then we have*

(2.2.22) $$\frac{K^r(h',2)}{K^{1-r}(h,2)}\Phi(A\#_p B) \le \frac{1}{K^{1-r}(h,2)}\Phi(A\nabla_p B) \le \Phi(A)\#_p \Phi(B) \le$$
$$\frac{1}{K^r(h',2)}\Phi(A\nabla_p B) \le \frac{K^{1-r}(h,2)}{K^r(h',2)}\Phi(A\#_p B).$$

where $p \in [0,1]$ and $r = min\{p, 1-p\}$.

Remark 2.2.15. It is well-known that the generalized Kantorovich constant *K*(*h*, *p*) [94] is defined by

(2.2.23) $$K(h,p) := \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p,$$

for all $p \in \mathbb{R}$, $p \ne 0,1$. By virtue of a generalized Kantorovich constant, in the matrix setting,

Bourin *et al.* in [24] gave the following reverse of Ando's inequality for a positive linear map: let *A* and *B* be positive operators such that *mA≤B≤MA*, and let Φ be a positive linear map. Then

(2.2.24) $$\Phi(A)\#_p \Phi(B) \le \frac{1}{K(h,p)}\Phi(A\#_p B),\ p \in [0,1],$$

where $h = \frac{M}{m}$. The above result naturally extends one proved in Lee [123] for $h = \frac{M}{m}$.

After discussion on inequalities related to the operator mean with positive linear map, we give a result on Tsallis relative operator entropy with a positive linear map. It is well-known that Tsallis relative operator entropy has the following information monotonicity:

(2.2.25) $$\Phi(T_p(A \mid B)) \le T_p(\Phi(A) \mid \Phi(B)),\ p \in [0,1],$$

Using relation (1.4.23), we have the following counterpart of (2.2.25):

Theorem 2.2.16 ([Moradi-Furuichi-Minculete,163]). *Let A, B be two invertible positive operators. Let* Φ *be normalized positive linear map on B(H), then*

(2.2.26) $$\frac{2r}{p}(\Phi(A\nabla B) - \Phi(A)\#\Phi(B)) + T_p(\Phi(A) \mid \Phi(B))$$
$$\le \Phi(B-A)$$

$$\leq \frac{2(1-r)}{p}(\Phi(A\nabla B)-\Phi(A\# B))+T_p(A\mid B),$$

where $p\in(0,1]$ and $r=min\{p,1-p\}$.

Tsallis relative entropy $D_p(A\|B)$ for two positive operators A and B is defined by:

$$D_p(A\|B):=\frac{1}{p}\left(Tr[A]-Tr\left[A^{1-p}B^p\right]\right),\ p\in(0,1]\ .$$

In information theory, relative entropy (divergence) is usually defined for density operators which are positive operators with unit trace. However, we consider Tsallis relative entropy defined for positive operators to derive the relation with Tsallis relative operator entropy. If A and B are positive operators, then

(2.2.27) $$Tr[A-B]\leq D_p(A\|B)\leq -Tr[T_p(A\mid B)],\ p\in(0,1]\ .$$

Note that the first inequality of (2.2.27) is due to Furuta [91] and the second inequality is due to Furuichi *et al.* [81].

As a direct consequence of Theorem 2.2.16, we have the following interesting relation, for $\Phi(X)=\frac{1}{dim\,H}Tr[X]$:

Theorem 2.2.17 ([Moradi-Furuichi-Minculete,163])**.** *Let A, B be two positive operators on a finite dimensional Hilbert space H , then*

(2.2.28) $$\frac{2(1-r)}{p}\left(Tr[A\# B]-\frac{Tr[A+B]}{2}\right)-Tr[T_p(A\mid B)]$$
$$\leq Tr[A-B]$$
$$\leq \frac{2r}{p}\left(\sqrt{Tr[A]Tr[B]}-\frac{Tr[A+B]}{2}\right)+D_p(A\|B),$$

where $p\in(0,1]$ *and* $r=min\{p,1-p\}$.

The inequalities in Theorem 2.2.5 are improvements of the inequalities (2.2.5). In the present section, we give the alternative tight bounds for the Tsallis relative operator entropy.

Theorem 2.2.18 ([Furuichi-Minculete, 79])**.** *Let A* and *B be strictly positive operators and let* $-1\leq p\leq 1$ *with* $p\neq 0$. *If* $A\leq B$, *then*

(2.2.29) $$S_{p/2}(A\mid B)\leq T_p(A\mid B)\leq \frac{S(A\mid B)+S_p(A\mid B)}{2}.$$

If $B\leq A$, *then*

(2.2.30) $$\frac{S(A\mid B)+S_p(A\mid B)}{2}\leq T_p(A\mid B)\leq S_{p/2}(A\mid B).$$

Proof. For $x\geq 1$ and $-1\leq p\leq 1$ with $p\neq 0$, we define the function $f(t)=x^{pt}\log x$ with $0\leq t\leq 1$. Since $\frac{d^2f(t)}{dt^2}=p^2x^{pt}(\log x)^3\geq 0$ for $x\geq 1$, the function $f(t)$ is convex on *t*, for the case $x\geq 1$. Therefore, we have

(2.2.31) $$x^{p/2}\log x\leq \frac{x^p-1}{p}\leq\left(\frac{x^p+1}{2}\right)\log x,$$

by Hermite-Hadamard inequality, since $\int_0^1 f(t)dt=\frac{x^p-1}{p}$. By Kubo-Ando theory [121], we have the following inequality

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{p/2} log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \le \frac{A\#_p B - A}{p}$$

$$\le \frac{A^{1/2} log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} + A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}}{2},$$

which is the inequality (2.2.29). The inequalities (2.2.30) can be similarly shown by the concavity of the function *f(t)* on *t*, for the case $0 < x \le 1$.

□

We note that both sides in the inequalities (2.2.29) and (2.2.30) converge to $S(A\,|\,B)$ in the limit $p \to 0$. From the proof of Theorem 2.2.18, for strictly positive operators A and B, we see

$$\int_0^1 S_{pt}(A\,|\,B)dt = T_p(A\,|\,B).$$

Remark 2.2.19 ([Furuichi-Minculete, 79])**.** For the case $0 \le p \le 1$ we see

$$S(A\,|\,B) \le S_{p/2}(A\,|\,B) \le T_p(A\,|\,B) \le \frac{S(A\,|\,B) + S_p(A\,|\,B)}{2} \le S_p(A\,|\,B) \tag{2.2.32}$$

from inequalities (2.2.29) since function $x^p \log x$ is monotone, increasing on $0 < p \le 1$ and $\frac{x^p + 1}{2} \log x \le x^p \log x$ for $x \ge 1$ and $0 < p \le 1$. For the case $-1 \le p < 0$, we also see

$$S_p(A\,|\,B) \le \frac{S(A\,|\,B) + S_p(A\,|\,B)}{2} \le T_p(A\,|\,B) \le S_{p/2}(A\,|\,B) \le S(A\,|\,B) \tag{2.2.33}$$

from inequalities (2.2.30) since function $x^p \log x$ is monotone, increasing on $-1 \le p < 0$ and $\frac{x^p + 1}{2} \log x \ge x^p \log x$ for $0 < x \le 1$ and $-1 \le p < 0$.

□

We will make some considerations about the generalized Kantorovich constant $K(h, p)$ given in relation (2.2.23), namely:

$$K(h,p) := \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p - 1}{h^p - h}\right)^p,$$

for all $p \in \mathbb{R}$, $p \ne 0,1$.

If we take $a = h^p - 1$ and $b = \frac{p}{p-1}\left(h^p - h\right)$, then we have $pa + (1-p)b = p\left(h^p - 1\right) - p\left(h^p - h\right) = p(h-1)$, so we deduce the following relation:

$$K(h,p) = \frac{a^p b^{1-p}}{pa + (p-1)b},$$

for all $p \in \mathbb{R}$, $p \ne 0,1$.

Taking into account the above remark, we can estimate the generalized Kantorovich constant using several inequalities related to Young's inequality. By exemple, using the inequality given by Kittaneh and Manasrah [116], in the following form

$$\frac{r\left(\sqrt{a} - \sqrt{b}\right)^2}{pa + (1-p)b} \le 1 - \frac{a^p b^{1-p}}{pa + (1-p)b} \le \frac{(1-r)\left(\sqrt{a} - \sqrt{b}\right)^2}{pa + (1-p)b},$$

where $p \in [0,1]$ and $r = min\{p, 1-p\}$, we find

(2.2.34)
$$\frac{r\left(\sqrt{(1-p)(h^p-1)}-\sqrt{p(h-h^p)}\right)^2}{p(1-p)(h-1)}\le 1-K(h,p)\le\frac{(1-r)\left(\sqrt{(1-p)(h^p-1)}-\sqrt{p(h-h^p)}\right)^2}{p(1-p)(h-1)},$$
where $h\ge 1$, $p\in(0,1)$ and $r=min\{p,1-p\}$.

In [Minculete, 151], we show another improvement of the Young inequality, see relation (1.4.23), thus:
$$\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r}\le\frac{pa+(1-p)b}{a^pb^{1-p}}\le\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)},$$
for the positive real numbers a, b and $p\in[0,1]$ and $r=min\{p,1-p\}$. This implies the following estimate for the generalized Kantorovich constant
$$\left(\frac{1}{p(1-p)}\frac{(p+ph+h^p-1)^2}{4(h-h^p)(h^p-1)}\right)^r\le\frac{1}{K(h,p)}\le\left(\frac{1}{p(1-p)}\frac{(p+ph+h^p-1)^2}{4(h-h^p)(h^p-1)}\right)^{1-r},$$
so, we obtain

(2.2.35) $$\left(p(1-p)\frac{4(h-h^p)(h^p-1)}{(p+ph+h^p-1)^2}\right)^{1-r}\le K(h,p)\le\left(p(1-p)\frac{4(h-h^p)(h^p-1)}{(p+ph+h^p-1)^2}\right)^r,$$
where $p\in(0,1)$ and $r=min\{p,1-p\}$.

Using inequality (1.4.24) which is given by Kantorovich constant, we have:
$$K^r(h',2)\le\frac{pa+(1-p)b}{a^pb^{1-p}}\le K^{1-r}(h',2),$$
where a, b>0, $p\in[0,1]$, $r=min\{p,1-p\}$, $K(h',2)=\frac{(h'+1)^2}{4h'}$ and $h'=\frac{b}{a}$. This inequality implies the following inequality

(2.2.36) $$K^{r-1}(h',2)\le K(h,p)\le K^{-r}(h',2),$$
where $p\in(0,1)$, $r=min\{p,1-p\}$, $K(h',2)=\frac{(h'+1)^2}{4h'}$ and $h'=\frac{p}{p-1}\frac{h^p-1}{h^p-1}$.

Chapter 3

Inequalities in an inner product space

The aim of this sections is to show new results about the Cauchy - Schwarz inequality in an inner product space and many other estimates of some classical inequalities.

We show a refinement of the triangle inequality in a normed space using integrals and the Tapia semi-product.

The theory of inequalities plays an important role in many areas of Mathematics. Among the most used inequalities we find the triangle inequality. We present several characterizations of it.

We also show another reverse inequality for the Cauchy-Schwarz inequality and for triangle inequality in an inner product space.

We find an improvement of Buzano's inequality and Richard's inequality, which are extensions of the Cauchy - Schwarz inequality.

Starting from a geometrical inequality, we present several inequalities concerning the Cauchy - Schwarz inequality and a characterization of an inner product space.

3.1 On the Cauchy - Schwarz inequality in an inner product space

In a beautiful presentation, Niculescu [167] makes a radiography of the inequalities that have played an important role in the Theory of Inequalities. The Cauchy Inequality is one of them.

In 1821, Cauchy [31] showed the following identity:

(3.1.1) $$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)=\left(\sum_{i=1}^{n} a_i b_i\right)^2+\sum_{1\le i<j\le n}\left(a_i b_j-a_j b_i\right)^2 .$$

In fact this is *Lagrange's identity*, because, in 1773, Lagrange proved the identity

$$\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right)=\left(\sum_{i=1}^{3} a_i b_i\right)^2+\sum_{1\le i<j\le 3}\left(a_i b_j-a_j b_i\right)^2 ,$$

used in the study of some problems about the triangular pyramids.

In fact, we have $\|a\|^2\|b\|^2=\langle a,b\rangle^2+\|a\times b\|^2$, for all $a,b\in\mathbb{R}^3$.

In a more compact vector notation, Lagrange's identity is expressed as:

(3.1.2) $$\|a\|^2\|b\|^2-\langle a,b\rangle^2=\left(\sum_{i=1}^{n} a_i b_i\right)^2+\sum_{1\le i<j\le n}\left(a_i b_j-a_j b_i\right)^2 ,$$

where a and b are n-dimensional vectors with components that are real numbers.

A direct consequence of Lagrange's identity is the Cauchy-Buniakovski-Schwarz Inequality (CBS).

(3.1.3) $$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

This inequality was studied in many papers [8], [12], [17].

If $X = (X, \langle\cdot,\cdot\rangle)$ is an inner product space, then we have the Cauchy-Schwarz inequality, given by the following:

(3.1.4) $$\|x\| \cdot \|y\| \geq |\langle x, y\rangle|.$$

For all $x, y \in X$ and $a, b \in \mathbb{R}$, we have that

$$\|ax+by\|^2 = \langle ax+by, ax+by\rangle = a^2\|x\|^2 + 2ab\langle x, y\rangle + b^2\|y\|^2$$

implies

(3.1.5) $$\|ax+by\|^2 = (a\|x\| + b\|y\|)^2 - 2ab(\|x\| \cdot \|y\| - \langle x, y\rangle).$$

In relation (3.1.5), for nonzero vectors x and y and $a = \|x\|^{-1}$ and $b = -\|y\|^{-1}$, we obtain

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 = \frac{2}{\|x\| \cdot \|y\|}(\|x\| \cdot \|y\| - \langle x, y\rangle),$$

it follows that

(3.1.6) $$\frac{1}{2}\|x\| \cdot \|y\| \cdot \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 = \|x\| \cdot \|y\| - \langle x, y\rangle.$$

Therefore, we have $\|x\| \cdot \|y\| \geq |\langle x, y\rangle|$, because $\frac{1}{2}\|x\| \cdot \|y\| \cdot \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 \geq 0$.

Remark 3.1.1. Another proof for equality (3.1.6) can be given using Lagrange's barycentric identity (see e.g. [167])

$$\frac{1}{M}\sum_{k=1}^{n} m_k\|z - x_k\|^2 = \left\|z - \frac{1}{M}\sum_{k=1}^{n} m_k x_k\right\|^2 + \frac{1}{M^2}\sum_{1\leq i<j\leq n} m_i m_j\|x_i - x_j\|^2.$$

For $z = 0$, $n = 2$, $x = m_1 x_1$, $y = m_2 x_2$, we obtain

(3.1.7) $$\frac{\|x\|^2}{m_1} + \frac{\|y\|^2}{m_2} = \frac{\|x+y\|^2}{m_1+m_2} + \frac{m_1 m_2}{m_1+m_2}\left\|\frac{x}{m_1} - \frac{y}{m_2}\right\|^2.$$

If we take $m_1 = \|x\|$ and $m_2 = \|y\|$ in relation (3.1.7), we deduce the equality (3.1.6). A consequence of this equality is the following:

(3.1.8) $$\frac{\|x+y\|^2}{m_1+m_2} \leq \frac{\|x\|^2}{m_1} + \frac{\|y\|^2}{m_2} \leq \frac{\|x+y\|^2}{m_1+m_2} + \frac{m_1+m_2}{4}\left\|\frac{x}{m_1} - \frac{y}{m_2}\right\|^2.$$

Maligranda proved in [130] the following:

Theorem C. *For nonzero vectors x and y in a normed space $X = (X, \|\cdot\|)$ it is true that*

(3.1.9) $$\|x+y\| \leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)min(\|x\|, \|y\|)$$

and

(3.1.10) $$\|x+y\| \geq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)max(\|x\|, \|y\|).$$

If either $\|x\| = \|y\| = 1$ or $y = cx$ with $c > 0$, then equality holds in both (3.1.9) and (3.1.10).

Theorem 3.1.2. *If* $X=(X,\langle\cdot,\cdot\rangle)$ *is an inner product space over the field of real numbers* $\mathbb{R}$ *and the norm* $\|\cdot\|$ *is generated by an inner product* $\langle\cdot,\cdot\rangle$*, then we have*

$$\text{(3.1.11)}\qquad \frac{\|x\|+\|y\|+\|x+y\|}{2}\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)min(\|x\|,\|y\|)\le\|x\|\cdot\|y\|-\langle x,y\rangle\le$$
$$\frac{\|x\|+\|y\|+\|x+y\|}{2}\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)max(\|x\|,\|y\|).$$

for nonzero vectors x and y in X *.*
Proof. In relation (3.1.5) for $a=1$ and $b=1$ we deduce

$$\|x+y\|^2=(\|x\|+\|y\|)^2-2(\|x\|\cdot\|y\|-\langle x,y\rangle).$$

So, we deduce the equality, for nonzero vectors *x* and *y* in a normed space, given by the following:

$$2(\|x\|\cdot\|y\|-\langle x,y\rangle)=(\|x\|+\|y\|)^2-\|x+y\|^2,$$

which means that

$$\frac{2(\|x\|\cdot\|y\|-\langle x,y\rangle)}{\|x\|+\|y\|+\|x+y\|}=\|x\|+\|y\|-\|x+y\|.$$

Using this equality and inequalities of Maligranda, we find the following inequality:

$$\text{(3.1.12)}\quad \left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)min(\|x\|,\|y\|)\le\frac{2(\|x\|\cdot\|y\|-\langle x,y\rangle)}{\|x\|+\|y\|+\|x+y\|}\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)max(\|x\|,\|y\|),$$

which is equivalent to the inequality of the statement.

□

Remark 3.1.3. From inequality (3.1.11) and using the triangle inequality, we have $2\|x+y\|\le\|x\|+\|y\|+\|x+y\|\ge 2(\|x\|+\|y\|)$. Therefore, we obtain a refined of the Cauchy-Schwarz inequality, given by:
(3.1.13)

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\|x+y\|min(\|x\|,\|y\|)\le\|x\|\cdot\|y\|-\langle x,y\rangle\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)(\|x\|+\|y\|)max(\|x\|,\|y\|)\ .$$

Corollary 3.1.4. *If* $X=(X,\langle\cdot,\cdot\rangle)$ *is an inner product space over the field of real numbers* $\mathbb{R}$ *and the norm* $\|\cdot\|$ *is generated by an inner product* $\langle\cdot,\cdot\rangle$*, then we have*

$$\text{(3.1.14)}\qquad \left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\|x\|\cdot\|y\|\le\|x\|\cdot\|y\|-\langle x,y\rangle,$$

for nonzero vectors x and y in X *.*
Proof. We show that for vectors *x* and *y* in a normed space $X=(X,\|.\|)$ it is true that

$$\text{(3.1.15)}\qquad \frac{\|x\|+\|y\|+\|x+y\|}{2}min(\|x\|,\|y\|)\ge\|x\|\cdot\|y\|.$$

We suppose that $\|x\|\le\|y\|$, so $min(\|x\|,\|y\|)=\|x\|$. Therefore, we have $\|x\|^2+\|x\|\|y\|+\|x\|\|x+y\|\ge 2\|x\|\cdot\|y\|$, which implies $\|x\|+\|x+y\|\ge\|y\|$, which is true. Combining relations (3.1.11) and (3.1.15), we obtain the relation (3.1.14).

□

In our paper [Minculete-Păltănea, 148], we obtain refined estimates of the triangle inequality in a normed space using integrals and the Tapia semi-product. The particular case of an inner product space is discussed in more detail.

The theory of inequalities plays an important role in many areas of Mathematics. Among the most used inequalities we find the triangle inequality. This inequality is the following:

$$\|x+y\| \le \|x\| + \|y\|.$$

for any vectors x and y in the normed linear space $X = (X, \|\cdot\|)$over the real numbers or complex numbers. Its continuous version is

$$\left\| \int_a^b f(x)dx \right\| \le \int_a^b \|f(x)\| dx.$$

where $f : [a,b] \subset \mathbb{R} \to \mathbb{X}$ is a strongly measurable function on the compact interval $[a,b]$ with values in a Banach space X and $\|f(\cdot)\|$ is the Lebesgue integrable on $[a,b]$. Diaz and Metcalf [49] proved a reverse of the triangle inequality in the particular case of spaces with inner product. Several other reverses of the triangle inequality were obtained by Dragomir in [50]. Also, in [51], there are given some inequalities for the continuous version of the triangle inequality using the Bochner integrable functions.

In [188], Rajić gives a characterization of the norm triangle equality in pre-Hilbert C^*-modules. In [130, 131], Maligranda proved a refinement of the triangle inequality. In [112] Kato, Saito and Tamura proved the sharp triangle inequality and reverse inequality in Banach space for nonzero elements $x_1, x_2, ..., x_n \in X$, which is in fact a generalization of Maligranda's inequality. Another extension of Maligranda's inequality for n elements in a Banach space was obtained in Mitani and Saito [154]. The problem of characterization of all intermediate values C satisfying $0 \le C \le \sum_{k=1}^{n} \|x_k\| - \left\| \sum_{k=1}^{n} x_k \right\|$, for $x_1, x_2, ..., x_n$ in a Banach space is studied by Mineno, Nakamura and Ohwada [153], Dadipour *et al.* [46], Sano *et al.* [195] and others. For other different results about the triangle inequality we mention only [178].

The main aim of this paper is to provide an improvement of the inequality due to Maligranda. Some other estimates which follow from the triangle inequality are also presented. Moreover, we can rewrite them as estimates for the so-called *norm-angular distance* or *Clarkson distance* (see *e.g.* [40]) between nonzero x and y as $\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$.

This distance was generalized to the p-angular distance in normed space in [130]. In [64], Dragomir characterizes this distance obtaining new bounds for it. A survey on the results for bounds for the angular distance, named Dunkl-Williams type theorems (see [65]), is given by Moslehian *et al.* [162].

In our paper [Minculete-Păltănea, 148] we show several estimates of the triangle inequality using integrals.

Let $X = (X, \|\cdot\|)$ be a real normed space.

Lemma 3.1.5. *For any* $x, y \in X$, *the function* $g(s) = \|x + sy\|, s \in \mathbb{R}$, *is convex.*

Applying Hermite-Hadamard's inequality and Hammer-Bullen's inequality, we found the following:

Theorem 3.1.6 ([Minculete-Păltănea, 148]). *For any $x,y\in X$, we have*

$$\|x+y\|\le 2\int_0^1\|(1-\lambda)x+\lambda y\|d\lambda\le\|x\|+\|y\|, \tag{3.1.16}$$

$$\|x\|+\|y\|+\|x+y\|\ge 4\int_0^1\|(1-\lambda)x+\lambda y\|d\lambda. \tag{3.1.17}$$

Corollary 3.1.7 ([Minculete-Păltănea, 148]). *For nonzero elements x, y from a space with inner product $X=(X,\langle\cdot,\cdot\rangle)$ and $a,b\in\mathbb{R}$, $a<b$, we have*

$$\frac{2(\|x\|\|y\|-\langle x,y\rangle)}{\|x\|+\|y\|+2\int_0^1\|(1-\lambda)x+\lambda y\|d\lambda}\le\|x\|+\|y\|-\|x+y\|\le\frac{2(\|x\|\|y\|-\langle x,y\rangle)}{\|x+y\|+2\int_0^1\|(1-\lambda)x+\lambda y\|d\lambda}. \tag{3.1.18}$$

Inequality (3.1.18) represents an improvement of the Cauchy-Schwarz inequality.

Next, we will study estimates of the triangle inequality using the Tapia semi-product. The Tapia semi-product on the normed space X (see [199]) is the function $(\cdot,\cdot)_T : X\times X\to\mathbb{R}$, defined by

$$(x,y)_T := \lim_{\substack{t\to 0\\ t>0}}\frac{\varphi(x+ty)-\varphi(x)}{t},$$

where $\varphi(x)=\frac{1}{2}\|x\|^2$, $x\in X$.

The above limit exists for any pair of elements $x,y\in X$. The Tapia semi-product is positive homogeneous in each argument and satisfies the inequality $|(x,y)_T|\le\|x\|\|y\|$ for all $x,y\in X$. In the case when the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then $(x,y)_T=\langle x,y\rangle$, for all $x,y\in X$.

The Maligranda inequality (see Theorem C) can be written as: for nonzero vectors x and y in a normed space $X=(X,\|\cdot\|)$ it is true that

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)min(\|x\|,\|y\|)\le\|x\|+\|y\|-\|x+y\|\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)max(\|x\|,\|y\|). \tag{3.1.19}$$

If in inequality (3.1.19) we replace y by ty with $t>0$, then we obtain

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)min(\|x\|,t\|y\|)\le\|x\|+t\|y\|-\|x+ty\|\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)max(\|x\|,t\|y\|),$$

which is equivalent to

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\frac{1}{t}min(\|x\|,t\|y\|)\le\|y\|-\frac{\|x+ty\|-\|x\|}{t}\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\frac{1}{t}max(\|x\|,t\|y\|),$$

so, by passing to limit for $t\to 0, t>0$, we deduce

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\lim_{\substack{t\to 0\\ t>0}}\frac{1}{t}min(\|x\|,t\|y\|)\le\|y\|-\lim_{\substack{t\to 0\\ t>0}}\frac{\|x+ty\|-\|x\|}{t}\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\lim_{\substack{t\to 0\\ t>0}}\frac{1}{t}max(\|x\|,t\|y\|)$$

.

Since $\lim\limits_{\substack{t\to 0\\ t>0}}\frac{\|x+ty\|-\|x\|}{t}=\lim\limits_{\substack{t\to 0\\ t>0}}\frac{\|x+ty\|^2-\|x\|^2}{t(\|x+ty\|+\|x\|)}=\frac{(x,y)_T}{\|x\|}$ and for $t\to 0, t>0$, we have $min(\|x\|,t\|y\|)=t\|y\|$, so $\frac{1}{t}min(\|x\|,t\|y\|)=\|y\|$, we deduce the inequality

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\|x\|\|y\|\le\|x\|\|y\|-(x,y)_T. \tag{3.1.20}$$

This inequality can be written as

$$\|x\|\|y\|+(x,y)_T\le\big\|x\|y\|+y\|x\|\big\|. \tag{3.1.21}$$

For nonzero elements $x,y\in X$, if we replace x by $\frac{x}{\|x\|}$ and y by $\frac{y}{\|y\|}$ in inequality (3.1.20), then we find the following inequality

$$\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)_T\le\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|-1. \tag{3.1.22}$$

If $X=(X,\langle\cdot,\cdot\rangle)$ is a space with inner product, then for nonzero elements x, y, inequality (3.1.20) becomes

$$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\|x\|\|y\|\le\|x\|\|y\|-\langle x,y\rangle. \tag{3.1.23}$$

This inequality represents an improvement of Cauchy-Schwarz's inequality.

For nonzero elements $x,y\in X$ denote $v(x,y)=\frac{x}{\|x\|}+\frac{y}{\|y\|}$. Then inequality (3.1.20) becomes:

Theorem. 3.1.8 ([Minculete-Păltănea, 148]). *Let $x,y\in X$ be nonzero vectors. Then, we have*

$$(x,y)_T\le\|x\|\|y\|(\|v(x,y)\|-1). \tag{3.1.24}$$

Theorem. 3.1.9 ([Minculete-Păltănea, 148]). *Let $x,y\in X$ be nonzero vectors such that $\|y\|\le\|x\|$ and $x\|y\|\ne -y\|x\|$. Then, we have*

$$\|x\|+\|y\|-\|x+y\|\ge(2-\|v(x,y)\|)\|x\|-\left(1-\left(\frac{x+y}{\|x+y\|},\frac{y}{\|y\|}\right)_T\right)(\|x\|-\|y\|), \tag{3.1.25}$$

$$\|x\|+\|y\|-\|x+y\|\le(2-\|v(x,y)\|)\|x\|-\left(1-\left(\frac{v(x,y)}{\|v(x,y)\|},\frac{y}{\|y\|}\right)_T\right)(\|x\|-\|y\|), \tag{3.1.26}$$

$$\|x\|+\|y\|-\|x+y\|\ge(2-\|v(x,y)\|)\|y\|-\left(1-\left(\frac{x+y}{\|x+y\|},\frac{x}{\|x\|}\right)_T\right)(\|x\|-\|y\|) \tag{2.1.27}$$

and

$$\|x\|+\|y\|-\|x+y\|\le(2-\|v(x,y)\|)\|y\|-\left(1-\left(\frac{v(x,y)}{\|v(x,y)\|},\frac{x}{\|x\|}\right)_T\right)(\|x\|-\|y\|). \tag{3.1.28}$$

It is easy to see that we can write $\alpha[x,y]=v(x,-y)$. Using inequalities (3.1.26) and (3.1.27) we deduce the following double inequality:

Corollary 3.1.10 ([Minculete-Păltănea, 148]). *For nonzero vectors x and y, such that* $x\|y\| \neq -y\|x\|$, *we have*

$$(3.1.29)\qquad \frac{\|x-y\|-\left|\|x\|-\|y\|\right|}{min(\|x\|,\|y\|)}+A \leq \alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq \frac{\|x-y\|+\left|\|x\|-\|y\|\right|}{max(\|x\|,\|y\|)}-B,$$

where

$$A=\left(1-\left(\frac{x-y}{\|x-y\|},\frac{x}{\|x\|}\right)_T\right)\frac{\left|\|x\|-\|y\|\right|}{min(\|x\|,\|y\|)} \geq 0,\ B=\left(1-\left(\frac{v(x,-y)}{\|v(x,-y)\|},-\frac{y}{\|y\|}\right)_T\right)\frac{\left|\|x\|-\|y\|\right|}{max(\|x\|,\|y\|)} \geq 0.$$

In [Minculete-Păltănea, 148] to section 4 we derive many inequalities in an inner product space from Theorem 3.1.9.

3.2 Reverse inequalities for the Cauchy-Schwarz inequality in an inner product space

Let X be an inner product space over the field of real numbers $\mathbb{R}$. The inner product $<\cdot,\cdot>$ induces an associated norm, given by $\|x\|=\sqrt{<x,x>}$, for all $x \in X$, thus X is a normed vector space.

For nonzero vectors x and y in X we define *the angular distance* $\alpha[x,y]$ between x and y by

$$\alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|,$$

(see [40]).

Therefore, using relation (3.1.6), we prove that

$$(3.2.1)\qquad \frac{1}{2}\|x\|\cdot\|y\|\cdot(\alpha[x,y])^2=\|x\|\cdot\|y\|-\langle x,y\rangle.$$

Theorem 3.2.1. *If* $X=(X,\langle\cdot,\cdot\rangle)$ *is an inner product space over the field of real numbers* $\mathbb{R}$ *and the norm* $\|\cdot\|$ *is generated by an inner product* $\langle\cdot,\cdot\rangle$, *then we have*

$$(3.2.2)\qquad \|x\|\cdot\|y\|-\langle x,y\rangle \leq \frac{2\cdot\|x\|\cdot\|y\|\cdot\|x-y\|^2}{[max(\|x\|,\|y\|)]^2},$$

for nonzero vectors x and y in X.

Proof. Massera-Schäffer proved in [134] the following inequality: for nonzero vectors x and y in X there is the inequality

$$(3.2.3)\qquad \alpha[x,y]\cdot max(\|x\|,\|y\|) \leq 2\|x-y\|.$$

Combining relations (3.2.1) and (3.2.3) we deduce the inequality of the statement, which is in fact a reverse inequality of Cauchy-Schwarz inequality.

Remark 3.2.2. Dunkl and Wiliams showed, in [65], the inequality

$$(3.2.4)\qquad \alpha[x,y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}.$$

Using this inequality, we obtain another reverse inequality of Cauchy-Schwarz inequality.

Lemma 2.2.3 ([Stoica-Minculete-Barbu, 197]). *In an inner product space X over the field of real numbers* $\mathbb{R}$, *we have*

(3.2.5) $$\|y\|\left\|x+\frac{1}{2}y\right\| \ge \sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}\text{, for all } x,y\in X.$$

Proof. For $y=0$ we obtain the equality in relation (3.2.5). For alln $x,y\in X$, $y\neq 0$, we have $\frac{\|x\|^2\|y\|^2-\langle x,y\rangle^2}{\|y\|^2}=\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2$, which means that

$$\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}=\|y\|\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|.$$

Therefore, the inequality of the statement is equivalently with $\left\|x+\frac{1}{2}y\right\|\ge\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|$, which is equivalent to

$$\left\|x+\frac{1}{2}y\right\|^2\ge\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2,$$

which implies

$$\|x\|^2+\langle x,y\rangle+\frac{1}{4}\|y\|^2\ge\|x\|^2-2\frac{\langle x,y\rangle^2}{\|y\|^2}+\frac{\langle x,y\rangle^2}{\|y\|^2},$$

so, it follows that $\left(\frac{\langle x,y\rangle}{\|y\|^2}+\frac{1}{2}\|x\|\right)^2\ge 0$, for all $x,y\in X$, $y\neq 0$.

Remark 3.2.4. It is easy to see that

(3.2.6) $$\|y\|\left\|x-\frac{1}{2}y\right\|\ge\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}\text{, for all } x,y\in X.$$

Theorem 3.2.5 ([Stoica-Minculete-Barbu, 197]). *In an inner product space X over the field of real numbers* $\mathbb{R}$*, we have*

(3.2.7) $$\|x\|^2+\|y\|^2+\|x-y\|^2\ge 2\sqrt{3}\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2},$$

for all $x,y\in X$.

Proof. From the parallelogram law, for every $x,y\in X$, we deduce the following equality:

$$2\left(\|x+y\|^2+\|x\|^2\right)=\|2x+y\|^2+\|y\|^2,$$

which is equivalent to

$$2\left(\|x+y\|^2+\|x\|^2\right)-\|y\|^2=4\left\|x+\frac{1}{2}y\right\|^2$$

so

(3.2.8) $$\left\|x+\frac{1}{2}y\right\|^2=\frac{\|x+y\|^2+\|x\|^2}{2}-\frac{\|y\|^2}{4}.$$

Therefore, combining the relations (3.2.5) and (3.2.8), we obtain the following

$$\|x\|^2+\|y\|^2+\|x+y\|^2=\frac{1}{2}\left[2\left(\|x+y\|^2+\|x\|^2\right)-\|y\|^2\right]+\frac{3}{2}\|y\|^2$$

$$=2\left\|x+\frac{1}{2}y\right\|^2+\frac{3}{2}\|y\|^2\ge 2\sqrt{3}\|y\|\left\|x+\frac{1}{2}y\right\|\ge 2\sqrt{3}\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}.$$

Replacing y by $-y$ in above inequality, we prove the inequality of the statement.

Corollary 3.2.6. *In an inner product space X over the field of real numbers* $\mathbb{R}$, *we have*

(3.2.9) $$\frac{1}{2\sqrt{3}}\left(\|x\|^2+\|y\|^2+\|x-y\|^2\right)\geq\|x\|\cdot\|y\|-\langle x,y\rangle,$$

for all $x,y\in X$.

Proof. It is easy to see that $\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}\geq\|x\|\cdot\|y\|-\langle x,y\rangle$ and using inequality (3.2.7), we have the inequality of the statement.

Corollary 3.2.7 ([Stoica-Minculete-Barbu, 197]). *In an inner product space X over the field of real numbers* $\mathbb{R}$, *we have*

(3.2.10) $$\|y\|\cdot\left\|x-\frac{1}{2}y\right\|\geq\|x\|\cdot\|y\|-\langle x,y\rangle,$$

for all $x,y\in X$.

Proof. From Lemma 3.2.3, we have $\|y\|\left\|x-\frac{1}{2}y\right\|\geq\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}$, so

$$\|y\|\cdot\left\|x-\frac{1}{2}y\right\|\geq\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}\geq\|x\|\cdot\|y\|-\langle x,y\rangle.$$

Remark 3.2.8. From Corollary 3.2.7, it is easy to see that

(3.2.11) $$min\left\{\|x\|\left\|y-\frac{1}{2}x\right\|,\|y\|\left\|x-\frac{1}{2}y\right\|\right\}\geq\|x\|\cdot\|y\|-\langle x,y\rangle,$$ for all $x,y\in X$.

This inequality represents another reverse inequality for the Cauchy-Schwarz inequality in an inner product space.

Several applications are given below:

1. *In triangle ABC the inequality*

$$\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{AC}\right\|^2+\left\|\overrightarrow{BA}\right\|^2\geq 4\sqrt{3}\Delta,$$

is true, where Δ *is the area of the triangle ABC.*

Proof. Let E_3 be the Euclidean punctual space. If we take the vectors $a=\overrightarrow{BC},b=\overrightarrow{AC},c=\overrightarrow{AB}$ in inequality (3.2.7), then using the Lagrange identity, $\|a\|^2\|b\|^2-\langle a,b\rangle^2=\|a\times b\|^2$, we obtain the following inequality:

$$\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{AC}\right\|^2+\left\|\overrightarrow{BA}\right\|^2\geq 2\sqrt{3}\left\|\overrightarrow{BC}\times\overrightarrow{AC}\right\|=4\sqrt{3}\Delta,$$

which is in fact *the Ionescu-Weitzenböck inequality.*

2. Using inequality (3.2.2) and the relation for $\alpha[x,y]$, we deduce de following inequality for the angular distance $\alpha[x,y]$: *for nonzero vectors x and y in X, we have a lower bound for the angular distance* $\alpha[x,y]$ *given by*

$$\sqrt{2\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)}\leq\alpha[x,y].$$

3. For the space $\left(R^n, \langle\cdot,\cdot\rangle\right)$, where $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$, we have $\langle x,y\rangle = x_1y_1+x_2y_2+...+x_ny_n$ and $\|x\|=\sqrt{x_1^2+x_2^2+...+x_n^2}$. We use inequality (3.2.10), $\|x\|\cdot\|y\|-\langle x,y\rangle \le \|y\|\cdot\left\|x-\frac{1}{2}y\right\|$, thus:

$$0\le\sqrt{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)}-\sum_{i=1}^n x_iy_i \le \sqrt{\left(\sum_{i=1}^n y_i^2\right)\left(\sum_{i=1}^n\left(x_i-\frac{1}{2}y_i\right)^2\right)}\ .$$

4. For the space $\left(C^0([a,b]),\langle\cdot,\cdot\rangle\right)$, where $f,g\in C^0([a,b])$, we have $\langle f,g\rangle=\int_a^b f(x)g(x)dx$ and $\|f\|=\sqrt{\int_a^b f^2(x)dx}$. We use inequality (3.2.10), thus:

$$0\le\sqrt{\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx}-\int_a^b f(x)g(x)dx\le\sqrt{\int_a^b g^2(x)dx\cdot\int_a^b\left(f(x)-\frac{1}{2}g(x)\right)^2dx}\ .$$

5. From inequality (3.2.9), we have $\frac{1}{2\sqrt{3}}\left(\|x\|^2+\|y\|^2+\|x-y\|^2\right)\ge\|x\|\cdot\|y\|-\langle x,y\rangle$, and replacing y by $-y$ in this inequality implies

$$\frac{1}{2\sqrt{3}}\left(\|x\|^2+\|y\|^2+\|x+y\|^2\right)\ge\|x\|\cdot\|y\|-\langle x,y\rangle.$$

3.3 Considerations about the several inequalities in an inner product space

The objective of this section is to show new results concerning the Cauchy - Schwarz inequality in an inner product space. We find an improvement of Buzano's inequality and Richard's inequality, which are extensions of the Cauchy - Schwarz inequality [Minculete, 141].

From Lagrange's identity, given above, we found the following inequality which states: if **a** = (a_1, …, a_n) and **b** = (b_1, …, b_n) are two n-tuples of real numbers, then

(3.3.1) $$\sqrt{\left(a_1^2+...+a_n^2\right)\left(b_1^2+...+b_n^2\right)}\ge\left|a_1b_1+...+a_nb_n\right|,$$

with equality holding if and only if **a**= λ **b**. This result is called the *Cauchy-Schwarz-Buniakowski inequality* or simply the *Cauchy inequality*.

Many refinements for Cauchy-Schwarz-Buniakowski inequality can be found in literature (see [8], [12], [17] and [154]). In particular, we mention one of them: Ostrowski [171], in 1952, proved the following: if **x** = (x_1, …, x_n), **y** = (y_1, …, y_n) and **z** = (z_1, …, z_n) are n-tuples of real numbers such that **x** and **y** are not proportional and

$$\sum_{k=1}^n y_kz_k=0,\ \text{and}\ \sum_{k=1}^n x_kz_k=1,\ \text{then}$$

(3.3.2) $$\sum_{k=1}^n y_k^2\Big/\sum_{k=1}^n z_k^2\le\sum_{k=1}^n x_k^2\sum_{k=1}^n y_k^2-\left(\sum_{k=1}^n x_ky_k\right)^2.$$

For all $x, y \in X$ in an inner product space $X = (X, \langle \cdot, \cdot \rangle)$ over the field of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$, then we have the Cauchy-Schwarz inequality, given by the following:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

The Cauchy-Schwarz inequality can be written, as in Aldaz [8] and Niculescu [167], in terms of the angular distance between two vectors, thus

$$\langle x, y\rangle = \|x\|\|y\|\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right), \tag{3.3.3}$$

for all nonzero vectors $x, y \in X$.

Buzano [28] showed an extension of the Cauchy-Schwarz inequality, given by the following:

$$|\langle a, x\rangle\langle x, b\rangle| \leq \frac{1}{2}\|x\|^2 (|\langle a, b\rangle| + \|a\| \cdot \|b\|). \tag{3.3.4}$$

for any $x, a, b \in X$.

It is easy to see that for $a = b$, the inequality (3.3.4) becomes the Cauchy-Schwarz inequality.

Another inequality which is included the Buzano inequality is mentioned by Precupanu [185] and Dragomir [62]:

$$\frac{1}{2}\|x\|^2 (|\langle a, b\rangle| - \|a\| \cdot \|b\|) \leq |\langle a, x\rangle\langle x, b\rangle| \leq \frac{1}{2}\|x\|^2 (|\langle a, b\rangle| + \|a\| \cdot \|b\|), \tag{3.3.5}$$

for any $x, a, b \in X$. In [95], Gavrea showed an extention of Buzano's inequality in inner product space.

For real inner spaces, Richard [192] found the following stronger inequality

$$\left|\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\|x\|^2 \langle a, b\rangle\right| \leq \frac{1}{2}\|x\|^2 \|a\| \cdot \|b\|, \tag{3.3.6}$$

for any $x, a, b \in X$.

In [183], Popa and Raşa showed that, for any $x, a, b \in X$, the inequality

$$\left|Re\left(\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\|x\|^2 \langle a, b\rangle\right)\right| \leq \frac{1}{2}\|x\|^2 \sqrt{\|a\|^2 \cdot \|b\|^2 - (Im\langle a, b\rangle)^2}, \tag{3.3.7}$$

holds.

Dragomir [61] presented the following refinement of the Richard inequality:

$$\left|\langle a, b\rangle\|x\|^2 - \alpha\langle a, x\rangle\langle x, b\rangle\right| \leq max\{1, |1 - \alpha|\}\|a\| \cdot \|b\| \cdot \|x\|^2, \tag{3.3.8}$$

for all vectors x, a, b in an inner product space X and $\alpha \in \mathbb{C}$.
This inequality was found in another way by Khosravi *et al.* [114].

In [129], Lupu and Schwarz proved the following inequality:

$$\left|\|a\|^2\langle b, c\rangle\right| + \left|\|b\|^2\langle c, a\rangle\right| + \left|\|c\|^2\langle a, b\rangle\right| \leq \|a\|^2\|b\|^2\|c\|^2 + 2|\langle a, b\rangle\langle b, c\rangle\langle c, a\rangle|, \tag{3.3.9}$$

for any vectors $a, b, c \in X$.

These inequalities are applied to the theory of Hilbert $\mathbb{C}^*$- modules over non-commutative $\mathbb{C}^*$- algebras, see Aldaz [8], Pečarić and Rajić [178] and Dragomir [61], [62].
In the beginning, we prove two lemmas:

Lemma 3.3.1 ([Minculete, 141])**.** *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

(3.3.10) $$\|x+\alpha y\|^2=\left|\alpha\|y\|+\frac{\langle x,y\rangle}{\|y\|}\right|^2+\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2,$$

for all $x,y\in X$, $y\neq 0$, and for every $\alpha\in\mathbb{C}$.

Proof. By several calculations, we deduce the following:

$$\|x+\alpha y\|^2=\langle x+\alpha y,x+\alpha y\rangle=\|x\|^2+\overline{\alpha}\langle x,y\rangle+\alpha\overline{\langle x,y\rangle}+|\alpha|^2\|y\|^2=$$

$$\left(\alpha\|y\|+\frac{\langle x,y\rangle}{\|y\|}\right)\left(\overline{\alpha}\|y\|+\frac{\overline{\langle x,y\rangle}}{\|y\|}\right)+\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}=\left|\alpha\|y\|+\frac{\langle x,y\rangle}{\|y\|}\right|^2+\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2,$$

because we have $\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2=\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}$.

□

Remark 3.3.2. Let $x,e\in X$ with $\|e\|=1$. If we take $y=e$ and $\alpha=-\lambda$ in relation (3.3.10), then we obtain $\|x-\lambda e\|^2=|\lambda-\langle x,e\rangle|^2+\|x-\langle x,e\rangle e\|^2$. Consequently, we deduce $\|x-\langle x,e\rangle e\|^2=\inf_{\lambda\in C}\|x-\lambda e\|^2$ which is a result found in [129].

Lemma 3.3.3 ([Minculete, 141])**.** *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

(3.3.11) $$\left\|\langle a,x\rangle x-\frac{1}{2}\|x\|^2a\right\|=\frac{1}{2}\|x\|^2\|a\|,$$

for all $x,a\in X$.

Proof. For $x=0$ the equality is true. For $x\neq 0$ inequality (3.3.11) becomes $\left\|a-2\frac{\langle a,x\rangle x}{\|x\|^2}\right\|=\|a\|$. If we take in equality (3.3.10) $\alpha=-2, y=\frac{\langle a,x\rangle}{\|x\|^2}x$, then by simple calculations, we deduce the following:

$$\left\|a-2\frac{\langle a,x\rangle x}{\|x\|^2}\right\|^2=\left|-2\frac{|\langle a,x\rangle|}{\|x\|}+\frac{|\langle a,x\rangle|}{\|x\|}\right|^2+\left\|a-\frac{\langle a,x\rangle}{\|x\|^2}x\right\|^2=\|a\|^2.$$

Consequently, inequality (3.3.11) is true.

□

Remark 3.3.4. A simple proof of Richard's inequality can be given by combining the Cauchy-Schwarz inequality and relation (3.3.11), thus:

$$\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right|=\left|\left\langle\langle a,x\rangle x-\frac{1}{2}\|x\|^2a,b\right\rangle\right|\leq\left\|\langle a,x\rangle x-\frac{1}{2}\|x\|^2a\right\|\|b\|=\frac{1}{2}\|x\|^2\|a\|\|b\|.$$

Theorem 3.3.5 ([Minculete, 141])**.** *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

(3.3.12) $$|\alpha|^2\|y\|^2\|z\|^2+\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq 2\,Re\left(\overline{\alpha}\left(\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right)\right),$$

for all $x,y,z\in X$, and for every $\alpha\in\mathbb{C}$.

Corollary 3.3.6 ([Min_JMI]). *In an inner product space X over the field of real numbers* $\mathbb{R}$, *we have*

(3.3.13) $$\frac{\|y\|^2}{\|z\|^2}\left(\frac{\langle x,y\rangle\langle y,z\rangle}{\|y\|^2}-\langle x,z\rangle\right)^2 \le \|x\|^2\|y\|^2-\langle x,y\rangle^2 ,$$

for all $x,y,z\in X$, $y\neq 0$, $z\neq 0$.

Proof I. If $y\neq 0$, $z\neq 0$, then we apply Theorem 3.3.5 for $\alpha\in\mathbb{R}$, and we have

$$\|y\|^2\|z\|^2\alpha^2-2\alpha\left(\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right)+\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\ge 0 ,$$

for all $x,y,z\in X$, and for every $\alpha\in\mathbb{R}$. Since $\|y\|^2\|z\|^2>0$, then the discriminant is negative, i.e., $\Delta=\left(\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right)^2-\|y\|^2\|z\|^2\left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)\le 0$. Therefore, we prove the statement.

Proof II. For $\alpha=\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2$ in relation (3.3.12), we have

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\ge\left(2-\|y\|^2\|z\|^2\right)\left|\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right|^2.$$

For $x=0$, inequality (3.3.13) is true. In the situation $x\neq 0$, $y\neq 0$, if we replace in the above relation *x* and *y* by $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, then we deduce the statement.

□

Remark 3.3.7. If we take $\langle x,z\rangle=1$ and $\langle y,z\rangle=0$, in inequality (3.3.13), then we find the inequality of Ostrowski for inner product spaces over the field of real numbers,

(3.3.14) $$\|y\|^2/\|z\|^2\le\|x\|^2\|y\|^2-\langle x,y\rangle^2 ,$$

for all $x,y,z\in X$, $y\neq 0$, $z\neq 0$.

It is easy to see that for $x,y,z\in X=R^n$ we obtain inequality (3.3.2).

Theorem 3.3.8 ([Minculete, 141]). *In an inner product space X over the field real or complex numbers, for any nonzero vectors* $x,a,b\in X$, *we have*

(3.3.15) $$\frac{1}{2}\|x\|^2\|a\|\cdot\|b\|-\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right|\ge\frac{A}{\|x\|^2\|a\|\cdot\|b\|}\ge 0 ,$$

where

$$A=\left(|\langle a,x\rangle|\left(\|x\|^2\|b\|^2-|\langle x,b\rangle|^2\right)-\frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\right)\right)^2.$$

Remark 3.3.9. a) For real or complex inner spaces, inequality (3.3.15) represents an improvement of Richard's inequality, given thus:

$$\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right|\le\frac{1}{2}\|x\|^2\|a\|\cdot\|b\|-\frac{A}{\|x\|^2\|a\|\cdot\|b\|} ,$$

where

$$A=\left(|\langle a,x\rangle|\left(\|x\|^2\|b\|^2-|\langle x,b\rangle|^2\right)-\frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\right)\right)^2.$$

b) Also, using above inequality, and from the continuity property of the modulus, i.e., $|\alpha-\beta|\ge\left||\alpha|-|\beta|\right|$, $\alpha,\beta\in\mathbb{C}$, we deduce the inequality

(3.3.16)

$$\frac{1}{2}\|x\|^2\left(|\langle a,b\rangle|-\|a\|\cdot\|b\|\right)+\frac{A}{\|x\|^2\|a\|\cdot\|b\|}\le|\langle a,x\rangle\langle x,b\rangle|\le\frac{1}{2}\|x\|^2\left(|\langle a,b\rangle|+\|a\|\cdot\|b\|\right)-\frac{A}{\|x\|^2\|a\|\cdot\|b\|},$$

which is in fact a refinement of Buzano's inequality.

In [120], we found the following result of Kouba:

Lemma 3.3.10. Let X be a real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$ and its corresponding norm $\|\cdot\|$. For any $x,y,z,u,v\in X$, with $\|u\|=\|v\|=1$, we have

$$\langle u,z\rangle^2+\langle v,z\rangle^2\le\left(1+|\langle u,v\rangle|\right)\|z\|^2, \tag{3.3.17}$$

$$\|x\|^2\langle y,z\rangle^2+\|y\|^2\langle x,z\rangle^2\le\|x\|\|y\|\|z\|^2\left(\|x\|\|y\|+|\langle x,y\rangle|\right). \tag{3.3.18}$$

Using AG inequality, we deduce $\|x\|^2\langle y,z\rangle^2+\|y\|^2\langle x,z\rangle^2\ge 2\|x\|\|y\||\langle x,z\rangle\langle z,y\rangle|$ and from inequality (ii) we obtain

$$|\langle x,z\rangle\langle z,y\rangle|\le\frac{1}{2}\|z\|^2\left(\|x\|\|y\|+|\langle x,y\rangle|\right),$$

for any $x,y,z\in X$.

This inequality has been studied at this section as Buzano's inequality [28]. We remark an improvement of the Buzano inequality given by:

$$|\langle x,z\rangle\langle z,y\rangle|\le\frac{\|x\|^2\langle y,z\rangle^2+\|y\|^2\langle x,z\rangle^2}{2\|x\|\|y\|}\le\frac{1}{2}\|z\|^2\left(\|x\|\|y\|+|\langle x,y\rangle|\right), \tag{3.3.19}$$

for any $x,y,z\in X$.

3.4 Several inequalities and a characterization of an inner product space

The aim of this section is to present several inequalities concerning the Cauchy - Schwarz inequality and a characterization of an inner product space.

We start from a geometrical interpretation in a triangle. In what follows, we will use the notations: *a, b, c* – the lengths of the sides; h_a - the length of the altitude of A; w_a- the length of the bisector of the angle A; and *R* is the circumradius.

In [18], we found the result of Ballieu (1949) given thus: in a triangle *ABC*, for every $t\in(0,1]$, the following inequality:

$$2^{t-1}\sin^t\frac{A}{2}\le\frac{a^t}{b^t+c^t}, \tag{3.4.1}$$

is true.

For $t=1$, the inequality of Ballieu becomes

$$\sin\frac{A}{2}\le\frac{a}{b+c}. \tag{3.4.2}$$

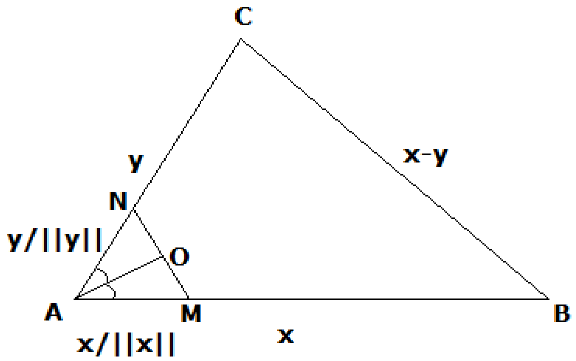
This is equivalent to the inequality

$$h_a\le w_a.$$

It is known that $h_a = \frac{2S}{a} = \frac{2abc}{4aR} = \frac{bc}{2R}$ and $w_a = \frac{2bc}{b+c} cos \frac{A}{2}$. By simple calculations, we have

$$h_a \le w_a \Leftrightarrow \frac{bc}{2R} \le \frac{2bc}{b+c} cos \frac{A}{2} \Leftrightarrow \frac{1}{2R} \le \frac{2}{b+c} cos \frac{A}{2} \Leftrightarrow \frac{a}{2R} \le \frac{2a}{b+c} cos \frac{A}{2} \Leftrightarrow$$

$$\frac{4R sin \frac{A}{2} cos \frac{A}{2}}{2R} \le \frac{2a}{b+c} cos \frac{A}{2} \Leftrightarrow sin \frac{A}{2} \le \frac{a}{b+c}.$$



In the above figure, we have $\overline{AB} = x$, $\overline{AC} = y$, $\overline{AM} = \frac{x}{\|x\|}$, $\overline{AN} = \frac{y}{\|y\|}$, $\overline{NM} = \frac{x}{\|x\|} - \frac{y}{\|y\|}$, and $\overline{CB} = x - y$. Therefore the inequality of Ballieu becomes:

(3.4.3) $$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{2\|x-y\|}{\|x\| + \|y\|}.$$

This inequality is in fact the inequality of Kirk and Smiley [115], for a real inner product space.
Using the inequality of Ballieu, for $t \in (0,1]$, we deduce

(3.4.4) $$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^t \le \frac{2\|x-y\|^t}{\|x\|^t + \|y\|^t}.$$

If we apply the cosine law for the angle A, then we have

(3.4.5) $$cos\, A = \frac{2AM^2 - MN^2}{2AM^2} = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC},$$

which is equivalent to the identity

$$2 - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{\|x\| \cdot \|y\|},$$

which implies the following relation:

(3.4.6) $$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \frac{\|x-y\|^2 - (\|x\| - \|y\|)^2}{\|x\| \cdot \|y\|}.$$

Next, we study the behavior of this equality in a real inner product space.

Theorem 3.4.1. *If* $X=(X,\langle\cdot,\cdot\rangle)$ *is an inner product space over the field of real numbers* $\mathbb{R}$ *and the norm* $\|\cdot\|$ *is generated by an inner product* $\langle\cdot,\cdot\rangle$, *then we have*

$$(b-a)\left(a\|x\|^2-b\|y\|^2\right)=ab\|x-y\|^2-\|ax-by\|^2 \tag{3.4.7}$$

for vectors x and y in X *and* $a,b\in\mathbb{R}$.

Proof. For all $x,y\in X$ and $a,b\in\mathbb{R}$, we have that

$$\|ax-by\|^2=\langle ax-by,ax-by\rangle=a^2\|x\|^2-2ab\langle x,y\rangle+b^2\|y\|^2.$$

It follows that

$$(b-a)\left(a\|x\|^2-b\|y\|^2\right)+\|ax-by\|^2=ab\left(\|x\|^2-2\langle x,y\rangle+\|y\|^2\right)=ab\|x-y\|^2.$$

Therefore, we obtain the statement.

□

Corollary 3.4.2. *If* $X=(X,\langle\cdot,\cdot\rangle)$ *is an inner product space over the field of real numbers* $\mathbb{R}$ *and the norm* $\|\cdot\|$ *is generated by an inner product* $\langle\cdot,\cdot\rangle$, *then we have*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2=\frac{\|x-y\|^2-(\|x\|-\|y\|)^2}{\|x\|\cdot\|y\|} \tag{3.4.8}$$

for nonzero vectors x and y in X.

Proof. For $a=\frac{1}{\|x\|}$ and $b=\frac{1}{\|y\|}$, in inequality (3.4.7), we deduce equality (3.4.8).

□

In 1964, Kirk and Smiley [115] showed that if the inequality

$$\alpha[x,y]\le\frac{2\|x-y\|}{\|x\|+\|y\|} \tag{3.4.9}$$

holds for all nonzero elements x and y of a normed linear space X, then X is an inner product space. In the same work, they also showed that the equality holds in (3.4.9) if and only if $\|x\|=\|y\|$ or $\|y\|x+\|x\|y=0$.

Theorem 3.4.3. *If X is a normed linear space over the field of real numbers* $\mathbb{R}$ *and we have the equality*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2=\frac{\|x-y\|^2-(\|x\|-\|y\|)^2}{\|x\|\cdot\|y\|} \tag{3.4.10}$$

for nonzero vectors x and y in X, *then X is an inner product space.*

Proof. If X is an inner product space, from Corollary 3.4.2, we deduce the equality. If X is a normed linear space and we have equality (3.4.10), then we show that

$$\alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{2\|x-y\|}{\|x\|+\|y\|}$$

for all nonzero elements x and y.

We have $\|x-y\|\le\|x\|+\|y\|$ so $\|x-y\|^2\le(\|x\|+\|y\|)^2$. Multiplying by $(\|x\|-\|y\|)^2$ we obtain $(\|x\|-\|y\|)^2\|x-y\|^2\le(\|x\|-\|y\|)^2(\|x\|+\|y\|)^2$.

It follows that

$$(\|x\|+\|y\|)^2\|x-y\|^2-(\|x\|-\|y\|)^2(\|x\|+\|y\|)^2\le 4\|x\|\|y\|\|x-y\|^2.$$

Therefore, dividing by $4\|x\|\|y\|((\|x\|+\|y\|))^2$, we deduce the inequality

$$\frac{\|x-y\|^2-(\|x\|-\|y\|)^2}{\|x\|\cdot\|y\|}\le\frac{4\|x-y\|^2}{(\|x\|+\|y\|)^2}\ ,$$

which is equivalent to $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\le\frac{4\|x-y\|^2}{(\|x\|+\|y\|)^2}$.

Consequently, we have $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{2\|x-y\|}{\|x\|+\|y\|}$. So, from Kirk and Smiley inequality, we deduce that X is an inner product space.

□

Maligranda's inequality, given above, can be written as the following:

(3.4.11) $$a=\frac{\|x-y\|-\big|\|x\|-\|y\|\big|}{min(\|x\|,\|y\|)}\le\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{\|x-y\|+\big|\|x\|-\|y\|\big|}{max(\|x\|,\|y\|)}=b.$$

If we replace y by $-y$ in Maligranda's inequality, we obtain the following

$$\|x-y\|\le\|x\|+\|y\|-\left(2-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\right)min(\|x\|,\|y\|)$$

which implies

$$\|x-y\|\le\|x\|+\|y\|-2min(\|x\|,\|y\|)+\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|min(\|x\|,\|y\|)$$

But $2min(\|x\|,\|y\|)=\|x\|+\|y\|-\big|\|x\|-\|y\|\big|$, so

$$\|x-y\|-\big|\|x\|-\|y\|\big|\le\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|min(\|x\|,\|y\|).$$

Similarly, since $2max(\|x\|,\|y\|)=\|x\|+\|y\|+\big|\|x\|-\|y\|\big|$, we deduce

$$\|x-y\|+\big|\|x\|-\|y\|\big|\ge\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|max(\|x\|,\|y\|).$$

Remark 3.4.4. It is easy to see that in an inner product space X, the inequality of Maligranda,

$$a=\frac{\|x-y\|-\big|\|x\|-\|y\|\big|}{min(\|x\|,\|y\|)}\le\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\le\frac{\|x-y\|+\big|\|x\|-\|y\|\big|}{max(\|x\|,\|y\|)}=b,$$

is very simple because

(3.4.12) $$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|=\sqrt{\frac{\|x-y\|-\big|\|x\|-\|y\|\big|}{min\{\|x\|,\|y\|\}}\cdot\frac{\|x-y\|+\big|\|x\|-\|y\|\big|}{max\{\|x\|,\|y\|\}}}=\sqrt{ab}$$

and $a\le\sqrt{ab}\le b$, so $\alpha[x,y]$ is the geometric mean of a and b.

Theorem 3.4.5. *If $X=(X,\langle\cdot,\cdot\rangle)$ is an inner product space over the field of real numbers $\mathbb{R}$ and the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then we have*

(3.4.13)
$$\frac{max(\|x\|,\|y\|)}{2min(\|x\|,\|y\|)}\cdot\left(\|x-y\|-\big|\|x\|-\|y\|\big|\right)^2\le\|x\|\cdot\|y\|-\langle x,y\rangle\le\frac{min(\|x\|,\|y\|)}{2max(\|x\|,\|y\|)}\cdot\left(\|x-y\|+\big|\|x\|-\|y\|\big|\right)^2.$$

Proof. Using relation (3.1.6) in the following form:

$$\frac{1}{2}\|x\|\cdot\|y\|\cdot\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2=\|x\|\cdot\|y\|-\langle x,y\rangle$$

and the inequality of Maligranda, we deduce the inequality

$$\frac{1}{2}\|x\|\cdot\|y\|\cdot\left(\frac{\|x-y\|-\big|\|x\|-\|y\|\big|}{min(\|x\|,\|y\|)}\right)^2\leq\|x\|\cdot\|y\|-\langle x,y\rangle\leq\frac{1}{2}\|x\|\cdot\|y\|\cdot\left(\frac{\|x-y\|+\big|\|x\|-\|y\|\big|}{max(\|x\|,\|y\|)}\right)^2,$$

which is equivalent to the inequality of the statement.

□

Remark 3.4.7. Inequality (3.4.13) shows a refinement of Cauchy-Schwarz's inequality and a reverse inequality for Cauchy-Schwarz's inequality.

(B-ii) The evolution and development plans for career development

4 Future directions for research

The purpose of this chapter is to present some of the lines that describe the present and future projects in scientific research and the teaching career.

I shall continue my research in the field of theory of inequalities related to inequalities for functionals, inequalities for invertible positive operators and inequalities in an inner product space. At the same time, I shall focus on certain types of inequalities and their applications in generalized entropies.

I shall continue to elaborate new scientific papers in all fields quated above, or other areas of mathematics, especially related to real and complex analysis.

I intend to write a scientific monograph related to my contributions in the theory of inequalities related to inequalities for functionals, inequalities for invertible positive operators and inequalities in an inner product space.

I would like to publish a book for students in computer science, mathematics, economics and finance. Several of my future research projects are described in the following.

4.1 Future directions for research related to Hermite-Hadamard's inequality and Hammer-Bullen's inequality

In this section, we intend to give two reverse inequalities of Bullen's inequality which represent the generalizations of results from [Minculete-Rațiu-Pečarić, 143]. We also present several applications about Stolarsky's mean, the logarithmic mean and the identric mean. The results obtained below are part of recent research.

In the monographs [166, 176] we find, for a convex function $f:[a,b]\to\mathbb{R}$, Bullen's inequality, namely:

(4.1.1) $$\frac{2}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right).$$

For a particularization of function *f*, Dragomir and Pearce in [51] obtained a refinement of Hammer-Bullen's inequality, given by the following:

Theorem B. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function for which there exists real constant m and M such that:* $m\le f''(x)\le M$, *for all* $x\in[a,b]$. *Then*

(4.1.2) $$m\frac{(b-a)^2}{24}\le\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)dx\le M\frac{(b-a)^2}{24}.$$

In [Minculete-Rațiu-Pečarić, 143] there were obtained two reverse inequalities of Bullen's inequality

(4.1.3) $$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)dx \le \frac{(b-a)(f'(b)-f'(a))}{16},$$

(4.1.4) $$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)dx-\frac{(b-a)(f'(b)-f'(a))}{24}\right| \le \frac{(M-m)(b-a)^2}{64}$$

and Acu and Gonska, in [5], extendend Bullen's inequality for continuous functions using the second order modulus of smoothness.

Inspired by the above work, I started a new joint project with F. C. Mitroi-Symeonidis and M. Niezgoda related to Hermite-Hadamard inequality. We would like to propose a new inequalities for Stolarsky's mean, logarithmic mean, and identric mean.

Lemma 4.1.1 ([Minculete-Niezgoda-Mitroi, 142]**).** *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function. Then we have the following:*

(4.1.5) $$\int_a^b (x-c)q_c(x)f''(x)dx=(b-a)f(c)+(c-a)f(a)+(b-c)f(b)-2\int_a^b f(x)dx,$$

where $a<c<b$ and

$$q_c(x):=\begin{cases} a-x, & x\in[a,c) \\ b-x & x\in[c,b] \end{cases}.$$

Proof. We make the calculations:

$$\int_a^b (x-c)q_c(x)f''(x)dx=\int_a^c (x-c)(a-x)f''(x)dx+\int_c^b (x-c)(b-x)f''(x)dx=$$

$$=\int_a^c [2x-(a+c)]f'(x)dx+\int_c^b [2x-(b+c)]f'(x)dx=$$

$$(b-a)f(c)+(c-a)f(a)+(b-c)f(b)-2\int_a^b f(x)dx.$$

Remark 4.1.2. a) It is easy to see that for $x\in[a,b]$, we have $(x-c)q_c(x)\ge 0$ and, by some elementary computations, we obtain:

(4.1.6) $$\int_a^b (x-c)q_c(x)dx=\frac{b-a}{6}\left(a^2+b^2+3c^2+ab-3bc-3ac\right).$$

Therefore, for $x\in[a,b]$ we can write:

(4.1.7) $$m(x-c)q_c(x)\le(x-c)q_c(x)f''(x)\le M(x-c)q_c(x).$$

Integrating from a to b and using Lemma 4.1.1, we find the relation:

$$\frac{m(b-a)}{6}\left(a^2+b^2+3c^2+ab-3bc-3ac\right)\le$$

(4.1.8) $$(b-a)f(c)+(c-a)f(a)+(b-c)f(b)-2\int_a^b f(x)dx\le$$

$$\frac{M(b-a)}{6}\left(a^2+b^2+3c^2+ab-3bc-3ac\right).$$

Theorem 4.1.3 ([Minculete-Niezgoda-Mitroi, 142]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable and convex function. Then we have the following inequality that holds:*
(4.1.9)

$$0\le (b-a)f(c)+(c-a)f(a)+(b-c)f(b)-2\int_a^b f(x)dx\le \frac{1}{4}max\{(a-c)^2,(b-c)^2\}[f'(b)-f'(a)],$$

where $a<c<b$.

Proof. Since f is a convex function, it follows that $f''(x)\ge 0$, for every $x\in[a,b]$. Because we have $0\le (x-c)q_c(x)\le \frac{1}{4}max\{(a-c)^2,(b-c)^2\}$, then we deduce the following inequality: $0\le (x-c)q_c(x)f''(x)\le \frac{1}{4}max\{(a-c)^2,(b-c)^2\}f''(x)$, for every $x\in[a,b]$. Therefore, by integrating, the last inequality from a to b, we obtain:

$$0\le \int_a^b (x-c)q_c(x)f''(x)dx\le \frac{1}{4}max\{(a-c)^2,(b-c)^2\}[f'(b)-f'(a)].$$

Using equality (4.1.5) in the previous inequality, we find the inequality from the statement.

Theorem 4.1.4 ([Minculete-Niezgoda-Mitroi, 142]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function for which there exists real constant m and M such that $m\le f''(x)\le M$, for all $x\in[a,b]$. Then*

(4.1.10)

$$\left|(b-a)f(c)+(c-a)f(a)+(b-c)f(b)-2\int_a^b f(x)dx-\frac{1}{6}\left(a^2+b^2+3c^2+ab-3bc-3ac\right)\frac{f'(b)-f'(a)}{b-a}\right|$$
$$\le \frac{M-m}{16}max\{(a-c)^2,(b-c)^2\}$$

where $a<c<b$.

Proof. Taking into account that $0\le (x-c)q_c(x)\le \frac{1}{4}max\{(a-c)^2,(b-c)^2\}$ and $m\le f''(x)\le M$, for all $x\in[a,b]$, and applying the inequality of Grüss (see e. g. [51, 166]), then we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b (x-c)q_c(x)f''(x)dx-\frac{1}{b-a}\int_a^b (x-c)q_c(x)dx\,\frac{1}{b-a}\int_a^b f''(x)dx\right|\le$$
$$\le \frac{M-m}{16}max\{(a-c)^2,(b-c)^2\}.$$

By simple calculations, we deduce the inequality of the statement.

Remark 4.1.5. a) If we choose $c=\lambda a+(1-\lambda)b$, with $\lambda\in(0,1)$ then inequalities (4.1.9) and (4.1.10) become:

(4.1.11) $$0\le (b-a)f(\lambda a+(1-\lambda)b)+(b-a)[(1-\lambda)f(a)+\lambda f(b)]-2\int_a^b f(x)dx$$

$$\leq \frac{(b-a)^2}{4} max\{(1-\lambda)^2, \lambda^2\}[f'(b)-f'(a)],$$

(4.1.12)

$$|(b-a)f(\lambda a+(1-\lambda)b)+(b-a)[(1-\lambda)f(a)+\lambda f(b)]-$$
$$2\int_a^b f(x)dx - \frac{(1-3\lambda(1-\lambda))(b-a)(f'(b)-f'(a))}{6}|$$
$$\leq \frac{(b-a)^2(M-m)}{16} max\{(1-\lambda)^2, \lambda^2\};$$

b) For $\lambda = \frac{1}{2}$ in inequalities (4.1.11) and (4.1.12), we deduce the inequalities (4.1.3) and (4.1.4).

Some applications can be identified, thus:

a) If we consider $f(x) = x^p$, with $p > 1$, then inequality (4.1.11) becomes:

(4.1.13)

$$\frac{(\lambda a+(1-\lambda)b)^p + (1-\lambda)a^p + \lambda b^p}{2} - \frac{p(b-a)(b^{p-1}-a^{p-1})}{8} max\{(1-\lambda)^2, \lambda^2\} \leq$$
$$[L_{p+1}(a,b)]^p \leq \frac{(\lambda a+(1-\lambda)b)^p + (1-\lambda)a^p + \lambda b^p}{2},$$

where $L_p(a,b) = \left[\frac{a^p - b^p}{p(a-b)}\right]^{\frac{1}{p-1}}$ is Stolarsky's mean.

b) We consider $f(x) = \frac{1}{x}$, with $x > 0$, in inequality (4.1.11), then we obtain:

(4.1.14)

$$\frac{2}{\frac{1}{\lambda a+(1-\lambda)b} + \frac{1-\lambda}{a} + \frac{\lambda}{b}} \leq L(a,b) \leq \frac{2}{\frac{1}{\lambda a+(1-\lambda)b} + \frac{1-\lambda}{a} + \frac{\lambda}{b} - \frac{(b-a)^2(a+b)}{4a^2b^2} max\{(1-\lambda)^2, \lambda^2\}},$$

where $L(a,b) = \frac{b-a}{ln b - ln a}$ is the logarithmic mean.

c) If we consider $f(x) = -ln x$, with $x > 0$, then inequality (4.1.11) becomes:

(4.1.15) $$(\lambda a+(1-\lambda)b)a^{1-\lambda}b^{\lambda} \leq I^2(a,b) \leq (\lambda a+(1-\lambda)b)a^{1-\lambda}b^{\lambda} e^{\frac{(b-a)^2}{4ab} max\{(1-\lambda)^2, \lambda^2\}},$$

where $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

I would like to start a new joint project with Shigeru Furuichi related to Hermite-Hadamard's inequality. We would like to propose a new improvements for Young's inequality.

We establish several inequalities using Hermite-Hadamard's inequality (if $f:[a,b]\to\mathbb{R}$ is convex function, then $\frac{f(a)+f(b)}{2}\geq\frac{1}{b-a}\int_a^b f(x)dx\geq f\left(\frac{a+b}{2}\right)$) for the function f_μ.

Since $f_\mu:[1,\infty)\to\mathbb{R}$ with $\mu\in\left[0,\frac{1}{2}\right]$ defined by

$$f_\mu(x)=(1-\mu)+\mu x+\frac{x}{(1-\mu)x+\mu}-2x^\mu$$

is convex function, for all $x\geq 1$, $\mu\in\left[0,\frac{1}{2}\right]$, we use Hermite-Hadamard's inequality on the interval $[1,a]$, with $a>1$. Thus, we obtain the inequalities:

$$\frac{f_\mu(1)+f_\mu(a)}{2}\geq\frac{1}{a-1}\int_1^a f_\mu(x)dx\geq f_\mu\left(\frac{a+1}{2}\right).$$

But $f_\mu(1)=0$, $f_\mu(a)=1-\mu+\mu a+\frac{a}{(1-\mu)a+\mu}-2a^\mu$,

$$f_\mu\left(\frac{a+1}{2}\right)=1-\mu+\mu\frac{a+1}{2}+\frac{a+1}{(1-\mu)(a+1)+2\mu}-\frac{(a+1)^\mu}{2^{\mu-1}}$$

and

$$\frac{1}{a-1}\int_1^a f_\mu(x)dx=1-\mu+\mu\frac{a+1}{2}+\frac{1}{1-\mu}-\frac{\mu}{(1-\mu)^2(a-1)}ln(a(1-\mu)+\mu)-\frac{2(a^{\mu+1}-1)}{(\mu+1)(a-1)}.$$

From Hermite-Hadamard's inequality, we deduce

$$\frac{1}{2}\left(1-\mu+\mu a+\frac{a}{(1-\mu)a+\mu}-2a^\mu\right)\geq$$

$$1-\mu+\mu\frac{a+1}{2}+\frac{1}{1-\mu}-\frac{\mu}{(1-\mu)^2(a-1)}ln(a(1-\mu)+\mu)-\frac{2(a^{\mu+1}-1)}{(\mu+1)(a-1)}\geq$$

$$1-\mu+\mu\frac{a+1}{2}+\frac{a+1}{(1-\mu)(a+1)+2\mu}-\frac{(a+1)^\mu}{2^{\mu-1}},$$

where $\mu\in\left[0,\frac{1}{2}\right]$ and $a>1$.

In this inequality, if we take $a=\frac{x}{y}>1$, so $x>y$, we will find an inequality of type Young. In the same way, we use Bullen inequality or Hammer-Bullen inequality, which states that:

$$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\geq\frac{2}{b-a}\int_a^b f(t)dt\ .$$

So, $\frac{f_\mu(1)+f_\mu(a)}{2}+f_\mu\left(\frac{a+1}{2}\right)\geq\frac{2}{a-1}\int_1^a f_\mu(x)dx$, which implies another inequality of type Young.

Now, we take the function $f_\mu : [0,1] \to R$ with $\mu \in \left[\frac{1}{2},1\right]$ defined by

$$f_\mu(x) = (1-\mu) + \mu x + \frac{x}{(1-\mu)x+\mu} - 2x^\mu$$

is convex function, for all $0 \le x \le 1$, $\mu \in \left[\frac{1}{2},1\right]$.

We use Hermite-Hadamard's inequality on the interval $[0,1]$. Thus, we obtain the inequalities:

$$\frac{f_\mu(0)+f_\mu(1)}{2} \ge \int_0^1 f_\mu(x)dx \ge f_\mu\left(\frac{1}{2}\right).$$

But $f_\mu(0) = 1-\mu$, $f_\mu(1) = 0$, $f_\mu(a) = 1-\mu+\mu a + \frac{a}{(1-\mu)a+\mu} - 2a^\mu$,

$$f_\mu\left(\frac{1}{2}\right) = 1 - \frac{\mu}{2} + \frac{1}{\mu+1} - 2^{1-\mu}$$

and

$$\int_0^1 f_\mu(x)dx = 1 - \frac{\mu}{2} + \frac{1}{1-\mu} + \frac{\mu}{(1-\mu)^2} ln(\mu) - \frac{2}{(\mu+1)}.$$

From Hermite-Hadamard's inequality, we deduce

$$\frac{1}{2}(1-\mu) \ge 1 - \frac{\mu}{2} + \frac{1}{1-\mu} + \frac{\mu}{(1-\mu)^2} ln(\mu) - \frac{2}{(\mu+1)} \ge 1 - \frac{\mu}{2} + \frac{1}{\mu+1} - 2^{1-\mu},$$

where $\mu \in \left[\frac{1}{2},1\right]$.

In the same way, we use Hammer-Bullen inequality and we will obtain another inequality of type Young.

Below, we propose another research idea related to the function gamma of Euler.

The function gamma is defined via a convergent improper integral as

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \text{ for all } t \ge 0,$$

it is known as Euler integral of the second kind. The following infinite product definition for the gamma function is due to Weierstrass,

$$\Gamma(t) = \frac{e^{-\gamma t}}{t}\prod_{n=1}^{\infty}\left(1+\frac{t}{n}\right)^{-1} e^{\frac{t}{n}},$$

where $\gamma = 0.577216..$ is the Euler-Mascheroni constant. This relation can be written as

(4.1.24) $$log\,\Gamma(t) = -\gamma t - log\,t - \sum_{n=1}^{\infty}\left(\frac{t}{n} - log\left(1+\frac{t}{n}\right)\right),$$

where the base of the logarithm is *e*, thus we obtain

$$log\,\Gamma(t+1) + \gamma t = -\gamma - log(t+1) - \sum_{n=1}^{\infty}\left(\frac{t+1}{n} - log\left(1+\frac{t+1}{n}\right)\right).$$

We consider the function $f : [0,\infty) \to \mathbb{R}$ defined by $f(t) = log\,\Gamma(t+1) + \gamma t$.

It easy to see that

$$f'(t) = -\frac{1}{t+1} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{t+n+1}\right)$$

and

$$f''(t) = \frac{1}{(t+1)^2} + \sum_{n=1}^{\infty}\frac{1}{(t+n+1)^2} = \sum_{n=0}^{\infty}\frac{1}{(t+n)^2}.$$

We remark that $f''(t) \geq 0$, which implies that f is convex and f' is increasing, so $f'(t) \geq f'(0) = 0$. Therefore, f is increasing, so we have $f(t) \geq f(0) = 0$.

We intend to study the properties of the functional Jensen and the functional Chebyshev for the function $f(t) = log\,\Gamma(t+1) + \gamma t$. For this function, we will apply Hermite-Hadamard's inequality or Hammer-Bullen's inequality. We will also study the functional Jensen and the functional Chebyshev for the functions log-convex ((A,G)- convex) or, more generally, for (M,N)- convex, where M and N are means.

Connected with the functional Jensen, in the future, we would like to study other properties of generalized entropies as the following:

a) the Tsallis entropy [201] defined by:

$$H_q(p_1, p_2, \ldots, p_n) = \sum_{j=1}^{n} p_j^q\, ln_q \frac{1}{p_j},\ (q \geq 0, q \neq 1),$$

where $\{p_1, p_2, \ldots, p_n\}$is a probability distribution with $p_j > 0$ for all $j = \overline{1,n}$ and the q-logarithmic function for $x > 0$;

b) the Rényi entropy [191] defined by

$$R_q(p_1, p_2, \ldots, p_n) \equiv \frac{1}{1-q} log\left(\sum_{j=1}^{n} p_j^q\right);$$

c) the quasilinear relative entropy defined by

$$D_1^{\psi}(p_1, p_2, \ldots, p_n \| r_1, r_2, \ldots, r_n) \equiv -log\,\psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right);$$

d) the Rényi relative entropy [3] defined by

$$R_q(p_1, p_2, \ldots, p_n \| r_1, r_2, \ldots, r_n) \equiv \frac{1}{q-1} log\left(\sum_{j=1}^{n} p_j^q r_j^{1-q}\right);$$

e) the Tsallis relative entropy defined by

$$D_q(p_1, p_2, \ldots, p_n \| r_1, r_2, \ldots, r_n) \equiv \sum_{j=1}^{n} p_j^q (ln_q\, p_j - ln_q\, r_j) = -\sum_{j=1}^{n} p_j\, ln_q \frac{r_j}{p_j};$$

f) the Tsallis quasilinear entropy (q-quasilinear entropy) defined by

$$I_q^{\psi}(p_1, p_2, \ldots, p_n) \equiv ln_q\, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right),$$

where $\{p_1, p_2, \ldots, p_n\}$ is a probability distribution with $p_j > 0$ for all $j = \overline{1,n}$,

g) the Tsallis quasilinear relative entropy defined by

$$D_q^{\psi}(p_1, p_2, \ldots, p_n \| r_1, r_2, \ldots, r_n) \equiv -ln_q\, \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right)\right).$$

4.2 Future directions for research related to Young's inequality and Hardy's inequality

Inspired by the method used like Elliott in proving Hardy's inequality, I started new work related to Young's inequality and its applications.

We consider the function $f(t)=t^{p-1}$, $p\in(0,1]$. It is easy to check that $f(t)$ is convex on $[1,\infty)$. Taking into account that

$$\int_1^x t^{p-1}dt=\frac{x^p-1}{p},$$

and using the right side of Hermite-Hadamard inequality, we deduce

$$\frac{x^p-1}{p}\le(x-1)\frac{x^{p-1}+1}{2}\le x-1.$$

In the above inequality, we replace x by $\frac{a}{b}$, with $a\ge b$, we obtain *Young's inequality*, which, in general form, says that, if $a,b>0$ and $p\in[0,1]$, then

$$a^p b^{1-p}\le pa+(1-p)b.$$

This inequality is equivalent to the following inequality, for $p=\frac{1}{u}, a=x^u, b=x^{\frac{u}{u-1}}$:

$$\frac{x^u}{u}+\frac{y^v}{v}\ge xy,$$

for all $x,y\ge 0$ and $u,v>1$ with $\frac{1}{u}+\frac{1}{v}=1$.

For $u=\alpha+1$, $v=\frac{\alpha+1}{\alpha}$, with $\frac{1}{u}+\frac{1}{v}=1$, $x\to x^{\alpha+1}$, $y\to y^{\alpha}$, and using Young's inequality we deduce the following relation:

(4.2.1) $$x^{\alpha+1}+\alpha y^{\alpha+1}\ge(\alpha+1)xy^{\alpha},$$

which is used by Elliott [67] in proving Hardy's inequality [173]: If $q>1$ and $a_n\ge 0$, then

(4.2.2) $$\sum_{n=1}^{\infty}\left(\frac{a_1+a_2+...+a_n}{n}\right)^q\le\left(\frac{q}{q-1}\right)^q\sum_{n=1}^{\infty}a_n^q$$

unless all the a_i are zero. The constant is the best possible.

In 1926, Copson [41] generalized Hardy inequality by replacing the arithmetic mean of a sequence by a weighted arithmetic mean, thus:

If $q>1,\lambda_n>0,a_n>0,n=1,2,...$, $\sum_{n=1}^{\infty}\lambda_n a_n^q$ converge, then

(4.2.3) $$\sum_{n=1}^{\infty}\lambda_n\left(\frac{\lambda_1 a_1+\lambda_2 a_2+...+\lambda_n a_n}{\lambda_1+\lambda_2+...+\lambda_n}\right)^q\le\left(\frac{q}{q-1}\right)^q\sum_{n=1}^{\infty}\lambda_n a_n^q.$$

unless all the a_i are zero. The constant is the best possible.

Inspired by the above work, I would like to start a new project related to Hardy inequality. We would like to propose a new refinement of Young's inequality which can be use in the proof of Hardy's inequality and Carleman's inequality.

Young's inequality was refined by Kittaneh and Manasrah in [116] or given as a particular case of Kober's inequality [119], thus:

(4.2.4) $\quad min\{p,1-p\}(\sqrt{a}-\sqrt{b})^2 \le pa+(1-p)b-a^p b^{1-p} \le max\{p,1-p\}(\sqrt{a}-\sqrt{b})^2$,

where a,b are nonnegative real numbers and $p\in[0,1]$.

For $p=\dfrac{1}{\alpha+1}$, we have $1-p=\dfrac{\alpha}{\alpha+1}$, and for, $a=x^{\alpha+1}$, $b=y^{\alpha}$, we use inequality (4.2.4), we deduce the following relation:

$$\left(x^{\frac{\alpha+1}{2}}-y^{\frac{\alpha}{2}}\right)^2 \le x^{\alpha+1}+\alpha y^{\alpha+1}-(\alpha+1)xy^{\alpha} \le \alpha\left(x^{\frac{\alpha+1}{2}}-y^{\frac{\alpha}{2}}\right)^2, \tag{4.2.5}$$

for all $\alpha\ge 1$, and $x,y\ge 0$.

Lemma 4.2.1. If $q>1$, $a_n\ge 0, n=1,2,...N$ and $M_n=\dfrac{a_1+a_2+...+a_n}{n}$, then

$$\sum_{n=1}^{N} M_n^q \le \left(\frac{q}{q-1}\right)^q \sum_{n=1}^{N} a_n^q - \frac{1}{q-1}\left(\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2\right)^q \left(\sum_{n=1}^{N} M_n^q\right)^{(1-q)}. \tag{4.2.6}$$

Proof. Using the same method as Elliott [67] in proving Hardy's inequality, we note $M_n=\dfrac{a_1+a_2+...+a_n}{n}$ and we make the following calculations:

$$M_n^q-\frac{q}{q-1}M_n^{q-1}a_n = M_n^q-\frac{q}{q-1}M_n^{q-1}\left(nM_n-(n-1)M_{n-1}\right)$$

$$=M_n^q\left(1-\frac{qn}{q-1}\right)+\frac{q(n-1)}{q-1}M_n^{q-1}M_{n-1}.$$

By convention, we take $M_0=1$.

If we apply inequality (4.2.5) for $x=M_{n-1}$, $y=M_n$ and $\alpha=q-1$, we deduce

$$\begin{aligned}\left(M_{n-1}^q+(q-1)M_n^q\right)-\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2 &\ge qM_n^{q-1}M_{n-1} \\ &\ge \left(M_{n-1}^q+(q-1)M_n^q\right)-n\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2.\end{aligned} \tag{4.2.7}$$

Therefore, we obtain

$$\begin{aligned}M_n^q\left(1-\frac{qn}{q-1}\right)+\frac{n-1}{q-1}\left(M_{n-1}^q+(q-1)M_n^q\right)-\frac{n-1}{q-1}\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2 &\ge M_n^q-\frac{q}{q-1}M_n^{q-1}a_n \\ \ge M_n^q\left(1-\frac{qn}{q-1}\right)+\frac{n-1}{q-1}\left(M_{n-1}^q+(q-1)M_n^q\right)-\frac{n-1}{q-1}n\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2&,\end{aligned}$$

which is equivalent to

$$\frac{1}{q-1}\left((n-1)M_{n-1}^q-nM_n^q\right)-\frac{n-1}{q-1}\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2 \ge M_n^q-\frac{q}{q-1}M_n^{q-1}a_n$$

$$\ge \frac{1}{q-1}\left((n-1)M_{n-1}^q-nM_n^q\right)-\frac{(n-1)n}{q-1}\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2,$$

Next, we pass the sum from *1* to *N*, thus:

(4.2.8) $$-\frac{NM_N^q}{q-1}-\frac{1}{q-1}\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2 \geq \sum_{n=1}^{N}M_n^q-\frac{q}{q-1}\sum_{n=1}^{N}M_n^{q-1}a_n$$
$$\geq -\frac{NM_N^q}{q-1}-\frac{1}{q-1}\sum_{n=1}^{N}n(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2,$$

which implies,

$$\sum_{n=1}^{N}M_n^q \leq \frac{q}{q-1}\sum_{n=1}^{N}M_n^{q-1}a_n-\frac{1}{q-1}\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2.$$

But, using Hölder's inequality with indices $q>1$ and $\frac{q}{q-1}$, we have

$$\sum_{n=1}^{N}M_n^{q-1}a_n \leq \left(\sum_{n=1}^{N}a_n^q\right)^{1/q}\left(\sum_{n=1}^{N}M_n^q\right)^{(q-1)/q},$$

which implies

$$\sum_{n=1}^{N}M_n^q \leq \frac{q}{q-1}\left(\sum_{n=1}^{N}a_n^q\right)^{1/q}\left(\sum_{n=1}^{N}M_n^q\right)^{(q-1)/q}-\frac{1}{q-1}\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2,$$

so, multiplying by $\left(\sum_{n=1}^{N}M_n^q\right)^{(1-q)/q}$, we deduce the inequality

$$\left(\sum_{n=1}^{N}M_n^q\right)^{1/q} \leq \frac{q}{q-1}\left(\sum_{n=1}^{N}a_n^q\right)^{1/q}-\frac{1}{q-1}\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2\left(\sum_{n=1}^{N}M_n^q\right)^{(1-q)/q}.$$

It follows that, by raising to the power *q*, the following inequality

$$\sum_{n=1}^{N}M_n^q \leq \left(\frac{q}{q-1}\right)^q\sum_{n=1}^{N}a_n^q-\frac{1}{q-1}\left(\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2\right)^q\left(\sum_{n=1}^{N}M_n^q\right)^{(1-q)}.$$

□

Theorem 4.2.2. If $q>1$, $a_n \geq 0, n=1,2,...$ and $M_n=\frac{a_1+a_2+...+a_n}{n}$, then

(4.2.9) $$\sum_{n=1}^{\infty}M_n^q \leq \left(\frac{q}{q-1}\right)^q\sum_{n=1}^{\infty}a_n^q-\frac{1}{q-1}\left(\sum_{n=1}^{\infty}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2\right)^q\left(\sum_{n=1}^{\infty}M_n^q\right)^{(1-q)}.$$

Proof. Using Lemma 4.2.1 and passing to limit for $N\to\infty$, we obtain the statement.

□

Remark 3.2.3. a) Inequality (4.2.9) represents an improvement of Hardy's inequality.

b) If the numerical series $\sum_{n=1}^{\infty}a_n$ converges, then $\lim_{N\to\infty}NM_N^q=0$.

Therefore, inequality (4.2.8) becomes

(4.2.10)

$$\sum_{n=1}^{\infty}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2 \le q\sum_{n=1}^{\infty}M_n^{q-1}a_n-(q-1)\sum_{n=1}^{\infty}M_n^q \le \sum_{n=1}^{\infty}n(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2.$$

c) Copson's inequality (see inequality (4.2.3)), which is a generalization of Hardy's inequality, can be refined by the same method. Thus, we identify a new research direction.

Another important result is Carleman's inequality [29], given by the following:

Let $a_1, a_2, a_3, ...$ be a sequence of non-negative real numbers, then

(4.2.11) $$\sum_{n=1}^{\infty}(a_1a_2...a_n)^{1/n} < e\sum_{n=1}^{\infty}a_n,$$

where the series $\sum_{n=1}^{\infty}a_n$ is convergent. The constant e in the inequality is optimal, that is, the inequality does not always hold if e is replaced by a smaller number. The inequality is strict if some elements in the sequence is non-zero.

Carleman discovered this inequality during his important work on quasi-analytical functions. This problem can be solved by using the Lagrange multiplier method.

But, below, we present a solution using the inequality from Lemma 4.2.1, thus: if $q>1$, $a_n \ge 0, n=1,2,...N$ and $M_n=\frac{a_1+a_2+...+a_n}{n}$, then

$$\sum_{n=1}^{N}M_n^q \le \left(\frac{q}{q-1}\right)^q\sum_{n=1}^{N}a_n^q-\frac{1}{q-1}\left(\sum_{n=1}^{N}(n-1)\left(M_{n-1}^{p/2}-M_n^{(p-1)/2}\right)^2\right)^q\left(\sum_{n=1}^{N}M_n^q\right)^{(1-q)}.$$

If we replace a_k, by $a_k^{1/q}$ in the above inequality, then we find the following inequality

(4.2.12) $$\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{n}a_k^{1/q}}{n}\right)^q \le \left(\frac{q}{q-1}\right)^q\sum_{n=1}^{N}a_n-A,$$

where

$$A=\frac{1}{q-1}\left(\sum_{n=1}^{N}(n-1)\left(\left(\frac{\sum_{k=1}^{n-1}a_k^{1/q}}{n-1}\right)^{q/2}-\left(\frac{\sum_{k=1}^{n}a_k^{1/q}}{n}\right)^{q-1/2}\right)^2\right)^q\left(\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{n}a_k^{1/q}}{n}\right)^q\right)^{(1-q)}.$$

In inequality (4.2.12) passing to limit for $q\to\infty$ and using the fundamental limit

$\lim_{q\to\infty}\left(\frac{\sum_{k=1}^{n}a_k^{1/q}}{n}\right)^q=(a_1a_2...a_n)^{1/n}$, we obtain the inequality

(4.2.13) $$\sum_{n=1}^{N}(a_1a_2...a_n)^{1/n} \le e\sum_{n=1}^{N}a_n-A(n),$$

where

$$A(n)=\lim_{q\to\infty}\frac{1}{q-1}\left(\sum_{n=1}^{N}(n-1)\left((a_1a_2...a_{n-1})^{1/2(n-1)}-(a_1a_2...a_n)^{1/2n}\right)^2\right)^q\left(\sum_{n=1}^{N}(a_1a_2...a_n)^{1/n}\right)^{(1-q)}.$$

Theorem 4.2.4. If $q>1$, $a_n\geq 0, n=1,2,...$ and $M_n=\dfrac{a_1+a_2+...+a_n}{n}$, then

$$\sum_{n=1}^{\infty}M_n^q\leq\left(\frac{q}{q-1}\right)^q\sum_{n=1}^{\infty}a_n^q-\frac{1}{q-1}\left(\sum_{n=1}^{\infty}(n-1)\left(M_{n-1}^{q/2}-M_n^{(q-1)/2}\right)^2\right)^q\left(\sum_{n=1}^{\infty}M_n^q\right)^{(1-q)} \tag{4.2.14}$$

Another proof is given by Redheffer [190] using the inequality:

$$NG_N+\sum_{n=1}^{N}n(b_n-1)G_n\leq\sum_{n=1}^{N}a_nb_n^n, \tag{4.2.15}$$

which holds for all $n=1,2,...$ and all positive sequences $\{b_n\}$ and where $G_n=(a_1a_2...a_n)^{1/n}$ is the geometric mean.

In particular case, for: a) $b_n=1$, for all $n=1,2,...$, we have

$$G_N=(a_1a_2...a_N)^{1/N}\leq\frac{1}{N}\sum_{n=1}^{N}a_n=A_N\ ,$$

i.e., we obtain the AG-inequality;

b) $b_n=1+\dfrac{1}{n}$, for all $n=1,2,...$, we have

$$N(a_1a_2...a_N)^{1/N}+\sum_{n=1}^{N}(a_1a_2...a_n)^{1/n}\leq\sum_{n=1}^{N}\left(1+\frac{1}{n}\right)^n a_n\ ,$$

which implies when $n\to\infty$, the Carleman inequality.

Next, we give another improvement of Young's integral inequality:

Theorem 4.2.5. *Suppose the conditions of Theorem 1.4.1 hold and more* $a<f^{-1}(b)$ and *f* is convex or $a>f^{-1}(b)$ and *f* is concave. *Then*

$$ab\leq ab+\frac{(b-f(a))(f^{-1}(b)-a)}{2}\leq\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx=Y(f;a,b). \tag{4.2.16}$$

Proof. The inequality (3.2.16) has a geometric interpretation involving the areas of the two functions, the rectangular area and the area of a triangle.

Minguzzi, in [140], proved a reverse Young's inequality in the following way:

$$0\leq\frac{a^p}{p}+\frac{b^q}{q}-ab\leq\left(b-a^{p-1}\right)\left(b^{q-1}-a\right), \tag{4.2.17}$$

for all $a,b\geq 0$ and $p,q>1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$.

This inequality is equivalent to the following inequality, for $p=\dfrac{1}{u}, a=x^u, b=x^{\frac{u}{u-1}}$:

If $a,b>0$ and $p\in(0,1)$, we change p and $\dfrac{1}{p}$, a by a^p and b by b^{1-p}, then inequality (3.2.17) becomes:

(4.2.18) $$0 \le pa + (1-p)b - a^p b^{1-p} \le \left(b^{1-p} - a^{1-p}\right)\left(b^p - a^p\right),$$

But, this is true $a, b > 0$ and $p \in [0,1]$.

In [Minculete, 151] we present another improvement of Young's inequality and a reverse inequality as follows

$$a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} \le pa + (1-p)b \le a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)},$$

for the positive real numbers a, b and $p \in [0,1]$ and $r = min\{p, 1-p\}$.

The first inequality can be found and in [121, Zuo] given by the Kantorovich's ratio defined by

$$K(h) = \frac{(h+1)^2}{4h}, h > 0,$$

and the second inequality is studied by Liao in [124], thus:

$$K^r\left(\frac{a}{b}\right)a^p b^{1-p} \le pa + (1-p)b \le K^{1-r}\left(\frac{a}{b}\right)a^p b^{1-p},$$

where $0 < b \le a$ and $p \in [0,1]$ and $r = min\{p, 1-p\}$.

This implies, the inequality

(4.2.19) $$a^p b^{1-p}\left[\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} - 1\right] \le pa + (1-p)b - a^p b^{1-p} \le a^p b^{1-p}\left[\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)} - 1\right],$$

where $0 < b \le a$ and $p \in [0,1]$ and $r = min\{p, 1-p\}$.

But, since

$$log\, t \le \frac{t^p - 1}{p} \le t^{p-1} log\, t, t \ge 1, p \in [0,1],$$

we have

$$2r\, log\left(\frac{a+b}{2\sqrt{ab}}\right) \le \left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} - 1$$

and

$$2(1-r)a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{1-2r} log\left(\frac{a+b}{2\sqrt{ab}}\right) \ge \left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)} - 1.$$

So, we have

(4.2.20)

$$2ra^p b^{1-p} log\left(\frac{a+b}{2\sqrt{ab}}\right) \le pa + (1-p)b - a^p b^{1-p} \le 2(1-r)a^p b^{1-p}\left(\frac{a+b}{2\sqrt{ab}}\right)^{1-2r} log\left(\frac{a+b}{2\sqrt{ab}}\right).$$

This inequality can be used to determine new inequalities for positive operators. Another idea to refine Young's inequality is the following: for $x > 1, p \ne 0$, we have

$$\frac{x^p - 1}{p} = \int_1^x t^{p-1} dt = \int_1^x t^p (log\, t)' dt = x^p log\, x - p\int_1^x t^{p-1} log\, t dt =$$

$$x^p log\, x - p\int_1^x t^p log\, t (log\, t)' dt = x^p log\, x - px^p log^2 x + p\int_1^x t^{p-1}(p\, log\, t + 1) log\, t dt$$

which implies the inequality

(4.2.21) $$x^p \log x - px^p \log^2 x \le \frac{x^p - 1}{p} \le x^p \log x,$$

for all $x > 1, p > 0$.

(4.2.22) $$x^p \log x \le \frac{x^p - 1}{p} \le x^p \log x - px^p \log^2 x,$$

for all $x > 1, p < 0$.

Inspired by the above work, I would like to start a new joint project with Shigeru Furuichi related to other inequalities of Young type. The results below have been developed together with Shigeru Furuichi in private communications:

Lemma 4.2.6. *If* $a > 0$ *, then the function* $h : \mathbb{R}^* \to \mathbb{R}$ *defined by* $h(x) = \frac{a^x - 1}{x}$ *is increasing.*

Proof. We consider $a > 0$ and the function $h : \mathbb{R}^* \to \mathbb{R}$ defined by $h(x) = \frac{a^x - 1}{x}$.

Taking into account that $u - 1 \ge \ln u$ for any $u > 0$, we deduce $\frac{1}{a^x} - 1 - \ln\frac{1}{a^x} \ge 0$, so $a^x (\ln a)x - a^x + 1 \ge 0$. Therefore, we have $h'(x) = \frac{a^x (\ln a)x - a^x + 1}{x^2} \ge 0$, i.e., the function h is increasing.

Proposition 4.2.7. *For* $a \ge b$ *and* $p,q,s \in \mathbb{R}$ *with* $q \le p \le s$, *the following inequality*

(4.2.23) $$\frac{1}{s}\left[sa + (1-s)b - a^s b^{1-s}\right] \le \frac{1}{p}\left[pa + (1-p)b - a^p b^{1-p}\right] \le \frac{1}{q}\left[qa + (1-q)b - a^q b^{1-q}\right],$$

holds.

Proof. For $q = 0$ or $p = 0$ or $s = 0$, the inequality is true.

We apply Lemma 4.2.6 for $t \ge 1$ and $p,q,s \in \mathbb{R}^*$ with $q \le p \le s$, and then we have the inequality $\frac{t^q - 1}{q} \le \frac{t^p - 1}{p} \le \frac{t^s - 1}{s}$, which is equivalent to

$$\frac{t^q - 1}{q} - (t-1) \le \frac{t^p - 1}{p} - (t-1) \le \frac{t^s - 1}{s} - (t-1).$$

If we take $t = \frac{a}{b} \ge 1$ in above inequality and multiplying by b, then we deduce the statement.

For $p = \frac{1}{2}$ and $q \le \frac{1}{2} \le s$ in inequality (4.2.23), we deduce

(4.2.24) $$\frac{1}{s}\left[sa + (1-s)b - a^s b^{1-s}\right] \le \left(\sqrt{a} - \sqrt{b}\right)^2 \le \frac{1}{q}\left[qa + (1-q)b - a^q b^{1-q}\right],$$

so, for $q \le \frac{1}{2} \le 1 - q$ in inequality (4.2.24), we deduce

(4.2.25) $$\frac{1}{1-q}\left[(1-q)a + qb - a^{1-q} b^q\right] \le \left(\sqrt{a} - \sqrt{b}\right)^2 \le \frac{1}{q}\left[qa + (1-q)b - a^q b^{1-q}\right],$$

which, in fact, proved the Kittaneh-Manasrah inequality.

Since $\int_1^x t^{p-1} dt = \frac{x^p - 1}{p}$, we have the following double integral:

$$(p-1)\int_1^t\int_1^x y^{p-2}dydx = \frac{t^p-1}{p}-(t-1), \text{ for } t,x\geq 1.$$

Lemma 4.2.8. *For the real numbers* $y\geq 1$ *and* $p\in\left[0,\frac{1}{2}\right]$, *the following inequality*

$$p(1-p)y^{p-\frac{1}{2}}\leq \frac{1}{2}max\{p,1-p\}, \tag{4.2.26}$$

and for the real numbers $y\geq 1$ *and* $p\in\left[\frac{1}{2},1\right]$, *the following inequality*

$$\frac{1}{2}min\{p,1-p\}\leq p(1-p)y^{p-\frac{1}{2}}. \tag{4.2.27}$$

Theorem 4.2.9. *For the real numbers* $t\geq 1$ *and* $p\in[0,1]$, *the following inequalities*

(4.2.28)

$$\int_1^t\int_1^x y^{-\frac{3}{2}}\left[\frac{1}{2}max\{p,1-p\}-p(1-p)y^{p-\frac{1}{2}}\right]dydx = t^p-pt-(1-p)+max\{p,1-p\}\left(\sqrt{t}-1\right)^2$$

and

(4.2.29)

$$\int_1^t\int_1^x y^{-\frac{3}{2}}\left[p(1-p)y^{p-\frac{1}{2}}-\frac{1}{2}min\{p,1-p\}\right]dydx = pt+(1-p)-t^p-min\{p,1-p\}\left(\sqrt{t}-1\right)^2$$

Proof. For $p\in\{0,1\}$, the equalities (4.2.28) and (4.2.29) are true. For $p\in(0,1)$, we have the following calculations:

$$\int_1^t\int_1^x y^{-\frac{3}{2}}\left[p(p-1)y^{p-\frac{1}{2}}+\frac{1}{2}max\{p,1-p\}\right]dydx =$$
$$= p(p-1)\int_1^t\int_1^x y^{p-2}dydx + max\{p,1-p\}\left(\sqrt{t}-1\right)^2 =$$
$$= t^p-1-p(t-1)+max\{p,1-p\}\left(\sqrt{t}-1\right)^2,$$

which is equivalent to inequality (4.2.28).

In an analogous way, we deduce the inequality (4.2.29).

Corollary 4.2.10. *For the real numbers* $t\geq 1$ *and* $p\in[0,1]$, *the following inequalities*

(4.2.30)

$$\int_1^t\int_1^x y^{-\frac{3}{2}}\left|\frac{1}{2}max\{p,1-p\}-p(1-p)y^{p-\frac{1}{2}}\right|dydx \leq t^p-pt-(1-p)+max\{p,1-p\}\left(\sqrt{t}-1\right)^2$$

and

(4.2.31)

$$\int_1^t\int_1^x y^{-\frac{3}{2}}\left|p(1-p)y^{p-\frac{1}{2}}-\frac{1}{2}min\{p,1-p\}\right|dydx \leq pt+(1-p)-t^p-min\{p,1-p\}\left(\sqrt{t}-1\right)^2.$$

Proof. Using the inequality Kittaneh and Manasrah in the form

$$min\{p,1-p\}\left(\sqrt{t}-1\right)^2 \leq pt+(1-p)-t^p \leq max\{p,1-p\}\left(\sqrt{t}-1\right)^2$$

and from $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$, we deduce the statement.

Theorem 4.2.11. *For the real numbers* $a \ge b$ *and* $p \in \left[0, \frac{1}{2}\right]$, *the following inequality:*

(4.2.32)

$$bp(1-p)\int_1^{a/b}\int_1^x y^{-\frac{3}{2}}\left[1-y^{p-\frac{1}{2}}\right]dydx \le a^p b^{1-p} - pa - (1-p)b + max\{p,1-p\}\left(\sqrt{a}-\sqrt{b}\right)^2$$

and for the real numbers $a \ge b$ *and* $p \in \left[\frac{1}{2}, 1\right]$, *the following inequality:*

(4.2.33)

$$\frac{b}{2}min\{p,1-p\}\int_1^{a/b}\int_1^x y^{-\frac{3}{2}}\left(y^{p-\frac{1}{2}}-1\right)dydx \le pa + (1-p)b - a^p b^{1-p} - min\{p,1-p\}\left(\sqrt{a}-\sqrt{b}\right)^2.$$

Proof. If $H(a,b) = \frac{2ab}{a+b}$ is the harmonic mean, then we have $min\{a,b\} \le H(a,b) = \frac{2ab}{a+b} \le max\{a,b\}$, which implies

$$\frac{1}{2}min\{p,1-p\} \le \frac{1}{2}H(p,1-p) = p(1-p) \le \frac{1}{2}max\{p,1-p\}.$$

But $\frac{1}{2}max\{p,1-p\} - p(1-p)y^{p-\frac{1}{2}} \ge p(1-p)\left(1-y^{p-\frac{1}{2}}\right)$ and using Theorem 4.2.9, we obtain

$$bp(1-p)\int_1^{a/b}\int_1^x y^{-\frac{3}{2}}\left[1-y^{p-\frac{1}{2}}\right]dydx \le a^p b^{1-p} - pa - (1-p)b + max\{p,1-p\}\left(\sqrt{a}-\sqrt{b}\right)^2$$

and since $p(1-p)y^{p-\frac{1}{2}} - \frac{1}{2}min\{p,1-p\} \ge \frac{1}{2}min\{p,1-p\}\left(y^{p-\frac{1}{2}}-1\right)$ and from Theorem 4.2.9 we deduce the statement.

4.3. Future directions for research related to inequalities in an inner product space

Maligranda [130] proved the following:

Theorem C. *For nonzero vectors x and y in a normed space* $X = (X, \|\cdot\|)$ *it is true that*

(4.3.1) $$\|x+y\| \le \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)min(\|x\|, \|y\|)$$

and

(4.3.2) $$\|x+y\| \ge \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)max(\|x\|, \|y\|).$$

Remark 4.3.1. If either $\|x\|=\|y\|=1$ or $y=cx$ with $c>0$, then equality holds in both (4.3.1) and (4.3.2).

We have that $\|ax+by\|^2=\langle ax+by,ax+by\rangle=a^2\|x\|^2+2ab\langle x,y\rangle+b^2\|y\|^2$, so implies

(4.3.3) $$\|ax+by\|^2=(a\|x\|+b\|y\|)^2-2ab(\|x\|\cdot\|y\|-\langle x,y\rangle).$$

In relation (4.3.3) for $a=1$ and $b=1$ we obtain

(4.3.4) $$\|x+y\|^2=(\|x\|+\|y\|)^2-2(\|x\|\cdot\|y\|-\langle x,y\rangle).$$

So, we deduce the equality, for nonzero vectors x and y in a normed space, given by the following:

(4.3.5) $$2(\|x\|\cdot\|y\|-\langle x,y\rangle)=(\|x\|+\|y\|)^2-\|x+y\|^2,$$

which means that

(4.3.6) $$\frac{2(\|x\|\cdot\|y\|-\langle x,y\rangle)}{\|x\|+\|y\|+\|x+y\|}=\|x\|+\|y\|-\|x+y\|.$$

This equality shows the equivalence between Cauchy-Schwarz's inequality and Minkowski's inequality.

Theorem 4.3.2. *For nonzero vectors x and y in a normed space* $X=(X,\|\cdot\|)$ *it is true that*

(4.3.7) $$\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)min(\|x\|,\|y\|)\le\frac{2(\|x\|\cdot\|y\|-\langle x,y\rangle)}{\|x\|+\|y\|+\|x+y\|}\le\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)max(\|x\|,\|y\|)$$

Remark. If either $\|x\|=\|y\|=1$ or $y=cx$ with $c>0$, then equality holds in (4.3,7).

From (4.3.6), for $x,y,z\in X$, we have

(4.3.8) $$\frac{2(\|x\|\cdot\|y+z\|-\langle x,y+z\rangle)}{\|x\|+\|y+z\|+\|x+y+z\|}=\|x\|+\|y+z\|-\|x+y+z\|.$$

We can reason and inversely: we find inferior and superior margin for Cauchy's inequality (see Radon's inequality [Rațiu-Minculete, 189]) and return to Minkowski's inequality written in norms.

In relation (4.3.3) for $a=1$ and $b=-1$ we obtain

(4.3.9) $$\|x-y\|^2=(\|x\|-\|y\|)^2+2(\|x\|\cdot\|y\|-\langle x,y\rangle).$$

In relation (4.3.3) for $a=\|x\|^{-1}$ and $b=\|y\|^{-1}$ we obtain

$$\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2=4-2\frac{1}{\|x\|\cdot\|y\|}(\|x\|\cdot\|y\|-\langle x,y\rangle)=2\left(1+\frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}\right).$$

In relation (4.3.3) for $a=\|x\|^{-1}$ and $b=-\|y\|^{-1}$ we obtain

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2=\frac{2}{\|x\|\cdot\|y\|}(\|x\|\cdot\|y\|-\langle x,y\rangle),$$

it follows that

(4.3.10) $$\frac{1}{2}\|x\|\cdot\|y\|\cdot\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2=\|x\|\cdot\|y\|-\langle x,y\rangle.$$

For nonzero vectors x and y in X we define the angular distance $\alpha[x,y]$ between x and y by

$$\alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|.$$

(see [40])
Therefore, we prove that

$$\frac{1}{2}\|x\|\cdot\|y\|\cdot(\alpha[x,y])^2=\|x\|\cdot\|y\|-\langle x,y\rangle. \tag{4.3.11}$$

Using the Massera-Schäffer inequality, proved in 1958 (see [134]): for nonzero vectors x and y in X there is the inequality

$$\alpha[x,y]\cdot max(\|x\|,\|y\|)\le 2\|x-y\|. \tag{4.3.12}$$

Combining relations (4.3.10) and (4.3.12) we deduce the inequality:

$$\|x\|\cdot\|y\|-\langle x,y\rangle\le\frac{2\cdot\|x\|\cdot\|y\|\cdot\|x-y\|^2}{[max(\|x\|,\|y\|)]^2}, \tag{4.3.13}$$

which is equivalent with

$$\|x\|\cdot\|y\|\left[[max(\|x\|,\|y\|)]^2-2\cdot\|x-y\|^2\right]\le\langle x,y\rangle[max(\|x\|,\|y\|)]^2.$$

Relation (4.3.10) can be written as

$$\frac{1}{2\|x\|\cdot\|y\|}\big\|\|y\|\cdot x-\|x\|\cdot y\big\|^2=\|x\|\cdot\|y\|-\langle x,y\rangle,$$

Therefore, we obtain

$$\frac{1}{2[max(\|x\|,\|y\|)]^2}\big\|\|y\|\cdot x-\|x\|\cdot y\big\|^2\le\|x\|\cdot\|y\|-\langle x,y\rangle\le\frac{1}{2[min(\|x\|,\|y\|)]^2}\big\|\|y\|\cdot x-\|x\|\cdot y\big\|^2. \tag{4.3.14}$$

We apply these inequalities in an inner product space:
a) $(R^n,\langle\cdot\rangle)$, where for $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$ we have

$$\langle x,y\rangle=x_1y_1+x_2y_2+...+x_ny_n \text{ and } \|x\|=\sqrt{x_1^2+x_2^2+...+x_n^2}.$$

But, we find

$$n\min_{i=1,n}(\|y\|\cdot x_i-\|x\|\cdot y_i)^2\le\big\|\|y\|\cdot x-\|x\|\cdot y\big\|^2=\sum_{i=1}^{n}(\|y\|\cdot x_i-\|x\|\cdot y_i)^2\le n\max_{i=1,n}(\|y\|\cdot x_i-\|x\|\cdot y_i)^2.$$

Combining this inequality with inequality (4.3.15), we deduce

$$\frac{n}{2}\frac{\min_{i=1,n}(\|y\|\cdot x_i-\|x\|\cdot y_i)^2}{[max(\|x\|,\|y\|)]^2}\le\|x\|\cdot\|y\|-\langle x,y\rangle\le\frac{n}{2}\frac{\max_{i=1,n}(\|y\|\cdot x_i-\|x\|\cdot y_i)^2}{[min(\|x\|,\|y\|)]^2}, \tag{4.3.15}$$

for all $x,y\in R^n$.

b) $(C^0[a,b],\langle\cdot\rangle)$, where for $f,g\in C^0[a,b]$ we have

$$\langle f,g\rangle=\int_a^b f(x)g(x)dx \text{ and } \|f\|=\sqrt{\int_a^b f^2(x)dx}.$$

If we replace in inequality (4.3.10), then we deduce

$$\frac{1}{2\sqrt{\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx}}\int_a^b\left(\sqrt{\int_a^b g^2(x)dx}\cdot f(x)-\sqrt{\int_a^b f^2(x)dx}\cdot g(x)\right)^2 dx=$$
$$=\sqrt{\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx}-\int_a^b f(x)g(x)dx.$$

Inspired by the above work, I would like to start a new joint project with Radu Păltănea related to inequalities in an inner product space. We would like to propose the extending of the notions of variance and covariance to vectors.

Vector projection is an important operation in the Gram-Schmidt orthonormalization of vector space bases.

The projection of a vector x onto a vector y is given by $proj_y x=\frac{\langle x,y\rangle}{\|y\|^2}y$.

If in $(R^n,\langle\cdot,\cdot\rangle)$, we denote by u, the vector $u=(1,1,...,1)$, then

$$proj_u x=\frac{\langle x,u\rangle}{\|u\|^2}u=\left(\frac{1}{n}\sum_{i=1}^n x_i,...,\frac{1}{n}\sum_{i=1}^n x_i\right).$$

In $(R^n,\langle\cdot,\cdot\rangle)$, we define *the variance of a vector x* by

$$var(x)=\frac{1}{\|u\|^2}\|x-proj_u x\|^2$$

and the covariance of a vectors x and y by

$$cov(x,y)=\frac{1}{\|u\|^2}\langle x-proj_u x,y-proj_u y\rangle.$$

$(C^0[a,b],\langle\cdot,\cdot\rangle)$, where for $f,g\in C^0[a,b]$ we have

$$\langle f,g\rangle=\int_a^b f(x)g(x)dx \text{ and } \|f\|=\sqrt{\int_a^b f^2(x)dx}.$$

The projection of a vector f onto a vector g is given by $proj_g f=\frac{\langle f,g\rangle}{\|g\|^2}g$.

If in $(C^0[a,b],\langle\cdot\rangle)$, we take $g(x)=1$, we have

$$proj_1 f=\frac{\langle f,1\rangle}{\|1\|^2}1=\frac{1}{b-a}\int_a^b f(x)dx.$$

Thus, in $(C^0[a,b],\langle\cdot\rangle)$, we define the variance of a function f by

$$var(f)=\frac{1}{\|1\|^2}\|f-proj_1 f\|^2$$

and the covariance of a vectors f and g by

$$cov(f,g)=\frac{1}{\|1\|^2}\langle f-proj_1 f,g-proj_1 g\rangle.$$

Another future direction for research in inequalities between the elements of an inner product space is related to Cauchy-Schwarz's inequality in an inner product space and its applications.

Next, we develop these inequalities for linear combinations of vectors.

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal system of vectors in unitary space $X = (X, \langle\cdot,\cdot\rangle)$ over the field of real numbers.

For $x \in X$, we put

$$\bar{x} = x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \text{ and } S_n(x, y) = \langle x, y \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, y \rangle,$$

where $x, y \in X$.

In [63], Dragomir proved the following inequality

(4.3.16) $$[S_n(x, y)]^2 \le S_n(x, x) S_n(y, y)$$

where $x, y \in X$. This inequality can be found in [113].

In relation (4.3.16) the equality holds if and only if $\{x, y, e_1, e_2, ..., e_n\}$ is linearly dependent. For $n = 1$, we apply inequality (4.3.16) on $L_2(a,b)$ for $\left\{e_1 = \frac{1}{\sqrt{b-a}}, x = \frac{1}{\sqrt{b-a}} f, y = \frac{1}{\sqrt{b-a}} g\right\}$, where $f, g \in L_2(a,b)$, and we obtain an inequality in terms of the Chebyshev functional, as follows:

(4.3.17) $$[T(f, g)]^2 \le T(f, f) T(g, g),$$

where $f, g \in L_2(a,b)$ and

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \, \frac{1}{b-a} \int_a^b g(x) dx.$$

This inequality proved the Grüss inequality, which for *f* and *g* two bounded functions defined on [a,b] with $\gamma_1 \le f(x) \le \Gamma_1$ and $\gamma_2 \le g(x) \le \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants, we have $T(f, f) \le \frac{1}{4}(\Gamma_1 - \gamma_1)^2$, so we obtain

$$T(f, g) \le \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2).$$

In terms of *h*-covariance inequality (2.4.15) becomes

$$[cov_h(f, g)]^2 \le var_h(f) var_h(g).$$

where $h : [a,b] \to [0,\infty)$ is a Riemann- integrable function with $\int_a^b h(x) dx > 0$.

From [113] we found the following identity.

$$\langle \bar{x}, \bar{y} \rangle = \langle x, y \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, y \rangle = S_n(x, y).$$

But, we remark that $\langle \bar{x}, \bar{y} \rangle = \langle \bar{x}, y \rangle = \langle x, \bar{y} \rangle$, so we deduce

$$\|\bar{x}\|^2 = \langle \bar{x}, \bar{x} \rangle = \langle \bar{x}, x \rangle = \langle x, \bar{x} \rangle = S_n(x, x) = \|x\|^2 - \sum_{k=1}^{n} \langle x, e_k \rangle^2.$$

It is easy to see that $\overline{\alpha x + \beta y} = \alpha \bar{x} + \beta \bar{y}$ for every real numbers α, β.

Inequality (2.4.14) is in fact the Cauchy-Schwarz inequality for vectors $\bar{x}, \bar{y}$, i.e., $\langle \bar{x}, \bar{y} \rangle^2 \le \|\bar{x}\|^2 \|\bar{y}\|^2$.

Proposition 4.3.3. *With above notations, we have*

(4.3.18) $$0 \le \frac{S_n(y,y)}{S_n(z,z)}\left(\frac{S_n(x,y)S_n(y,z)}{S_n(y,y)} - S_n(x,z)\right)^2 \le S_n(x,x)S_n(y,y) - [S_n(x,y)]^2,$$

for all $x,y,z \in X$, $\{y,e_1,e_2,\dots,e_n\}$, $\{z,e_1,e_2,\dots,e_n\}$ *are linearly independent.*

Proof. Using Corollary 2.3.6, we have

$$\frac{\|\bar{y}\|^2}{\|\bar{z}\|^2}\left(\frac{\langle \bar{x},\bar{y}\rangle\langle \bar{y},\bar{z}\rangle}{\|\bar{y}\|^2} - \langle \bar{x},\bar{z}\rangle\right)^2 \le \|\bar{x}\|^2\|\bar{y}\|^2 - \langle \bar{x},\bar{y}\rangle^2,$$

for all $x,y,z \in X$, $\{y,e_1,e_2,\dots,e_n\}$, $\{z,e_1,e_2,\dots,e_n\}$ are linearly independent. By substitution we deduce the statement.

□

This inequality represents an improvement of inequality (4.3.16).

Similarly to the ones mentioned above for $n=1$, we apply inequality (4.3.16) on $L_2(a,b)$ for $\left\{e_1 = \frac{1}{\sqrt{b-a}}, x = \frac{1}{\sqrt{b-a}} f, y = \frac{1}{\sqrt{b-a}} g, z = \frac{1}{\sqrt{b-a}} h\right\}$, where $f,g \in L_2(a,b)$, and we obtain an inequality in terms of the Chebyshev functional, as follows:

(4.3.19) $$0 \le \frac{T(g,g)}{T(h,h)}\left(\frac{T(f,g)T(g,h)}{T(g,g)} - T(f,h)\right)^2 \le T(f,f)T(g,g) - [T(f,g)]^2,$$

where $f,g,h \in L_2(a,b)$, $T(g,g), T(h,h) > 0$, and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \, \frac{1}{b-a}\int_a^b g(x)dx.$$

This inequality is an improvement of inequality (4.3.17).

Let $X = (X, \langle \cdot,\cdot \rangle)$ be a inner product space over the field of real numbers.

For $n=1$ in inequality (4.3.16) and the vector $e \in X$ with $\|e\| = 1$, we have

(4.3.20) $$[\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle]^2 \le \left(\|x\|^2 - \langle x,e\rangle^2\right)\left(\|y\| - \langle y,e\rangle^2\right).$$

Next, we obtain a refinement of inequality (4.3.20), thus:

Corollary 4.3.4. *For all* $e,x,y,z \in X$ *with* $\|e\| = 1$ *and* $\{y,e\}$, $\{z,e\}$ *are linearly independent, we have*

(4.3.21) $$0 \le A \le \left(\|x\|^2 - \langle x,e\rangle^2\right)\left(\|y\| - \langle y,e\rangle^2\right) - [\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle]^2,$$

where $$A = \frac{\|y\|^2 - \langle y,e\rangle^2}{\|z\|^2 - \langle z,e\rangle^2}\left(\frac{(\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle)(\langle y,z\rangle - \langle y,e\rangle\langle e,z\rangle)}{\|y\|^2 - \langle y,e\rangle^2} - \langle x,z\rangle - \langle x,e\rangle\langle e,z\rangle\right)^2.$$

Proof. Using Proposition 4.3.3 for $n=1$, we obtain the statement.

□

Theorem 4.3.5. *For all* $e,x,y,z \in X$ *with* $\|e\| = 1$, *we have*

(4.3.22)
$$\left[\langle x,y\rangle\left(\|z\|^2 - \langle z,e\rangle^2\right) + \left(\langle z,e\rangle(\langle x,z\rangle\langle y,e\rangle + \langle y,z\rangle\langle x,e\rangle) - \langle x,z\rangle\langle y,z\rangle - \|z\|^2\langle x,e\rangle\langle e,y\rangle\right)\right]^2$$

$$\leq \left[\|x\|^2\left(\|z\|^2-\langle z,e\rangle^2\right)-\left(\langle x,z\rangle-\langle x,e\rangle\langle z,e\rangle\right)^2\right]\left[\|y\|^2\left(\|z\|^2-\langle z,e\rangle^2\right)-\left(\langle y,z\rangle-\langle y,e\rangle\langle z,e\rangle\right)^2\right].$$

Proof. We consider the vectors $e,x,y,z\in X$ with $\|e\|=1$, we take $w=z-\langle z,e\rangle e$, $\{z,e\}$ are linearly independent. It follows that $\|w\|^2=\|z\|^2-\langle z,e\rangle^2$ and $\langle e,w\rangle=0$. For $u=\frac{w}{\|w\|}$, we have $\|e\|=\|u\|=1, \langle e,u\rangle=0$, so, applying inequality (4.3.16), we obtain

$$S_2(x,y)=\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle-\langle x,u\rangle\langle u,y\rangle=\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle$$
$$-\frac{1}{\|w\|^2}\left(\langle x,z\rangle-\langle x,e\rangle\langle e,z\rangle\right)\left(\langle y,z\rangle-\langle y,e\rangle\langle e,z\rangle\right)$$
$$=\langle x,y\rangle+\frac{1}{\|w\|^2}\left(\langle z,e\rangle\left(\langle x,z\rangle\langle y,e\rangle+\langle y,z\rangle\langle x,e\rangle\right)-\langle x,z\rangle\langle y,z\rangle-\|z\|^2\langle x,e\rangle\langle e,y\rangle\right).$$

Therefore, we have

$$S_2(x,y)=\langle x,y\rangle+\frac{1}{\|w\|^2}\left(\langle z,e\rangle\left(\langle x,z\rangle\langle y,e\rangle+\langle y,z\rangle\langle x,e\rangle\right)-\langle x,z\rangle\langle y,z\rangle-\|z\|^2\langle x,e\rangle\langle e,y\rangle\right).$$

It follows that

$$S_2(x,x)=\|x\|^2+\frac{1}{\|w\|^2}\left(2\langle x,z\rangle\langle x,e\rangle\langle z,e\rangle-\langle x,z\rangle^2-\|z\|^2\langle x,e\rangle^2\right).$$

But, using the Cauchy-Schwarz inequality, $\langle z,e\rangle^2\leq\|z\|^2$, we deduce

$$S_2(x,x)\leq\|x\|^2+\frac{1}{\|w\|^2}\left(2\langle x,z\rangle\langle x,e\rangle\langle z,e\rangle-\langle x,z\rangle^2-\langle z,e\rangle^2\langle x,e\rangle^2\right)$$
$$=\|x\|^2-\frac{1}{\|w\|^2}\left(\langle x,z\rangle-\langle x,e\rangle\langle z,e\rangle\right)^2.$$

Consequently, we obtain the inequality

$$S_2(x,x)\leq\|x\|^2-\frac{\left(\langle x,z\rangle-\langle x,e\rangle\langle z,e\rangle\right)^2}{\|z\|^2-\langle z,e\rangle^2}.$$

Similarly, we deduce $S_2(y,y)\leq\|y\|^2-\frac{\left(\langle y,z\rangle-\langle y,e\rangle\langle z,e\rangle\right)^2}{\|z\|^2-\langle z,e\rangle^2}$.

According with inequality (4.3.16), we find $[S_2(x,y)]^2\leq S_2(x,x)S_2(y,y)$ and combining with above inequalities, we obtain the statement.

□

We intend to study other applications of inequality (4.3.16) and we will investigate another improvement of this inequality.

Conclusions

In the present work we have described results related to mathematical inequalities and its applications. We obtained a series of inequalities related to inequalities for functionals, inequalities for invertible positive operators and inequalities in an inner product space. We presents several applications to the inequalities found to probability and statistics.

This habilitation thesis contains a number of new and basic inequalities related to Hermite-Hadamard's inequality, Grüss's inequality, Hammer- Bullen's inequality and Cauchy-Schwarz's inequality (in an inner product space) investigated in order to achieve a diversity of desired goals.

We conclude with a list of items that are part of our current and future directions research. The domain of mathematical inequalities is quite lively, but it can always generate novelty elements and interesting applications.

We summarize the list of our current and future directions research as follows:

- A first direction of research refers to the reconsideration of Hermite-Hadamard's inequality and with a new approach we can find an improvement and new applications of it. We would like to propose a new inequalities for Stolarsky's mean, logarithmic mean, identric mean, etc.
- Connected with the functional Jensen, in the future, we would like to study other properties of generalized entropies as the following: the Tsallis entropy, the Rényi entropy, the quasilinear relative entropy, the Rényi relative entropy, the Tsallis relative entropy, the Tsallis quasilinear entropy (q-quasilinear entropy), the Tsallis quasilinear relative entropy.
- Inspired by the above work and the recent results, I would like to start a new project related to Hardy inequality. We would like to propose a new refinement of Young's inequality which can be use in the proof of Hardy's inequality and Carleman's inequality.
- Another future direction for research in inequalities between the elements of an inner product space is related to Cauchy-Schwarz's inequality in an inner product space and its applications. As the main starting point, we refer to the inequality, $[S_n(x,y)]^2 \le S_n(x,x)S_n(y,y)$, given by Dragomir in [63].

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