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HABILITATION THESIS
SUMMARY

Title: Contributions to the Theory of Inequalities

Domain: Mathematics

Author: Assoc. Prof. PhD. Nicuşor Minculete
Transilvania University of Braşov

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SUMMARY

”*All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove.*” – G.H. Hardy

In this habilitation thesis we have described the significant results achieved by the author after obtaining his PhD degree in Mathematics from the Simion Stoilow Institute of Mathematics of the Romanian Academy, in 2012. The Theory of Inequalities represents an old topic of many mathematical areas which still remains an attractive research domain, with many applications. The study of convex functions has occupied and occupies a central role in the Theory of Inequalities, because the convex functions develop a number of known inequalities.

The research results presented here are concerned with the improvement of classical inequalities resulting from convex functions and highlighting their applications.

Regarding the probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, known as Jensen's inequality, underlies many important inequalities. Convexity generates a significant number of inequalities in different frameworks: in normed spaces, in inner product spaces, in 2-inner product spaces etc.

The classic book of Hardy, Littlewood and Pólya [112] played an important role in the popularization of the topic of convex functions.

Another important result related to convex functions is the *Hermite–Hadamard inequality* (see [177]), which asserts that for every convex function $f : [a, b] \to \mathbb{R}$ the following inequality holds:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Related to the Hermite–Hadamard inequality, many mathematicians have worked with great interest to generalize, refine and extend it for different classes of functions such as: quasi-convex functions, log-convex, r-convex functions, etc and apply it to special means (logarithmic mean, Stolarsky mean etc). For example, in [77], Dragomir and Pearce state that the Hermite–Hadamard double inequality is the first fundamental result for convex functions defined on an interval of real numbers with a natural geometrical interpretation. In this monograph they present the basic facts related to Hermite–Hadamard inequalities for convex

functions and a large number of results for special means. In [80], Dragomir, Pečarić and Persson proved a form of the Hermite–Hadamard inequality for the class of functions introduced by Godunova and Levin. In [128], Kunt, Işcan, Yazici and Gözütok showed new Hermite–Hadamard type inequalities for harmonically convex functions in fractional integral forms, obtained an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for harmonically convex functions in fractional integral forms.

Regarding the study of inequalities and their applications, using the convexity, Niculescu and Persson [178] say the following:

"*What motivates the constant interest for this subject?*

First, its elegance and the possibility to prove deep results even with simple mathematical tools. Second, many problems raised by science, engineering, economics, informatics, etc. fall in the area of convex analysis."

Related to the study of the inequalities Tanner [208] says that:

"*The equality indicates a boundary, but we are really concerned with what lies inside and outside. The equality is like a fine ceremonial dress, beautiful for show; but you get into your shirt-sleeves for the real work. In fact, that seems to be the keynote of the situation; we like to present our finished mathematics, mathematics for show to the public, as much as possible in equality form, but in the mathematical workshop, inequalities are the standard tools.*"

The present habilitation thesis is focused on the study of important inequalities from the Theory of Inequalities and on their impact in some applications.

It consists of six chapters. It also includes a list of notations and a bibliography with 220 references.

In the first part of this thesis we present the scientific and professional achievements and it contains several original results, many of them published in Web of Science journals, namely:

- *Mediterranean Journal of Mathematics*
- *Annals of Functional Analysis*
- *Bulletin of the Malaysian Mathematical Sciences Society*
- *Mathematical Inequalities* & *Applications*
- *Aequationes Mathematicae*
- *Physica A: Statistical Mechanics and its Applications*

After conferring the title of doctor, other original scientific results, which do not fall within the theme of the habilitation thesis, were published in the journal:

- *Expositiones Mathematicae*

So far, my scientific results have recorded 140 citations, of which 47 are recognized by the specialized CNATDCU commission.

Chapter 1 is dedicated to the contributions of the author of this thesis to the theory of inequalities developed from convex functions.

In *Section* 1.1 we recall some results related to the Hermite–Hadamard inequality given by Hardy, Littlewood and Pólya in [112], Dragomir and Pearce in [77]. The Hermite–Hadamard inequality is the starting point to Choquet theory [177]. Motivated by the above results we have shown in [156] other inequalities of the Hermite–Hadamard type and of the Fejér type. One of the results is the following: Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq f'' \leq M$. Assume $g : [a, b] \to \mathbb{R}_+$ is integrable and symmetric about $\frac{a+b}{2}$. Then the following inequalities hold:

$$\frac{m}{2}\int_a^b (t-a)(b-t)g(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)dt - \int_a^b f(t)g(t)dt$$
$$\leq \frac{M}{2}\int_a^b (t-a)(b-t)g(t)dt$$

and

$$\frac{m}{8}\int_a^b (2t-a-b)^2 g(t)dt \leq \int_a^b f(t)g(t)dt - f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$
$$\leq \frac{M}{8}\int_a^b (2t-a-b)^2 g(t)dt.$$

In *Section* 1.2 we give bounds on the difference between the weighted arithmetic mean and the weighted geometric mean. These imply a refined Young inequality and the reverses of the Young inequality. We also studied some properties of the difference between the weighted arithmetic mean and the weighted geometric mean. Applying the newly obtained inequalities, we show some results on the Tsallis divergence ($J_q(\mathbf{p}|\mathbf{r})$), the Rényi divergence, the Jeffreys–Tsallis divergence and the Jensen–Shannon–Tsallis divergence. We have shown several inequalities on generalized entropies. One of the results is the following: For two probability distributions $\mathbf{p} := \{p_1, \cdots, p_n\}$ and $\mathbf{r} := \{r_1, \cdots, r_n\}$ with $p_j > 0$ and $r_j > 0$ for all $j = 1, \cdots, n$, and $0 \leq q < 1$, we have

$$\frac{4r}{1-q}h^2(\mathbf{p}|\mathbf{r}) + \frac{2A\left(q\right)}{1-q}\sum_{j=1}^n p_j r_j \cdot \log^2\left(\frac{p_j}{r_j}\right) \leq J_q(\mathbf{p}|\mathbf{r})$$
$$\leq \frac{4\left(1-r\right)}{1-q}h^2(\mathbf{p}|\mathbf{r}) + \frac{2B\left(q\right)}{1-q}\sum_{j=1}^n p_j r_j \cdot \log^2\left(\frac{p_j}{r_j}\right),$$

where $r = \min\{q, 1-q\}$, $A\left(q\right) = \frac{q(1-q)}{2} - \frac{r}{4}$, $B\left(q\right) = \frac{q(1-q)}{2} - \frac{1-r}{4}$ and the *Hellinger distance* ([133], [215]) is defined by

$$h(\mathbf{p}|\mathbf{r}) := \frac{1}{\sqrt{2}}\sqrt{\sum_{j=1}^n (\sqrt{p_j} - \sqrt{r_j})^2}.$$

Generalized entropies have been studied by many researchers. Rényi [199] and Tsallis [210] entropies are well known as one-parameter generalizations of Shannon's entropy, being intensively studied not only in the field of classical statistical physics [211], [212], [213], but also in the field of quantum physics [206].

In *Section* 1.3 we present a refinement of Grüss inequality via Cauchy–Schwarz inequality for discrete random variables in finite case. We analyze the bounds of several statistical indicators and we give a generalized form of Grüss inequality and we obtain other integral inequalities. Here we obtain some bounds of several statistical indicators, namely: the variance, the standard deviation, the coefficient of variation and the convariance for discrete random variables in finite case. Among the obtained results we have: If X, Y and Z are finite discrete random variables, with $X \neq \lambda Z$, then we have the inequality

$$0 \leq \frac{[\mathrm{Cov}(X,Y)\mathrm{Cov}(X,Z) - \mathrm{Cov}(Y,Z)\mathrm{Var}(X)]^2}{\mathrm{Var}(X)\mathrm{Var}(Z) - [\mathrm{Cov}(X,Z)]^2} \leq \mathrm{Var}(X)\mathrm{Var}(Y) - [\mathrm{Cov}(X,Y)]^2 .$$

We also give a generalized form of Grüss inequalities and other integral inequalities. For example: Let f and g be two Riemann-integrable functions defined on $[a,b]$ with $\gamma_1 \leq f(x) \leq \Gamma_1$ and $\gamma_2 \leq g(x) \leq \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants and we have a Riemann-integrable function $h : [a,b] \to [0,\infty)$ with $\int_a^b h(x)dx > 0$. Then we have

$$|\mathrm{cov}_h(f,g)| \leq \sqrt{(\Gamma_1 - M_h[f])(M_h[f] - \gamma_1)(\Gamma_2 - M_h[g])(M_h[g] - \gamma_2)}$$

$$\leq \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2),$$

where $M_h[f]$ is the h-integral arithmetic mean for a Riemann-integrable function f.

Chapter 2 is devoted to the contributions of the author of this thesis to the study of inequalities in an inner product space (pre-Hilbert space). We remark the study of the Cauchy–Schwarz inequality in an inner product space and some reverse inequalities for the Cauchy–Schwarz inequality in an inner product space. We also make considerations about several inequalities and we mention a characterization of an inner product space.

In *Section* 2.1, motivated by the results obtained by Maligranda [136] and [137] we proved an improvement of the Maligranda inequality given in [157]. Here we showed some estimates of the triangle inequality in normed spaces using the Tapia semi-product. The main result is the following: Let $x, y \in X$ be nonzero vectors such that $\|y\| \le \|x\|$ and $\|x\|y \ne -\|y\|x$. Then

$$\begin{aligned}
\|x\|+\|y\|-\|x+y\| &\ge (2-\|v(x,y)\|)\,\|x\| \\
&\quad -\left(1-\left(\frac{x+y}{\|x+y\|},\frac{y}{\|y\|}\right)_T\right)(\|x\|-\|y\|),\\
\|x\|+\|y\|-\|x+y\| &\le (2-\|v(x,y)\|)\,\|x\| \\
&\quad -\left(1-\left(\frac{v(x,y)}{\|v(x,y)\|},\frac{y}{\|y\|}\right)_T\right)(\|x\|-\|y\|),\\
\|x\|+\|y\|-\|x+y\| &\ge (2-\|v(x,y)\|)\,\|y\| \\
&\quad +\left(1-\left(\frac{x+y}{\|x+y\|},\frac{x}{\|x\|}\right)_T\right)(\|x\|-\|y\|),\\
\|x\|+\|y\|-\|x+y\| &\le (2-\|v(x,y)\|)\,\|y\| \\
&\quad +\left(1-\left(\frac{v(x,y)}{\|v(x,y)\|},\frac{x}{\|x\|}\right)_T\right)(\|x\|-\|y\|),
\end{aligned}$$

where the Tapia semi-product on the normed space X, (see [209]), is the function $(\cdot,\cdot)_T : X \times X \to \mathbb{R}$, defined by

$$(x,y)_T = \lim_{t\searrow 0}\frac{\varphi(x+ty)-\varphi(x)}{t},$$

where $\varphi(x) = \frac{1}{2}\|x\|^2$, $x \in X$.

In *Section* 2.2 we introduce the notation:

$$\Gamma(a,b;c,d) = \langle a,c\rangle\langle b,d\rangle - \langle a,d\rangle\langle b,c\rangle.$$

for $a,b,c,d \in X$, where X is an inner product space over the field of real numbers. Here we establish several properties of $\Gamma(a,b;c,d)$. We also obtained new results related to the inequality given by Harvey [113] and Choi [46]. One of the results is the following: Let X be an inner product space. For any elements $a,b,c,d \in X$ with $\{a,b\}$ linearly independent and c nonzero vector, we have

$$\begin{aligned}
&\Gamma(a,b;a,b)\cdot\Gamma(c,d;c,d)-\Gamma^2(a,b;c,d)\\
&\ge \frac{1}{\|c\|^4\Gamma^2(a,b;a,b)}\Big[\Gamma(a,b;b,c)\Gamma(a,c;c,d)+\Gamma(a,b;c,a)\Gamma(b,c;c,d)\Big]^2.
\end{aligned}$$

We present an application of the above result to S_n numbers. Recall that if $(X,\langle\cdot,\cdot\rangle)$ is a finite-dimensional inner product space and $\{e_1,\ldots,e_n\}$ is an orthonormal system of vectors of X, then for any vectors $x,y \in X$, we define as in [119]:

$$S_n(x,y) = \langle x,y\rangle - \sum_{k=0}^{n}\langle x,e_k\rangle\langle y,e_k\rangle,$$

and $\hat{x} = x - \sum_{k=0}^{n}\langle x,e_k\rangle e_k$. Let $\{e_1,\ldots,e_n\}$ be an orthonormal system in the inner product space X. Let $\ell_1,\ell_2 \in X$ be linearly independent, such that $\langle e_k,\ell_i\rangle = 0$, for $k \in \{1,...,n\}$ and $i \in \{1,2\}$. Then, for any vectors $x,y \in X$ we have

$$S_n(x,x)S_n(y,y) - S_n^2(x,y) \ge \frac{\Gamma^2(x,y;\ell_1,\ell_2)}{\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)} + T(\hat{x},\hat{y},\ell_1,\ell_2),$$

where

$$T(\hat{x},\hat{y},\ell_1,\ell_2) = \frac{\Big[\Gamma(\hat{x},\hat{y};\hat{y},\ell_1)\Gamma(x,\ell_1;\ell_1,\ell_2) + \Gamma(\hat{x},\hat{y};\ell_1,\hat{x})\Gamma(y,\ell_1;\ell_1,\ell_2)\Big]^2}{\|\ell_1\|^4\Gamma^2(\hat{x},\hat{y};\hat{x},\hat{y})\Gamma(\ell_1,\ell_2;\ell_1,\ell_2)}.$$

In *Section* 2.3, we study some interesting characterizations of real inner product spaces expressed in terms of angular distances, where the norm-angular distance or *the Clarkson distance* (see e.g. [136]) between two nonzero vectors x and y is given by $\alpha[x,y] = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. We first discuss the equivalence of characterizing an inner product space via the usual angular distance and the p-angular distance, which for p in the interval $[0,\infty)$ and for nonzero x and y in X is defined as $\alpha_p[x,y] = \left\|\|x\|^{p-1}x - \|y\|^{p-1}y\right\|$, with $\alpha_0[x,y] = \alpha[x,y]$, [136]. Then, we establish a parametric family of upper bounds for the usual angular distance which also serves as a characterization of an inner product space. As an application, bounds for the usual angular distance are utilized in obtaining improvements of the real Cauchy–Schwarz inequality. Finally, we give several comparative relations for angular distances in inner product spaces. In 2013, Dehghan [57], established an interesting characterization of an inner product space relying on the relationship between $\alpha[x,y]$ and $\beta[x,y]$. More precisely, he proved that a normed space X is an inner space if and only if $\alpha[x,y] \le \beta[x,y]$ holds for all nonzero elements $x,y \in X$, where the skew p-angular distance between nonzero elements x,y in a normed linear space X is given as $\beta_p[x,y] = \|\|x\|^{p-1}x - \|y\|^{p-1}y\|$, $p \in \mathbb{R}$, [202]. We show a generalization of the inequality of Kirk and Smiley. If $X = (X,\langle\cdot,\cdot\rangle)$ is an Euclidean space and the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then we have

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^t \le \frac{2\|x-y\|^t}{\|x\|^t + \|y\|^t},$$

for all nonzero vectors x and y in X and $t \in (0,1]$.

In *Section* 2.4, we show some results regarding the Cauchy–Schwarz inequality and the triangle inequality. We will also present some characterizations of the relationship between these two inequalities. If $X = (X,\langle\cdot,\cdot\rangle)$ is an Euclidean space and the norm $\|\cdot\|$ is generated by the inner product $\langle\cdot,\cdot\rangle$, then we have

$$\max\{\|x\|,\|y\|\}(\|x\|+\|y\|-\|x+y\|) \le \|x\|\cdot\|y\| - \langle x,y\rangle,$$

for every vectors x and y in X. Other inequalities related to the p-angular distance are the following:

$$\alpha_p[x,y] \ge \|x\|^p + \|y\|^p - \min\{\|x\|^p,\|y\|^p\}\left(1 + \frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right),$$

for every nonzero vectors x and y in X and

$$\alpha_p[x,y] \ge |\|x\|^p - \|y\|^p| + \frac{1}{2}\min\{\|x\|^p,\|y\|^p\}\alpha^2[x,y],$$

for every vectors x and y in X, $p \in \mathbb{R}$.

In **Chapter 3**, we present the contributions of the author of this thesis to prove certain properties of a 2-inner product space. In [103], Gähler investigated the concept of linear 2-normed spaces and 2-metric spaces. In [59, 60], Diminnie, Gähler and White studied the 2-inner product spaces and their properties. A classification of the results which are related to the theory of 2-inner product spaces can be found in [47, 127]. Here, several properties of 2-inner product spaces are given. In [76], Dragomir *et al.* showed the corresponding version of the Boas–Bellman inequality in 2-inner product spaces. Najati *et al.* [175] showed the generalized Dunkl-Williams inequality in 2-normed spaces.

In *Section* 3.1, we obtain some refinements of Ostrowski's inequality in an inner product space. We establish an extension of Ostrowski's inequality in a 2-inner product space. We also show certain types of Ostrowski's inequality in a 2-inner product space and we present some applications which are related to the Chebyshev function and the Grüss inequality.

In an Euclidean space $X = (X, \langle\cdot,\cdot\rangle)$, with $dim_{\mathbb{R}} X \geq 2$, the following inequality

$$0 \leq \frac{\|y\|^2\|z\|^2 - \langle y,z\rangle^2}{\|u\|^2\|z\|^2 - \langle u,z\rangle^2}\|z\|^4 \leq \left(\|x\|^2\|z\|^2 - \langle x,z\rangle^2\right)\left(\|y\|^2\|z\|^2 - \langle y,z\rangle^2\right) - \left(\langle x,y\rangle\|z\|^2 - \langle x,z\rangle\langle z,y\rangle\right)^2$$

holds for all $x, y, z, u \in X$, with $\langle x,u\rangle\|z\|^2 = \|z\|^2 + \langle x,z\rangle\langle z,u\rangle$, $\langle y,u\rangle\|z\|^2 = \langle y,z\rangle\langle z,u\rangle$ and $\{u, z\}$ linearly independent.

The *Section* 3.2 is devoted to the study of some identities in a 2-pre-Hilbert space and we prove new results related to several inequalities in a 2-pre-Hilbert space. We mention the Cauchy–Schwarz inequality. The novelty of this section is the introduction, for the first time, of the concepts of average, variance, covariance, standard deviation and correlation coefficient for vectors, using the standard 2-inner product and some of its properties. We also present a brief characterization of a linear regression model for the random variables in discrete case.

Chapter 4 contains the contributions of the author of this thesis to the study of another type of n-inner product. Misiak [162] generalizes this concept of a 2-inner product space, in 1989, to an n-inner product space, $n \geq 2$.

In *Section* 4.2, we define the weak n-inner product and the n-iterated 2-inner product and we study its properties. Suppose that $(X, (\cdot,\cdot|\cdot,...,\cdot))$ is a weak n-inner product space over the field of real numbers $\mathbb{R}$. Let $x_2, \ldots, x_n \in X$, $n \geq 2$ be fixed. Denote by Y the $span\{x_2, \ldots, x_n\}$. Then the function $\psi : (X/Y)^2 \to \mathbb{R}$, $\psi(\hat{x}, \hat{y}) := (x, y|x_n, ..., x_2)$, $\hat{x}, \hat{y} \in X/Y$ is well defined and is a semi-inner product on X/Y. Moreover, if $x_2, \ldots, x_n$ are linearly independent, then ψ is an inner product. Let $(X, (\cdot,\cdot|\cdot,...,\cdot))$ be a weak n-inner product space. For any $x, y, x_2, ..., x_n \in X$ we have

$$|(x, y|x_n, ..., x_2)| \leq \sqrt{(x, x|x_n, ..., x_2)}\sqrt{(y, y|x_n, ..., x_2)}.$$

In the case when $x_2, \ldots, x_n$ are linearly independent, equality holds in the above relation if and only if there exist $\mu \in \mathbb{R}_+$ and $u \in Y := span\{x_2, \ldots, x_n\}$ such that $y = \mu x + u$.

In *Section* 4.3, we obtain a representation of the n-iterated 2-inner product, given in Definition 4.2.4 in terms of the standard k-inner products $k \le n$. For this we use Dodgson's identity for determinants, [62], [63]. Historical notes about this identity, in connection with Chió's formula can be found in [1]. To express this identity we adopt the compact notation used by Eves [86]. If $A = (a_{i,j})_{1 \le i,j \le n}$ is a square matrix, denote the determinant of A by $|a_{1,1} \ \ldots a_{n,n}|$ and the sub-determinant involving rows $i_1, \ldots, i_s$ and columns $j_1, \ldots, j_s$ by $|a_{i_1,j_1} \ \ldots a_{i_s,j_s}|$. In [86] - Theorem 3.6.3, the following Dodgson type identity ($n \ge 3$) is proved:

$$\begin{aligned}&|a_{1,1} \ \ldots \ a_{n-2,n-2}| \cdot |a_{1,1} \ \ldots \ a_{n,n}| \\ = &\begin{vmatrix} |a_{1,1} \ \ldots \ a_{n-2,n-2} \ a_{n-1,n-1}| & |a_{1,1} \ \ldots a_{n-2.n-2} \ a_{n-1,n}| \\ |a_{1,1} \ \ldots \ a_{n-2,n-2} \ a_{n,n-1}| & |a_{1,1} \ \ldots a_{n-2,n-2} \ a_{n,n}| \end{vmatrix}.\end{aligned}$$

For us it is more convenient to use the following identity ($n \ge 3$):

$$\begin{aligned}&|a_{2,2} \ \ldots \ a_{n-1,n-1}| \cdot |a_{1,1} \ \ldots \ a_{n,n}| \\ = &\begin{vmatrix} |a_{1,1} \ \ldots \ a_{n-2,n-2} \ a_{n-1,n-1}| & |a_{1,2} \ \ldots a_{n-2,n-1} \ a_{n-1,n}| \\ |a_{2,1} \ \ldots \ a_{n-1,n-2} \ a_{n,n-1}| & |a_{2,2} \ \ldots a_{n-1,n-1} \ a_{n,n}| \end{vmatrix}.\end{aligned}$$

For $n = 3$ one has:

$$a_{2,2} \cdot |a_{1,1} \ a_{2,2} \ a_{3,3}| = \begin{vmatrix} |a_{1,1} \ a_{2,2}| & |a_{1,2} \ a_{2,3}| \\ |a_{2,1} \ a_{3,2}| & |a_{2,2} \ a_{3,3}| \end{vmatrix}.$$

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. For $n \ge 2$, consider the vectors $x, y, x_2, \ldots, x_n \in X$. Then

$$(x, y|x_n, \ldots, x_2)_* = E_n \cdot \langle x, y|x_n, \ldots, x_2 \rangle,$$

where $E_2 = 1$ and

$$E_n = \prod_{k=2}^{n-1} \langle x_k, x_k|x_{k-1}, \ldots, x_2 \rangle^{2^{n-k-1}}, \ (n \ge 3).$$

In *Section* 4.4, we give several applications of the n-iterated 2-inner product and we generalize the Chebyshev functional to the n-Chebyshev functional. In [179], the *Chebyshev functional* is defined by

$$T_z(x, y) = \|z\|^2 \langle x, y \rangle - \langle x, z \rangle \langle y, z \rangle,$$

for all $x, y \in X$, where $z \in X$ is a given nonzero vector.

It is easy to see that if the standard 2-inner product $(\cdot, \cdot|\cdot)$ is defined by the inner product $\langle \cdot, \cdot \rangle$, then we have $T_z(x, y) = (x, y|z)_* = (x, y|z)$.

Therefore, we generalize this Chebyshev functional to the following functional:

$$T_{x_n, \ldots, x_2}(x, y) := (x, y|x_n, \ldots, x_2)_*,$$

which we will call the *n-Chebyshev functional*, so

$$\begin{aligned} T_{x_n,...,x_2}(x,y) &= T_{x_{n-1},...,x_2}(x,y)T_{x_{n-1},...,x_2}(x_n,x_n) \\ &\quad -T_{x_{n-1},...,x_2}(x,x_n)T_{x_{n-1},...,x_2}(x_n,y), \end{aligned}$$

for all $x, y \in X$, where $x_2, ..., x_n \in X$ are given nonzero vectors.

In a particular case, when $n = 3$, we have

$$T_{w,z}(x,y) = (x,y|w,z)_* = (x,y|z)_*(w,w|z_*) - (x,w|z)_*(w,y|z)_*$$

so, we have

$$\begin{aligned} T_{w,z}(x,x) &= (x,x|w,z)_* = (x,x|z)_*(w,w|z)_* - (x,w|z)_*(w,x|z)_* \\ &= \|x|z\|^2\|w|z\|^2 - (x,w|z)^2 \\ &= (\|x\|^2\|w\|^2\|z\|^2 + 2\langle w,z\rangle\langle z,x\rangle\langle x,w\rangle - \|x\|^2\langle w,z\rangle^2 \\ &\quad -\|w\|^2\langle z,x\rangle^2 - \|z\|^2\langle x,w\rangle^2)\|z\|^2. \end{aligned}$$

In **Chapter 5**, we present the contributions of the author of this thesis to inequalities for operators.

In *Section* 5.1, the purpose is to study several inequalities related to the orthogonal projections, which involves the problem of minimization. Among these results we established an inequality which characterizes Bessel's inequality and we will mention Ostrowski's inequality as a consequence of our results.

Let $U \subset X$ be a subspace of a complex Hilbert space and $x \in X$. If $T \in \mathbb{B}(X)$, such that $\|x - Tx\| \leq \|x - y\|$, for every $y \in U$, then we have

$$\|(T - P_U)(x)\| \leq \|T\|\|x - y\|,$$

for every $y \in U$. Let z be a nonzero vector of a complex Hilbert space X and $x \in X$. For $U = \text{span}\{z\}$, we have $P_U x = P_z x$, where P_z is the orthoprojector from X onto $\text{span}\{z\}$. Therefore, using the above inequality, we deduce for an operator $T \in \mathbb{B}(X)$, such that $\|x - Tx\| \leq \|x - z\|$, the following inequality:

$$\|(T - P_z)(x)\| \leq \|T\|\|x - z\|.$$

If $T \in \mathbb{B}(X)$, then the following inequality holds

$$\|T - I\| \geq \|T\| - \omega(T).$$

We present an improvement of the inequality between the numerical radius of an operator and the norm of an operator and we also show other inequalities for a bounded linear operator. We also study some inequalities for invertible positive operators that have applications in operator equations, network theory and in quantum information theory.

In *Section* 5.2, motivated by the results given above, we study a refinement of the Cauchy–Bunyakowski–Schwarz inequality in the framework of Euclidean spaces and several inequalities involving two bounded linear operators on the

Hilbert space H. Among them, we mention Bohr's inequality and Bergström's inequality for operators.

In *Section* 5.3, we present several Cauchy–Bunyakowski–Schwarz type results, for bounded linear operators, by the monotony of a sequence technique. Finally, we prove a refinement of Aczél's inequality for bounded linear operators on the Hilbert space H. For any vectors $x_1, x_2, ..., x_n$ in an Euclidean space H and for arbitrary real numbers $\lambda_1, \lambda_2, ..., \lambda_n$, with $\lambda_i \neq 0$, $i = \overline{1, n}$, we have

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{i=1}^{n} \|x_i\|^2 - \Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\|^2 \geq \max_{i,j \in \{1,...,n\}} \frac{\|\lambda_i x_j - \lambda_j x_i\|^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^{n} \lambda_i^2,$$

for any $n \geq 2$. For any operators $T_1, T_2, ..., T_n$ in $\mathbb{B}(H)$ and for arbitrary real numbers $\lambda_1, \lambda_2, ..., \lambda_n$, with $\lambda_i \neq 0$, $i = \overline{1, n}$, $n \geq 2$, we have

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{i=1}^{n} |T_i|^2 - \Big| \sum_{i=1}^{n} \lambda_i T_i \Big|^2 \geq \frac{|\lambda_i T_j - \lambda_j T_i|^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^{n} \lambda_i^2,$$

for all $i, j \in \{1, ..., n\}$.

We also give some identities for real numbers obtained from some identities for Hermitian operators.

In the last part of this habilitation thesis we have presented the evolution and development plans for career development.

Chapter 6, the last one, presents some future plans regarding my professional and scientific career. It describes the following three aspects:

- the research directions that I intend to follow, namely:
  As short term research projects:
  - the study of new aspects of the connections between various inequalities, namely: Hermite–Hadamard's inequality, Hammer–Bullen's inequality, Young's inequality and Hardy's inequality;
  - the study of the other properties of the function $\Gamma(a, b; c, d)$ given in Section 2.2.
  - the search of the future directions for research related to inequalities in an inner product space. We recall that the norm-angular distance was generalized to the p-angular distance in a normed space in [136], namely:
  $$\alpha_p[x, y] = \Big\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \Big\|,$$
  where the nonzero vectors x and y are in a normed space $X = (X, \|\cdot\|)$ and $p \geq 0$. In fact, we can take $p \in \mathbb{R}$. It is easy to see that $\alpha_0[x, y] = \alpha[x, y]$.
  In [202], Rooin, Habibzadeh and Moslehian defined the notion of skew p-angular distance between nonzero vectors x and y in a normed space $X = (X, \|\cdot\|)$ and $p \in \mathbb{R}$, thus:
  $$\beta_p[x, y] = \Big\| \frac{x}{\|y\|^{1-p}} - \frac{y}{\|x\|^{1-p}} \Big\|.$$

This notion generalizes the concept of skew angular distance between nonzero vectors x and y, $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$, given by Dehghan in [57], since $\beta_0[x,y] = \beta[x,y]$. In [57] we found an estimate of skew angular distance between nonzero vectors x and y.

In [72], Dragomir characterizes this distance, obtaining new bounds for the p-angular distance. A survey on the results for bounds for the angular distance, named Dunkl–Williams type theorems (see [82], [189]), is given by Moslehian *et al.* [171].

Motivated by the above results I am going to prove other results for bounds for the p-angular distance as in [124] and [125]. Another goal for the future research directions is the establishment of new inequalities in an inner product space and in a normed space, similar to the inequalities given by Maligranda [136] and Dehghan [57]. Other estimates which follow from the triangle inequality will be studied in connection with the p-angular distance and the skew p-angular distance.

As long term research project:

– In [52], the Fermi–Dirac–Tsallis entropy was introduced by

$$I_q^{FD}(\mathbf{p}) := \sum_{j=1}^{n} p_j \ln_q \frac{1}{p_j} + \sum_{j=1}^{n} (1-p_j) \ln_q \frac{1}{1-p_j}$$

for $\mathbf{p} \in \Delta_n$ and also the Bose–Einstein–Tsallis entropy was introduced in [100] as

$$I_q^{BE}(\mathbf{p}) := \sum_{j=1}^{n} p_j \ln_q \frac{1}{p_j} - \sum_{j=1}^{n} (1+p_j) \ln_q \frac{1}{1+p_j},$$

where $\ln_q(x) := \dfrac{x^{1-q}-1}{1-q}$ is q-logarithmic function defined for $x > 0$ and $q > 0$ with $q \neq 1$. It is known that $\lim_{q\to 1} R_q(\mathbf{p}) = \lim_{q\to 1} H_q(\mathbf{p}) = H(\mathbf{p})$.

An interesting differential relation of the Rényi entropy is

$$\frac{dR_q(\mathbf{p})}{dq} = -\frac{1}{(1-q)^2} \sum_{j=1}^{n} v_j \log \frac{v_j}{p_j}.$$

It is proportional to the Kullback-Leibler divergence, where $v_j = p_j^q / \sum_{j=1}^{n} p_j^q$. Taking the limit $q \to 1$, we have

$$\lim_{q\to 1} I_q^{FD}(\mathbf{p}) = I_1^{FD}(\mathbf{p}) := -\sum_{j=1}^{n} p_j \log p_j - \sum_{j=1}^{n} (1-p_j) \log(1-p_j)$$

and

$$\lim_{q\to 1} I_q^{BE}(\mathbf{p}) = I_1^{BE}(\mathbf{p}) := -\sum_{j=1}^{n} p_j \log p_j + \sum_{j=1}^{n} (1+p_j) \log(1+p_j),$$

where $I_1^{FD}(\mathbf{p})$ and $I_1^{BE}(\mathbf{p})$ are the Fermi–Dirac entropy and the Bose–Einstein entropy, respectively, see [100]. Motivated by the above results, the future research directions can include the establishment of new bounds for these entropies.

- I would like to attract students to the study of the theory of inequalities by means of the topics that I teach.

- I intend to publish a monography related to classical inequalities and their new versions.

The study of the theory of inequalities is initiated so as to improve some results on classical inequalities.

As a final consideration about the structure of this thesis, I should mention that most results are presented without proofs. The proofs of the above results can be found in my papers [96], [97], [99], [124], [146], [147], [148], [159], [150], [151], [152], [153], [155], [154], [156], [157], [158] and [159].

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