Geometric Methods of Finsler-Based Field Theory

Habilitation Thesis

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Contents

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Rezumat

Lucrarea are ca scop principal dezvoltarea unui aparat geometric unitar şi riguros pentru teoriile câmpurilor fizice bazate pe geometria Finsler. Însă, deoarece o parte din metodele nou introduse sunt utile în teoriile generale de câmp, aplicaţiile prezentate nu se vor limita la geometria Finsler.

Studiul a fost motivat de una din provocările majore ale fizicii moderne: obţinerea unei extinderi a teoriei generale a relativitătii, care să abordeze problemele acesteia la scară foarte mare sau foarte mică - probleme ce au dat naştere noţiunilor de materie/energie întunecată, respectiv, tensiunii cu mecanica cuantică, [204].

Geometria Finsler este cea mai generală geometrie ce admite o noţiune bine definită de lungime de arc, ea incluzând drept caz particular, geometria riemanniană. În fizica gravitaţională, ea apare ca model natural în cel puţin două situaţii: fenomenologia cuantică a gravitaţiei (*relaţiile de dispersie modificate,* [4] [159], [166]), respectiv, în *teoria cinetică a gazelor*, [94], [95] (ce permite descrierea câmpului gravitaţional generat de surse multiple, ce se mişcă cu viteze diferite).

În plus, din punct de vedere pur matematic, geometria Lorentz-Finsler este un domeniu puţin explorat, surprinzător de diferit de geometria Finsler pozitiv definită şi cu aplicaţii uneori spectaculoase, v. Sec. 2.4.

Teza este structurată astfel. Cap. 1 este dedicat metodelor geometrice în calculul variaţional; Cap. 2 discută noţiunea de spaţiu-timp Finsler şi problemele asociate, iar Cap. 3 introduce un cadru geometric general pentru teoriile de câmp finsleriene, împreună cu o aplicaţie: un model concret pentru câmpul gravitaţional.

Limbajul modern pentru calculul variaţional, utilizat în lucrare, este bazat pe fibratele de jeturi asociate varietăţilor fibrate. Câmpurile fizice sunt tratate ca secţiuni, lagrangienii - ca forme diferenţiale, iar variaţiile, ca derivate Lie. Mai mult, noţiunea de *echivalent Lepage* al unui lagrangian permite o descriere geometrică concisă, bazată exclusiv pe operaţii cu forme diferenţiale, a întregului aparat al calculului variaţional. Sec. 1.1. prezintă pe scurt acest formalism.

Sec. 1.2 introduce noţiunea de *completare variaţională canonică,* [202], ce transformă un sistem arbitrar de ecuaţii diferenţiale într-unul variaţional, prin adăugarea unui termen corecţie. Acest termen, construit cu ajutorul aşa-numitului lagrangian Vainberg-Tonti, "corectează", de exemplu, tensorul Ricci al unei varietăţi riemanniene, în tensorul Einstein; o altă aplicaţie prezentată aici este in teoria Gauss-Bonnet a gravitaţiei, [98].

În Sec. 1.3, dedicată tensorilor energie-impuls, demonstrăm că, pe fibrate naturale arbitrare de index 1, orice lagrangian natural conduce la o *lege de echilibru,* [201], ce extinde legea de conservare covariantă a tensorului energie-impuls din cazul teoriilor metrice ale gravitaţiei. Ca aplicaţie, în teoriile metric-afine generale ale gravitaţiei, am obţinut o lege de echilibru simplă, invariantă la transformări de coordonate.

Sec. 1.4. discută o proprietate a echivalenţilor Lepage ai lagrangienilor, numită *proprietatea închiderii;* aceasta asigură că, trecând la formalismul hamiltonian bazat pe forme Lepage, lagrangienii ce conduc la aceleaşi ecuaţii Euler-Lagrange vor conduce şi la aceleaşi ecuaţii Hamilton. Pentru lagrangienii de ordin superior, un echivalent Lepage cu proprietatea închiderii a fost definit pentru prima dată în [198].

Secţiunile 2.1. şi 2.2 prezintă noţiunea de *spaţiu-timp Finsler* introdusă în [97] şi structurile geometrice asociate. O atenţie specială o acordăm noţiunii de *dependenţă omogenă de vectorii tangenţi* la varietatea spaţiu-timp a obiectelor geometrice finsleriene - noţiune esenţială în a asigura existenţa unei lungimi de arc corect definite. Sec. 2.3. face o scurtă comparaţie între spaţiile-timp Finsler şi spaţiile Finsler pozitiv definite, respectiv, varietăţile lorentziene, [76], [200]. Sec. 2.4. discută o aplicaţie a geometriei Lorentz-Finsler în obţinerea de inegalităţi pe $\mathbb{R}^n$, [140].

Sec. 3.1-3.2 introduc un cadru geometric general pentru problemele variaţionale ale căror variabile dinamice au o dependenţă omogenă de direcţie, [97]. *Spaţiile configuraţiilor* construite aici, ce admit obiectele geometrice omogene drept secţiuni şi totodată, permit aplicarea corectă a tehnicilor calculului variaţional, au ca varietate bază fibratul tangent proiectivizat pozitiv (fibratul sferă proiectiv) asociat varietăţii spaţiu-timp. Lagrangienii naturali conduc, în acest caz, la un *tensor de distribuţie a energiei şi impulsului* dependent de direcţie, ce respectă o lege de conservare sub formă integrală.

Un *model finslerian concret* pentru câmpul gravitaţional este construit în Sec. 3.3. Ecuaţiile în cazul vidului, [93], sunt obţinute prin completare variaţională, pornind de la ideea (aparţinând lui Pirani) că, în vid, urma operatorului de deviaţie a geodezicelor trebuie să se anuleze. Apoi, pentru materia descrisă ca un gaz cinetic, deducem ecuaţiile de câmp şi tensorul de distribuţie a energiei şi impusului, [94].

În Sec. 3.4, [96], pornind de la o definiţie axiomatică a *simetriei cosmologice*, determinăm generatorii acesteia în cazul finslerian. Forma generală rezultată pentru metricile Finsler cu simetrie cosmologică este folosită apoi pentru a obţine o clasificare completă, în cazul spaţiilor-timp Berwald.

Teza se bazează pe câteva lucrări ce le-am publicat ca autor sau coautor, dupa sustinerea tezei de doctorat: [76], [93]-[98], [140], [198]-[204]. Rezultate mai vechi, ca: [205]-[208], respectiv, [13]-[19], [22]-[25], [39]-[44],[167], [209]-[218], au fost lăsate la o parte, însă au contribuit la evoluţia mea ştiinţifică.

Cu excepţia Sec. 1.1, rezultatele prezentate în teză sunt, în absenţa altor specificaţii, rezultate originale, la care contribuţia mea a fost una esenţială.

Summary

The main goal of this thesis is to develop a general geometric apparatus allowing for mathematically rigorous Lagrangian field theories based on Finsler geometry. But, as some of the new tools can be used in basically any field theory, I will also explore some of these applications.

This effort is motivated by one of the main quests of modern physics: extending general relativity so as to address the problems arising at either the largest, or the smallest scales - and which gave rise to the so-called dark phenomenology and to tensions with quantum mechanics, [204].

Finsler geometry is the most general geometry admitting a well defined notion of arc length, thus including Riemannian geometry as a subcase. In gravitational physics, it arises as a natural model in at least two situations: *modified dispersion relations* occurring in quantum gravity phenomenology, [4] [159], [166], and the *kinetic description of gases*, [94], [95] (which allows one to describe the gravitational field generated by multiple sources, moving with different velocities).

Yet, even from a purely mathematical point of view, Lorentz-Finsler geometry is a still very little explored, strikingly different realm from its positive definite counterpart, with sometimes beautiful applications to other areas of mathematics, see, e.g., Section 2.4.

The work is structured as follows. Chapter 1 presents a general geometric toolkit for the calculus of variations; Chapter 2 introduces Finsler spacetimes and discusses the arising subtleties and challenges. Finally, Chapter 3 combines the tools in the previous chapters to create a general framework for Finsler-based Lagrangian field theories and introduces, within this framework, a concrete model for the gravitational field generated by a kinetic gas, [93], [94].

In a modern language, the natural stage for the calculus of variations are jet bundles of fibered manifolds. Thus, physical fields are treated as sections, Lagrangians are seen as differential forms and variations, as Lie derivatives. Going a step further and using the notion of *Lepage equivalent* of a Lagrangian, one can describe Euler-Lagrange equations, Noether currents and Hamilton equations in a concise, coordinate-free manner, solely in terms of operations with differential forms. This formalism is briefly reviewed in Section 1.1.

Adopting this standpoint, Section 1.2 introduces the notion of *canonical variational completion,* [202], which is a way of turning an arbitrary system of differential equations into a variational one, by adding a correction term built via the so-called Vainberg-Tonti Lagrangian. When applied to the Ricci tensor of a Riemannian manifold, this method provides the Einstein tensor; another application presented here is in Gauss-Bonnet gravity theory, [98].

Section 1.3. explores energy-momentum tensors in Lagrangian field theories and shows that, on arbitrary natural bundles of index 1, any natural Lagrangian leads to an *energy-momentum*

balance law, [201], which generalizes the energy-momentum conservation law known from metric field theories. The algorithm is then applied to obtain a simple, explicitly covariant energy-momentum balance law in the case of general metric-affine gravity theories.

Section 1.4. discusses the so-called *closure property* of Lepage equivalents of Lagrangians, which ensures that, passing to a the Lepage form-based Hamiltonian formalism, one obtains a unique set of Hamilton equations for all Lagrangians sharing the same dynamics. For general higher order Lagrangians, Lepage equivalents with this property were determined for the first time in my joint paper with former students S. Garoiu and B. Vasian, [198].

The first two sections of Chapter 2 present the notion of *Finsler spacetime* as introduced in [97] and the associated structures. A special attention is paid to the *homogeneous* dependence on tangent vectors to spacetime, of the typical Finslerian geometric objects - which is key to ensuring the existence of a well defined arc length. Section 2.3. makes a brief comparison between Finsler spacetimes and positive definitely Finsler spaces, respectively, Lorentzian spacetimes, [76], [200]. Section 2.4. shows an application of Lorentz-Finsler geometry to inequalities on $\mathbb{R}^n$, [140].

Sections 3.1-3.2 introduce the general framework for variational problems whose dynamical variables depend homogeneously on tangent vectors of spacetime, [97]. The configuration bundles introduced here, which admit these objects as sections and allow one to consistently apply the tools of the calculus of variations, sit over the positively projectivized tangent bundle of spacetime. On such spaces, general covariance of Lagrangians leads to the novel, direction-dependent notion of *energy-momentum distribution tensor*, obeying an *averaged conservation law.*

A concrete model for the gravitational field is then constructed in Section 3.3 as follows. A vacuum action is built, [93] using Pirani's idea that, in vacuum, the trace of the geodesic deviation operator should vanish, together with the variation completion technique; then, assuming that matter is described as a kinetic gas, we deduce the resulting field equation and energy-momentum distribution tensor, [94].

In Section 3.4, [96], we find the generators of Finslerian *cosmological symmetry,* starting from an axiomatic definition. Then, the resulting most general form of cosmologically symmetric Finsler spacetimes is used to obtain a complete classification in the particular case of Berwald spacetimes.

The thesis is based on several papers I have published as an author or a coauthor, after my Ph.D. defense: [76], [93]-[98], [140], [198]-[204]. Older results, such as: [205]-[208], respectively, [13]-[19], [22]-[25], [39]-[44],[167], [209]-[218], have been left aside, though they all contributed to my scientific evolution.

Except for Section 1.1 and unless elsewhere specified, the presented results are original ones, to which my contribution was essential.

Chapter 1

A geometric toolkit for the calculus of variations

1.1 Preliminaries

The language of differential forms allows a concise, coordinate-free formulation of variational calculus on arbitrary manifolds. In this approach, Lagrangians are regarded as differential forms on certain jet bundles, rather than as functions; this allows their variations to be understood as Lie derivatives and, accordingly, Euler-Lagrange expressions and Noether currents to be described solely in terms of operations with differential forms.

This section, which combines parts of our papers [97] and [198], briefly reviews the known results in the literature that are necessary for a further understanding of the text; for a more complete and in-depth exposition, we refer to the monographs by D. Krupka [114] and, respectively, Giachetta, Mangiarotti and Sardanashvili, [79].

1.1.1 Fibered manifolds and their jet prolongations

A *fibered manifold* is a triple (Y, π, X), where X, Y are $\mathcal{C}^\infty$-smooth manifolds with $\dim X = n$, $\dim Y = n + m$ and $\pi : Y \to X$ is a surjective submersion; in the following, we will always assume that the *base manifold* X is connected. The level sets $Y_x = \pi^{-1}(x)$ are called the *fibers* of Y.

Any fibered manifold admits an atlas consisting of *fibered charts*. A local chart (V, ψ), $\psi = (x^A, y^\sigma)$, $(A = 0, ..., n-1,\ \sigma = 1, ..., m)$ on Y is called a fibered chart if there exists a local chart (U, φ), $\varphi = (x^A)$ on X, with $\pi(V) = U$, in which π has the coordinate representation $(\varphi \circ \pi \circ \psi^{-1})\left(x^A, y^\sigma\right) = \left(x^A\right)$. In the following, if the coordinate charts are fixed, we will typically omit φ and ψ and designate by a colon " : " the coordinate representations of mappings between manifolds; e.g., the above relation will be simply written as: $\pi : (x^A, y^\sigma) \mapsto (x^A)$.

Local sections $\gamma : U \to Y$ (with $U \subset X$ open) of a fibered manifold (Y, π, X), which will be called, briefly, *sections*, are smooth maps such that $\pi \circ \gamma = id_X$; in a fibered chart, they will have a representation of the form $\gamma : (x^A) \mapsto (x^A, y^\sigma(x^A))$. The set of sections of (Y, π, X) will be denoted by $\Gamma(Y)$.

Here are some more notations and conventions to be used throughout the thesis:

- $\Omega_k(Y)$ and $\Omega(Y)$, will mean the set of differential k-forms, respectively, the set of all differential forms defined over open subsets $W \subset Y$; the symbol $\mathbf{i}$ will denote interior product.
- By $\mathcal{F}(Y)$, we mean the set of all smooth functions $f : W \to \mathbb{R}$ defined on open subsets $W \subset Y$.
- $\mathcal{X}(Y)$ denotes the $\mathcal{F}(Y)$-module of vector fields on Y.
- Commas $_{,A}$ will designate partial differentiation with respect to x^A.

Also, by "smooth", unless elsewhere specified, we will mean $\mathcal{C}^\infty$-smooth.

Physical interpretations.

1. In *field theory*, the base manifold X is usually (but not always, see Chapter 3) interpreted as spacetime manifold and the manifold Y is called the *configuration space*. Sections $\gamma : U \to Y$ are interpreted as *fields*.

   For instance: in metric theories of gravity, $Y = Met(X)$ is the bundle of symmetric nondegenerate tensors of type (0,2) over X, whereas in classical electromagnetic field theory (whose fundamental variable is the 4-potential 1-form $A \in \Omega_1(X)$), one has $Y = T^*X$ etc..

   In the cases when X is interpreted as spacetime, we will typically re-denote it as M and the local coordinates x^A on M will be labeled by lowercase Latin indices i, j, k etc..

2. *Mechanics* is characterized by $X \subset \mathbb{R}$, that is, $\dim X = 1$; in this case, the (unique) coordinate $x^0 =: t$ is interpreted as time and $y^\sigma =: q^\sigma$, as generalized coordinates; local sections of the configuration space Y are identified with *curves* on Y.

Most examples of fibered manifolds used in physics actually belong to the more restrictive class of *fiber bundles*. By a *fiber bundle* (in a broad sense), we will understand, as in [155], a fibered manifold that is smoothly locally trivial, i.e., in the above, each fibered chart domain V is equal to $\pi^{-1}(U)$ and is diffeomorphic to a Cartesian product $U \times \mathcal{F}$, where $\mathcal{F}$ is a manifold called the typical fiber[1]. Yet, unless elsewhere specified, we will prefer to keep full generality, i.e., work in the wider class of fibered manifolds.

The natural setting for all applications involving partial derivatives of field variables are *jet bundles* of fibered manifolds (Y, π, X).

Two local sections $\gamma, \tilde{\gamma} : U \to Y$ (where $U \subset X$ is open) are said to have a *contact of order* $r \geq 0$ at a point $x_0 \in U$ if $\gamma(x_0) = \tilde{\gamma}(x_0)$ and there exists a fibered chart (V, ψ) around $\pi^{-1}(x_0)$, in which the coordinate representations $\left(x^A\right) \mapsto (x^A, f^\sigma(x^A))$, $\left(x^A\right) \mapsto (x^A, \tilde{f}^\sigma(x^A))$ of γ and $\tilde{\gamma}$ have the property that all the partial derivatives of the functions f^σ and $\tilde{f}^\sigma$, up to order r, agree at the coordinate representation $\left(x_0^A\right)$ of x_0. The notion of contact of order r at x_0 is actually independent of the choice of fibered charts and provides an equivalence relation on the set of sections defined on U. The equivalence class $J^r_{x_0}\gamma$ of $\gamma \in \Gamma(Y)$ at x_0, called the r-jet of γ at x_0, is thus uniquely determined by the values $(x_0^A, f^\sigma(x_0^A), \dfrac{\partial f^\sigma}{\partial x^C}(x_0^A), ..., \dfrac{\partial^r f^\sigma}{\partial x^{C_{i_1}} ... \partial x^{C_{i_r}}}(x_0^A))$, with $C_{i_1} \leq C_{i_2} \leq ... \leq C_{i_r}$.

The jet bundle J^rY is defined as the set of r-jets $J^r_x\gamma$ of local sections $\gamma \in \Gamma(Y)$ of class at least $\mathcal{C}^r$, at points $x \in X$, i.e.,

$$J^rY = \{J^r_x\gamma \mid \gamma \in \Gamma(Y),\ x \in X\}.$$

[1] That is, we will not *a priori* impose any structural group (apart from the automatic one, which is, [155], the group $Diff(\mathcal{F})$ of diffeomorphisms of the typical fiber).

The set J^rY is naturally equipped with a manifold structure; an atlas consisting of fibered charts (V^r, ψ^r) on J^rY, where $V^r = J^rV$ and the coordinate map $\psi^r = (x^A, y^\sigma, y^\sigma{}_C, y^\sigma{}_{C_1C_2}, ..., y^\sigma{}_{C_1C_2...C_r})$ (with $C_{i_1} \leq C_{i_2} \leq ... \leq C_{i_r}$) is induced by the fibered charts (V, ψ) on Y, as follows. If the section γ is represented in coordinates as $\gamma : (x^A) \mapsto (x^A, y^\sigma(x^A))$, then the value of the coordinate function $y^\sigma{}_{C_1C_2...C_k}$ at the point $J^r_x\gamma \in J^rY$ is defined as the partial derivative:

$$y^\sigma{}_{C_1...C_k}(J^r_x\gamma) = \frac{\partial^k y^\sigma}{\partial x^{C_1}...\partial x^{C_k}}(x^A). \tag{1.1}$$

In the following, by charts on J^rY, we will always mean induced fibered charts (V^r, ψ^r) as above.

Any section $\gamma \in \Gamma(Y)$ is naturally prolonged into a section $J^r\gamma \in \Gamma(J^rY)$, by the rule: $J^r\gamma(x) := J^r_x\gamma$. That is, in any fibered chart,

$$J^r\gamma : (x^A) \mapsto \left(x^A, y^\sigma(x^A), \frac{\partial y^\sigma}{\partial x^A}(x^B), ..., \frac{\partial^r y^\sigma}{\partial x^{A_1}...\partial x^{A_r}}(x^B)\right).$$

The jet bundle J^rY is a fibered manifold over all lower order jet bundles J^sY, $0 \leq s < r$ (where $J^0Y := Y$) and over X, with canonical projections:

$$\pi^{r,s} : J^rY \to J^sY, \quad J^r_x\gamma \mapsto J^s_x\gamma, \qquad \pi^r : J^rY \to X, \quad J^r_x\gamma \mapsto x;$$

in fibered coordinates, these are given by

$$\begin{aligned} \pi^{r,s} &: (x^A, y^\sigma, y^\sigma{}_{C_1}, ..., y^\sigma{}_{C_1C_2...C_r}) \mapsto (x^A, y^\sigma, y^\sigma{}_{C_1}, ..., y^\sigma{}_{C_1C_2...C_s}), \\ \pi^r &: (x^A, y^\sigma, y^\sigma{}_{C_1}, ..., y^\sigma{}_{C_1C_2...C_r}) \mapsto (x^A). \end{aligned}$$

1.1.2 Horizontal and contact forms

In the following, we discuss differential forms on jet bundles J^rY. For the simplicity of writing, if there is no risk of confusion, we will sometimes identify forms ρ with their pullbacks $(\pi^{s,r})^*\rho$, $s \geq r$; that is, instead of $(\pi^{s,r})^*\rho = \theta$, we may simply write $\rho = \theta$.

Horizontal forms and horizontalization. A differential form $\rho \in \Omega_k(J^rY)$ is said to be π^r-*horizontal* (*semi-basic* with *respect to* π^r, [45], or simply, *horizontal*) if $\mathbf{i}_\Xi\rho = 0$ whenever $\Xi \in \mathcal{X}(J^rY)$ is π^r-vertical - i.e., whenever $d\pi^r(\Xi) = 0$. In a fibered chart, any π^r-horizontal form is expressed as:

$$\rho = \frac{1}{k!}\rho_{A_1A_2...A_k} dx^{A_1} \wedge dx^{A_2} \wedge ... \wedge dx^{A_k}, \tag{1.2}$$

where $\rho_{A_1A_2...A_k}$ are smooth functions of the local coordinates $x^A, y^\sigma, y^\sigma{}_{C_1}, ..., y^\sigma{}_{C_1C_2...C_r}$ on J^rY. Also, for a more compact writing, it is advantageous to use the following locally defined forms:

$$d^nx := dx^1 \wedge ... \wedge dx^n, \tag{1.3}$$

$$\omega_A := \mathbf{i}_{\partial_A} d^nx = (-1)^{A-1} dx^1 \wedge ... \wedge \widehat{dx^A} \wedge ... \wedge dx^n, \tag{1.4}$$

$$\omega_{A_1...A_k} := \mathbf{i}_{\partial_{A_k}} \mathbf{i}_{\partial_{A_{k-1}}} ... \mathbf{i}_{\partial_{A_1}} d^nx. \tag{1.5}$$

This way, any π^r-horizontal k-form on J^rY $(k < n)$ will have a coordinate expression:

$$\rho = \frac{1}{k!}\tilde{\rho}^{A_1...A_{n-k}} \omega_{A_1...A_{n-k}}, \tag{1.6}$$

where $\tilde{\rho}^{A_1...A_{n-k}}$ are smooth functions of $x^A, y^\sigma, y^\sigma{}_{C_1}, ..., y^\sigma{}_{C_1C_2...C_r}$. The set of all π^r-horizontal k-forms on J^rY will be denoted by $\Omega_{k,X}(J^rY)$.

Similarly, $\pi^{r,s}$-*horizontal* forms, $0 \leq s \leq r$, are locally generated by $dx^A, dy^\sigma, ..., dy^\sigma{}_{C_1...C_s}$.

Any differential form $\rho \in \Omega(J^rY)$ of order r can be transformed into a horizontal one, of order $r+1$. This is achieved via the *horizontalization* operator, which is the unique morphism of exterior algebras $h : \Omega^r(Y) \to \Omega^{r+1}(Y)$ obeying, in any fibered chart:

$$hf = f \circ \pi^{r+1,r}, \qquad hdf = d_A f dx^A, \tag{1.7}$$

for all $f \in \mathcal{F}(J^rY)$; here, $d_A f := \partial_A f + \dfrac{\partial f}{\partial y^\sigma} y^\sigma{}_A + ... \dfrac{\partial f}{\partial y^\sigma{}_{C_1...C_r}} y^\sigma{}_{C_1...C_r A}$ denotes the total derivative (of order $r+1$) with respect to x^A.

For instance, on the natural basis 1-forms, h acts as:

$$hdx^A := dx^A, \;\; hdy^\sigma = y^\sigma{}_A dx^A, ..., hdy^\sigma{}_{C_1...C_k} = y^\sigma{}_{C_1...C_k A} dx^A, \qquad k = 1, ..., r. \tag{1.8}$$

Another useful property is the following. For any $f \in \mathcal{F}(J^rY)$ and any $\gamma \in \Gamma(Y)$:

$$\partial_A(f \circ J^r\gamma) = J^{r+1}\gamma^* d_A f. \tag{1.9}$$

Contact forms. A differential form $\rho \in \Omega(J^rY)$ is a *contact form* if it vanishes whenever pulled back by prolonged sections: $J^r\gamma^*\rho = 0$, $\forall \gamma \in \Gamma(Y)$. For example,

$$\begin{aligned} &\theta^\sigma = dy^\sigma - y^\sigma{}_C dx^C, \quad \theta^\sigma{}_{A_1} = dy^\sigma{}_{A_1} - y^\sigma{}_{A_1 C} dx^C, ..., \\ &\theta^\sigma{}_{A_1A_2...A_{r-1}} = dy^\sigma{}_{A_1A_2...A_{r-1}} - y^\sigma{}_{A_1A_2...A_{r-1}C} dx^C, \end{aligned} \tag{1.10}$$

are contact forms on a given chart domain $V^r \subset J^rY$; they are elements of a local basis $\{dx^A, \theta^\sigma, ..., \theta^\sigma{}_{A_1...A_{r-1}}, dy^\sigma{}_{A_1...A_r}\}$ of $\Omega_1(J^rY)$, called the *contact basis.*

The wedge product between a contact form and any other form $\alpha \in \Omega(J^rY)$ is a contact form.

Raising to the next "floor" $J^{r+1}Y$, any differential form $\rho \in \Omega_k(J^rY)$ can be uniquely split into a horizontal part $h\rho$ and a contact one $p\rho$:

$$\left(\pi^{r+1,r}\right)^* \rho = h\rho + p\rho. \tag{1.11}$$

Intuitively, the horizontal component $h\rho \in \Omega_k(J^{r+1}Y)$ is what will survive of ρ when pulled back to X by prolonged sections $J^r\gamma$, where $\gamma \in \Gamma(Y)$, while $p\rho \in \Omega_k(J^{r+1}Y)$ vanishes:

$$J^{r+1}\gamma^*(h\rho) = J^r\gamma^*\rho, \qquad J^{r+1}\gamma^*(p\rho) = 0. \tag{1.12}$$

In its turn, the contact component $p\rho$ is split as:

$$p\rho = p_1\rho + ... + p_k\rho, \tag{1.13}$$

where the *l-th contact component* p_l ($l \leq k$) has the property that the interior product $\mathbf{i}_{\Xi_1}...\mathbf{i}_{\Xi_l}\rho$ with any l vertical vector fields $\Xi_1, ..., \Xi_l \in \mathcal{X}(Y)$, is horizontal; in the contact basis, an l-contact k-form is written as linear combination of wedge products of k basis elements, each such wedge product containing precisely l of the contact forms $\theta^\sigma, ..., {\theta^\sigma}_{A_1...A_{r-1}}, {\theta^\sigma}_{A_1...A_r}$ and $k-l$ of the forms dx^A.

Exterior differentiation of forms does not decrease the degree of contactness, i.e., if ρ is k-contact, then $d\rho$ is at least k-contact (where $p_0 := h$).

Divergence expressions. In particular, if $\rho \in \Omega_{n-1}(J^rY)$ is an $(n-1)$-form, then $hd\rho$ is obtained by differentiation of $h\rho$ only, as the contact component $p\rho$ will not contribute to it. Denoting, in any fibered chart $h\rho = \rho^i\omega_i \in \Omega_{n,X}(J^{r+1}Y)$, then:

$$hd\rho = (d_i\rho^i)d^nx, \tag{1.14}$$

is given by a divergence expression (of order $r+2$). The latter relation will be extremely useful in discussing both Noether currents and trivial Lagrangians.

Source forms. A $\pi^{r,0}$-horizontal, 1-contact $(n+1)$-form $\eta \in \Omega_{n+1}(J^rY)$ is called a *source form* or a *dynamical form.* In local coordinates, any source form is expressed as:

$$\eta = \eta_\sigma\theta^\sigma \wedge d^nx, \tag{1.15}$$

where $\eta_\sigma = \eta_\sigma(x^A, y^\mu, ...{y^\mu}_{A_1...A_r})$. A prominent example of source forms are Euler-Lagrange forms of Lagrangians, to be discussed in the next subsection.

Behavior under coordinate changes. Denoting by $\left(x^A, y^\sigma\right)$ and $(x^{A'}, y^{\sigma'})$ the coordinates in two overlapping fibered charts on Y, the coordinate transformation rule is always of the form:

$$x^A = x^A(x^{B'}), \quad y^\sigma = y^\sigma(x^{B'}, y^{\mu'}).$$

Here are some direct consequences (see, e.g., Section 2.1 of [114]), to be used in the following:

$$dx^A = \frac{\partial x^A}{\partial x^{B'}}dx^{B'}, \qquad \theta^\sigma = \frac{\partial y^\sigma}{\partial y^{\mu'}}\theta^{\mu'}, \tag{1.16}$$

$$d^nx = \det\left(\frac{\partial x^A}{\partial x^{B'}}\right)d^nx'. \tag{1.17}$$

For the components of a source form $\eta \in \Omega_{n+1}(J^rY)$, we get:

$$\eta_{\mu'} = \frac{\partial y^\sigma}{\partial y^{\mu'}}\det\left(\frac{\partial x^A}{\partial x^{B'}}\right)\eta_\sigma \tag{1.18}$$

1.1.3 Fibered automorphisms and deformations of sections

In variational calculus, deformations, or variations, of sections are given by 1-parameter groups of *fibered automorphisms* of the configuration manifold.

A *fibered morphism* is a diffeomorphism $\Phi : Y \to \tilde{Y}$ between the total spaces of two fibered manifolds (Y, π, X), $(\tilde{Y}, \tilde{\pi}, \tilde{X})$ such that exists a diffeomorphism $\phi : X \to \tilde{X}$, with $\pi \circ \Phi = \phi \circ \pi$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & \tilde{Y} \\ \pi\downarrow & & \downarrow\tilde{\pi}. \\ X & \xrightarrow{\phi} & \tilde{X} \end{array}$$

In this case, Φ is said to *cover* ϕ. In fibered coordinates, any fibered morphism is represented as:

$$\phi \quad : \quad (x^A) \mapsto \tilde{x}^A(x^B) \tag{1.19}$$
$$\Phi \quad : \quad (x^A, y^\sigma) \mapsto (\tilde{x}^A(x^B), \tilde{y}^\sigma(x^B, y^\mu)). \tag{1.20}$$

A fibered morphism with $(Y, \pi, X) = (\tilde{Y}, \tilde{\pi}, \tilde{X})$ is called an *automorphism of the fibered manifold* (Y, π, X), or briefly, an *automorphism of* Y; the group of automorphisms of (Y, π, X) will be denoted by $Aut(Y)$.

In particular, an automorphism $\Phi \in Aut(Y)$ is called *strict,* or *vertical*, if $\phi = id_X$. The set $Aut_s(Y)$ consisting of strict automorphisms is a subgroup of $Aut(Y)$.

Given $\Phi \in Aut(Y)$ as above, any section $\gamma \in \Gamma(Y)$ will be deformed by Φ into another section $\tilde{\gamma} \in \Gamma(Y)$, by the rule:

$$\tilde{\gamma} := \Phi \circ \gamma \circ \phi^{-1}. \tag{1.21}$$

On a diagram (where, we have specified for simplicity, X as the domain of definition of γ; of course, γ may in principle be defined on an open subset $U \subset X$ only), this is:

$$\begin{tikzcd} Y \arrow[r, "\Phi"] & Y \\ X \arrow[u, "\gamma"] & X \arrow[l, "\phi^{-1}"'] \arrow[u, "\tilde{\gamma}"'] \end{tikzcd}$$

Automorphisms $\Phi \in Aut(Y)$ are prolonged into automorphisms $J^r\Phi$ of J^rY by the rule:

$$J^r\Phi(J^r_x\gamma) := J^r_{\phi(x)}(\Phi \circ \gamma \circ \phi^{-1}). \tag{1.22}$$

A differential form $\rho \in \Omega(J^rY)$ is called Φ-*invariant*, if it is invariant under the jet prolongation $J^r\Phi$, i.e.,

$$J^r\Phi^*\rho = \rho. \tag{1.23}$$

Passing to the infinitesimal level, any generator $\Xi \in \mathcal{X}(Y)$ of a 1-parameter group $\{\Phi_\varepsilon\}$ of automorphisms of Y is a π-*projectable* vector field, i.e., the pushforward $\pi_*\Xi$ is a well defined vector field on X. In any fibered chart, projectable vector fields are represented as:

$$\Xi = \xi^A(x^B)\partial_A + \Xi^\sigma(x^B, y^\mu)\partial_\sigma, \tag{1.24}$$

where ∂_σ is a shorthand notation for $\partial/\partial y^\sigma$. In particular, 1-parameter groups of strict automorphisms are generated by π-*vertical* vector fields, i.e., $d\pi(\Xi) = 0$; in coordinates:

$$\Xi = \Xi^\sigma(x^B, y^\mu)\partial_\sigma. \tag{1.25}$$

The generator of the 1-parameter group $\{J^r\Phi_\varepsilon\}$ is called the *r-th prolongation of the vector field* Ξ and denoted by $J^r\Xi$. For instance, the first jet prolongation is expressed in the natural local basis of $\mathcal{X}(J^1Y)$ as:

$$J^1\Xi = \xi^A\partial_A + \Xi^\sigma\partial_\sigma + \Xi^\sigma{}_A\frac{\partial}{\partial y^\sigma{}_A}, \qquad \Xi^\sigma{}_A = d_A\Xi^\sigma - y^\sigma{}_B\xi^B{}_{,A}, \tag{1.26}$$

or, equivalently, in terms of the first order total derivative $d_A^{(1)} := \partial_A + y^\sigma{}_A \dfrac{\partial}{\partial y^\sigma}$:

$$J^1\Xi = \xi^A d_A^{(1)} + \tilde{\Xi}^\sigma \partial_\sigma + \Xi^\sigma{}_A \frac{\partial}{\partial y^\sigma{}_A}, \qquad \tilde{\Xi}^\sigma = \Xi^\sigma - y^\sigma{}_A \xi^A. \tag{1.27}$$

The role of the local components $\tilde{\Xi}^\sigma$ is seen as follows. The 1-parameter group $\{\Phi_\varepsilon\}$ generated by Ξ deforms any section $\gamma \in \Gamma(Y)$ into a section $\gamma_\varepsilon \in \Gamma(Y)$, by the rule (1.28); denoting the coordinate representations of γ and, respectively, of γ_ε as: $\gamma : (x^A) \mapsto (x^A, y^\sigma(x^A))$, $\gamma_\varepsilon : (x^A) \mapsto (x^A, \tilde{y}^\sigma(x^A))$, the first order approximation in ε of $\tilde{y}^\sigma(x^A)$ is then given by:

$$\tilde{y}^\sigma(x^A) = y^\sigma(x^A) + \varepsilon(\tilde{\Xi}^\sigma \circ J^1\gamma)(x^A) + \mathcal{O}(\varepsilon^2). \tag{1.28}$$

The functions $\tilde{\Xi}^\sigma \circ J^1\gamma$ are typically denoted in the literature (though, in a bit imprecise way), by δy^σ and called the *variations* of the field components $y^\sigma \circ \gamma$.

1.1.4 Lagrangians, Lepage equivalents and first variation formula

Lagrangians, action and first variation formula.

Let, again, (Y, π, X) denote a fibered manifold.

Definition 1 *[114]: A Lagrangian of order r on Y is a π^r-horizontal form $\lambda \in \Omega_{n,X}(J^rY)$ of top degree $n = \dim X$.*

In fibered coordinates, Lagrangians are described as:

$$\lambda = \mathcal{L}d^n x, \qquad \mathcal{L} = \mathcal{L}(x^A, y^\sigma, ..., y^\sigma_{A_1...A_r}). \tag{1.29}$$

The *action* attached to the Lagrangian (1.29) and to a *piece* $D \subset X$ is the function:

$$S_D : \Gamma(Y) \to \mathbb{R}, \qquad S_D(\gamma) = \int_D J^r\gamma^*\lambda; \tag{1.30}$$

by a *piece* $D \subset X$, we understand, [114], a compact n-dimensional submanifold with boundary of X. In coordinates, (1.30) reduces to the familiar expression

$$S_D(\gamma) = \int_D \mathcal{L}\left(x^A, y^\sigma(x^B), ..., y^\sigma{}_{,A_1...A_r}(x^B)\right) d^n x,$$

where commas denote partial differentiation.

Consider an arbitrary 1-parameter group $\{\Phi_\varepsilon\} \subset Aut(Y)$, with generator $\Xi \in \mathcal{X}(Y)$ and denote by $\gamma_\varepsilon = \Phi_\varepsilon \circ \gamma \circ \phi_\varepsilon^{-1}$ the corresponding deformations of sections $\gamma \in \Gamma(Y)$. The *variation of the action* $S_D(\gamma)$ under $\{\Phi_\varepsilon\}$ is defined as:

$$\delta_\Xi S_D(\gamma) := \left(\frac{d}{d\varepsilon} S_{\phi_\varepsilon(D)}(\gamma_\varepsilon)\right)|_{\varepsilon=0}. \tag{1.31}$$

A brief computation ([114], Ch. 4) leads to

Lemma 2 ***[114], [79]:*** *The variation $\delta_\Xi S_D(\gamma)$ is expressed as the Lie derivative:*

$$\delta_\Xi S_D(\gamma) = \int_D J^r\gamma^* \mathfrak{L}_{J^r\Xi}\lambda. \tag{1.32}$$

A section $\gamma \in \Gamma(Y)$ is called a *critical section* for S, if for any piece $D \subset X$ and for any $\Xi \in \mathcal{X}(Y)$ such that *supp*$(\Xi \circ \gamma) \subset D$, there holds: $\delta S_D(\gamma) = 0$.

Using the above Lemma, one finds:

Theorem 3 ***(First variation formula, [114], [79]):*** *For any Lagrangian $\lambda \in \Omega_{n,X}(J^rY)$, there exists a unique source form $\mathcal{E}(\lambda) \in \Omega_{n+1}(J^{s+1}Y)$ of order $s+1 \le 2r$ and an $(n-1)$-form $\mathcal{J}^\Xi \in \Omega_{n-1}(J^sY)$ such that, for any $\Xi \in \mathcal{X}(Y)$:*

$$J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda) = J^{s+1}\gamma^* \mathbf{i}_{J^{s+1}\Xi}\mathcal{E}(\lambda) - J^s\gamma^* d\mathcal{J}^\Xi. \tag{1.33}$$

The interpretation of these two forms is the following:

- The source form[2] $\mathcal{E}(\lambda) \in \Omega_{n+1}(J^{s+1}Y)$ is called the *Euler-Lagrange form* of λ; in coordinates, if $\lambda = \mathcal{L}d^nx$, then:
$$\mathcal{E}(\lambda) = \mathcal{E}_\sigma \theta^\sigma \wedge d^nx,$$
with:
$$\mathcal{E}_\sigma = \frac{\delta\mathcal{L}}{\delta y^\sigma} = \frac{\partial\mathcal{L}}{\partial y^\sigma} - d_A\frac{\partial\mathcal{L}}{\partial y^\sigma{}_A} + ... + (-1)^r d_{A_1}...d_{A_r}\frac{\partial\mathcal{L}}{\partial y^\sigma{}_{A_1...A_r}}. \tag{1.34}$$
The section $\gamma \in \Gamma(Y)$ is critical for λ if and only if $E_\sigma \circ J^{s+1}\gamma = 0$.

- The $(n-1)$-form $\mathcal{J}^\Xi$ is called the *Noether current* associated to λ and to the vector field Ξ. If Ξ is a symmetry generator for λ, i.e., if $\mathfrak{L}_{J^r\Xi}\lambda = 0$, then, *Noether's first theorem* states that the Noether current is conserved along critical sections:
$$J^s\gamma^* d\mathcal{J}^\Xi \approx 0, \tag{1.35}$$
where $\approx$ denotes equality *on-shell*, i.e., along critical sections γ.

In integral form, the first variation formula reads:

$$\int_D J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda) = \int_D J^{s+1}\gamma^* \mathbf{i}_{J^{s+1}\Xi}\mathcal{E}(\lambda) - \int_{\partial D} J^s\gamma^* \mathcal{J}^\Xi. \tag{1.36}$$

Remark.

1. The fact that $\mathcal{E}(\lambda) = \mathcal{E}_\sigma\theta^\sigma \wedge d^nx$ is a source form implies that locally, only the zeroth order components ξ^i and $\Xi^\sigma$of $J^r\Xi$ will appear in the expression $\mathbf{i}_{J^s\Xi}\mathcal{E}(\lambda)$:
$$\mathbf{i}_{J^s\Xi}\mathcal{E}(\lambda) = (\tilde{\Xi}^\sigma\mathcal{E}_\sigma)d^nx, \qquad \tilde{\Xi}^\sigma = \Xi^\sigma - y^\sigma{}_C\xi^C. \tag{1.37}$$

[2] The notation $s+1$ for the order of $\mathcal{E}(\lambda)$ will be explained below, in the paragraph discussing Lepage forms.

2. In order to identify the Euler-Lagrange form, it is sufficient to use π-*vertical* variation vector fields $\Xi \in \mathcal{X}(Y)$. Yet, general variation vector fields Ξ are needed in discussing general covariance and its consequence, energy-momentum conservation.

Here is one more property which will be very useful in the following.

Proposition 4 ***(Symmetries of the Euler-Lagrange form,*** *[114], p. 118****):*** *For any automorphism $\Phi \in Aut(Y)$ and any Lagrangian $\lambda \in \Omega_{n,X}(J^rY)$,*

$$J^{s+1}\Phi^*\mathcal{E}(\lambda) = \mathcal{E}(J^r\Phi^*\lambda); \tag{1.38}$$

in particular, if λ is Φ-invariant, then so is its Euler-Lagrange form $\mathcal{E}(\lambda)$.

Identification of $E(\lambda)$ and J^Ξ. Lepage equivalents.

Identification of the Euler-Lagrange form $\mathcal{E}(\lambda)$ and of Noether currents $\mathcal{J}^\Xi$ can be of course, done via integration by parts, which is the straightforward, elementary method - but requires the explicit use of coordinates. In the following, we will briefly present an alternative method, which allows one to write Euler-Lagrange forms, Noether currents (and further on, Hamiltonians) in a coordinate-free manner, solely in terms of operations with differential forms. This method is based on the notion of *Lepage equivalent of a Lagrangian* defined by D. Krupka, [115], [114], which is a higher order, field-theoretical generalization of the notion of Poincaré-Cartan form from mechanics.

Definition 5 *A* ***Lepage equivalent*** *of a Lagrangian $\lambda \in \Omega_{n,X}(J^rY)$ is an n-form ρ_λ on some jet prolongation J^sY, with the following properties:*
(i) $h\rho_\lambda = \lambda$ (where the equality should be understood up to the corresponding jet projections);
(ii) $p_1 d\rho_\lambda$ is a source form.

The first property above implies: $J^s\gamma^*\rho_\lambda = J^r\gamma^*\lambda$, for any $\gamma \in \Gamma(Y)$, which means that we can substitute ρ_λ for λ into the action $S_D(\gamma)$. Then, from Cartan's formula $\mathfrak{L}_{J^s\Xi}\rho_\lambda = \mathbf{i}_{J^s\Xi}d\rho_\lambda + d(\mathbf{i}_{J^s\Xi}\rho_\lambda)$ one finds:

$$J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda) = J^s\gamma^*(\mathbf{i}_{J^s\Xi}d\rho_\lambda) + J^s\gamma^*d(\mathbf{i}_{J^s\Xi}\rho_\lambda). \tag{1.39}$$

The only component which produces a nonzero result in the term $J^s\gamma^*(\mathbf{i}_{J^s\Xi}d\rho_\lambda)$ is the horizontal one $h\mathbf{i}_{J^s\Xi}d\rho_\lambda$. But, since the degree of ρ_λ is $n = \dim X$, we have $hd\rho_\lambda = 0$, i.e., $d\rho_\lambda \in \Omega_{n+1}(J^sY)$ is at least 1-contact; that is, $h\mathbf{i}_{J^s\Xi}d\rho_\lambda = \mathbf{i}_{J^{s+1}\Xi}(p_1 d\rho_\lambda)$, which gives:

$$J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda) = J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}(p_1 d\rho_\lambda) + J^s\gamma^*d(\mathbf{i}_{J^s\Xi}\rho_\lambda). \tag{1.40}$$

Setting:

$$p_1 d\rho_\lambda =: \mathcal{E}(\lambda) \in \Omega_{n+1}(J^{s+1}Y), \tag{1.41}$$

a direct computation using *(ii)*, see [114], shows that, for any ρ_λ, $\mathcal{E}(\lambda)$ is given in coordinates by the Euler-Lagrange expressions (1.34). Thus, a local section $\gamma \in \Gamma(Y)$ is critical for the Lagrangian λ if and only if, for any π-vertical vector field $\Xi \in \mathcal{X}(Y)$, there holds:

$$J^{s+1}\gamma^*(\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}(\lambda)) = 0, \tag{1.42}$$

which in coordinates, becomes equivalent to the Euler-Lagrange equations $\mathcal{E}_\sigma(\lambda) \circ J^{s+1}\gamma = 0$.

The remaining (boundary) term is, up to a sign, the Noether current:

$$\mathbf{i}_{J^s\Xi}\rho_\lambda = -\mathcal{J}^{\Xi}. \tag{1.43}$$

Example: The principal (Poincaré-Cartan) Lepage equivalent. Denote by (V, ψ) a fibered coordinate chart on Y and by (V^r, ψ^r), $V^r = J^rV$, the induced fibered chart on J^rY. Then, every Lagrangian defined on V^r admits Lepage equivalents. The most frequently used one, called the *principal Lepage equivalent,* is a 1-contact form of order $\leq 2r-1$ and it is given, in the fibered coordinate chart (V^{2r-1}, ψ^{2r-1}), by:

$$\begin{aligned} \Theta_\lambda &= \mathcal{L}d^nx + \Big(\sum_{k=0}^{r-1} f_\sigma^{AB_1...B_k}\theta^\sigma_{B_1...B_k}\Big)\wedge \mathbf{i}_{\partial_A}d^nx, \\ f^{B_1...B_{r+1}} = 0, &\qquad f_\sigma^{B_1...B_k} = \frac{\partial\mathcal{L}}{\partial y^\sigma_{B_1...B_k}} - d_A f_\sigma^{AB_1...B_k}. \end{aligned} \tag{1.44}$$

Generally, the principal Lepage equivalent Θ_λ is defined only *locally*; yet, for first and second order Lagrangians, it is globally defined on $J^{2r-1}Y$ whenever λ itself is globally defined on J^rY, [114], [170].

Here are some particular cases:

- In the case of *first order mechanics* $(n = 1, r = 1)$, this reduces to the famous *Poincaré-Cartan form*:

$$\Theta_\lambda = \mathcal{L}dt + \frac{\partial\mathcal{L}}{\partial\dot{q}^\sigma}\theta^\sigma. \tag{1.45}$$

- For second order Lagrangians $\lambda = \mathcal{L}(x^i, y^\sigma, y^\sigma{}_i, y^\sigma{}_{ij})d^nx$, (1.44) gives:

$$\Theta_\lambda = \mathcal{L}d^nx + B^A_\sigma\theta^\sigma\wedge\omega_A + B^{AC}_\sigma\theta^\sigma{}_C\wedge\omega_A, \tag{1.46}$$

  where:

$$B^A_\sigma = \frac{\partial\mathcal{L}}{\partial y^\sigma{}_A} - d_C\Big(\frac{\partial\mathcal{L}}{\partial y^\sigma{}_{AC}}\Big), \qquad B^{AC}_\sigma = \frac{\partial\mathcal{L}}{\partial y^\sigma{}_{AC}}. \tag{1.47}$$

Proposition 6 *, [198]: If $\lambda = \mathcal{L}d^nx \in \Omega_{n,X}(V^r)$ is affine in the highest order variables $y^\sigma{}_{A_1...A_r}$, then the order of Θ_λ is actually, at most $2r-2$.*

Proof. The statement follows immediately by inspecting the highest order term $d_{C_1}...d_{C_l}\dfrac{\partial\mathcal{L}}{\partial y^\sigma{}_{B_1...B_kC_1...C_lA}}$ of Θ_λ. ■

Local decomposition of Lepage equivalents. Any Lepage equivalent ρ_λ of a Lagrangian λ can be locally decomposed, [114], as:

$$\rho_\lambda = \Theta_\lambda + d\nu + \mu, \tag{1.48}$$

where ν is at least 1-contact and μ is at least 2-contact. In particular, any *1-contact* Lepage equivalent of λ can be expressed, [170], up to the corresponding jet projections, as:

$$\rho_\lambda = \Theta_\lambda + p_1 d\nu. \tag{1.49}$$

Relation (1.48) makes it clear that the Euler-Lagrange form $\mathcal{E}(\lambda) = p_1 d\rho_\lambda$ is unique, whereas Noether currents (1.43) will depend on the choice of ρ_λ.

1.1.5 Natural bundles and natural Lagrangians

This subsection is a very brief introduction to the topic of natural (generally covariant) Lagrangians and sets the stage for discussing, later, in Sections 1.3 and 3.2, a core concept in field theory: the energy-momentum tensor. As, for energy-momentum tensors, the relevant base manifold X is interpreted as *spacetime*, we will denote it throughout the subsection as M.

Let $\mathcal{M}_n$ denote the category of smooth n-dimensional manifolds, with smooth embeddings[3] as morphisms and $\mathcal{FB}$, the category of smooth fiber bundles, whose morphisms are smooth fibered morphisms.

A *natural bundle functor over n-manifolds* is, [155], a functor $\mathfrak{F} : \mathcal{M}_n \to \mathcal{FB}$, such that:

1. For each $M \in Ob(\mathcal{M}_n)$, $\mathfrak{F}(M)$ is a fiber bundle over M;
2. For each embedding $\alpha_0 : M \to M' \in Morf(\mathcal{M}_n)$, the fibered manifold morphism $\mathfrak{F}(\alpha_0)$: $\mathfrak{F}(M) \to \mathfrak{F}(M')$ covers α_0.

Natural lifts of diffeomorphisms. Fix $M \in Ob(\mathcal{M}_n)$ and denote $Y := \mathfrak{F}(M)$. Then, any diffeomorphism ϕ of M admits a *canonical (or natural) lift* $\Phi := \mathfrak{F}(\phi)$ to Y. These natural lifts encode the transformations of fields - more precisely, their local expressions are similar to *transition functions* on the fibers of Y, [79], [70].

For example, if $Y = T^p_q(M)$ is the bundle of tensors of type (p, q) over M, diffeomorphisms $\phi \in Diff(M)$ are canonically lifted into automorphisms Φ of $T^p_q(M)$ by pushforward/pullback; in natural fibered coordinates $(x^i, y^{i_1...i_p}_{j_1...j_q})$ on $T^p_q(M)$ (obtained by decomposing elements of $T^p_q M$ with respect to the natural local basis $dx^{j_1} \otimes ... \otimes dx^{j_q} \otimes \partial_{i_1} \otimes .... \otimes \partial_{i_p}$), we get:

$$\Phi : \quad \tilde{x}^i = \tilde{x}^i(x^j), \qquad \tilde{y}^{i_1...i_p}_{j_1...j_q} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} ... \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} ... \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}} y^{k_1...k_p}_{l_1...l_q}; \tag{1.50}$$

Passing to infinitesimal generators, any vector field $\xi \in \mathcal{X}(M)$ admits a canonical lift $\Xi := l(\xi) \in \mathcal{X}(Y)$, expressed in a fibered chart as:

$$\Xi = \xi^i(x^j) \frac{\partial}{\partial x^i} + \Xi^\sigma(x^j, y^\mu) \frac{\partial}{\partial y^\sigma}, \tag{1.51}$$

where the components Ξ^σ can always be expressed in terms of the components ξ^i of ξ and a finite number $k \in \mathbb{N}$ of partial derivatives thereof. The number k (which is assumed to be minimal with this property) is called the *index,* [84]*,* or the *order,* [155], of the lifting.

Liftings of index 1. Tensor lifting. For $k = 1$, the components of the lifted vector field (1.51) can be written in any fibered chart, as:

$$\Xi^\sigma = C^\sigma{}_i \xi^i + C^{\sigma j}_{\ \ i} \xi^i_{,j},$$

[3] A smooth embedding between two manifolds $M, N \in \mathcal{M}_n$ is, [128], p. 85, a smooth immersion $f : M \to N$ which is also a topological embedding, i.e., a homeomorphism onto its image $f(M) \subset N$ (where $f(M)$ is equipped with the subspace topology).

for some functions C_i^σ, $C_i^{\sigma j}$ of (x^k, y^μ) only. A direct calculation shows that, with respect to fibered coordinate changes $x^i = x^i(x^{i'})$, $y^\sigma = y^\sigma(x^{i'}, y^{\sigma'})$, the top degree coefficients $C_i^{\sigma j} = C_i^{\sigma j}(x^k, y^\mu)$ transform by the rule:

$$C_{i'}^{\sigma' j'} = \frac{\partial y^{\sigma'}}{\partial y^\sigma}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^i}{\partial x^{i'}}C_i^{\sigma j}. \tag{1.52}$$

The most common example of natural lifting of index 1 is obtained for the bundle of tensors $T_q^p M$. In this case, the lifting of vector fields

$$\xi = \xi^i \partial_i \in \mathcal{X}(M) \quad \mapsto \quad \Xi = \xi^i \partial_i + \Xi^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial y^{i_1 \dots i_p}_{j_1 \dots j_q}} \in \mathcal{X}(T_q^p M)$$

corresponding to (1.50) is given, [80], by:

$$\Xi^{i_1 \dots i_p}_{j_1 \dots j_q} = \xi^{i_1}_{,l} y^{l i_2 \dots i_p}_{j_1 \dots j_q} + \dots + \xi^{i_p}_{,l} y^{i_1 \dots i_{p-1} l}_{j_1 \dots j_q} - \xi^l_{,j_1} y^{i_1 \dots i_p}_{l j_2 \dots j_q} - \dots - \xi^l_{,j_q} y^{i_1 \dots i_p}_{j_1 \dots j_{q-1} l}. \tag{1.53}$$

Natural (generally covariant) Lagrangians. A globally defined Lagrangian λ on $J^r\mathfrak{F}(M)$ is called *natural*, or *generally covariant*, [70], if it is invariant under canonical lifts of arbitrary diffeomorphisms of spacetime, i.e.,

$$J^r\mathfrak{F}(\phi)^*\lambda = \lambda, \quad \forall \phi \in Diff(M).$$

Using the formal similarity between lifts $\mathfrak{F}(\phi)$ of (*a priori*, active) diffeomorphisms $\phi \in Diff(M)$ and fibered *coordinate transformations* on $\mathfrak{F}(M)$ induced by ϕ regarded as a coordinate transformation on M, naturality amounts to the fact that λ must be invariant to any such coordinate changes on $\mathfrak{F}(M)$ - for *any* possible base manifold $M \in Ob(\mathcal{M}_n)$.

In coordinates, this is translated as: in the writing $\lambda = \mathcal{L}d^n x$, $\mathcal{L}$ transforms as a density of weight 1 or, equivalently, in the writing $\lambda = \mathbb{L}dV$ in terms of an invariant volume form dV, $\mathbb{L}$ is invariant under any coordinate transformations on $Y = \mathfrak{F}(M)$ induced by coordinate transformations on M (such a function $\mathbb{L}$ is called an *invariant scalar,* or a *differential invariant).*

In terms of infinitesimal generators, λ is generally covariant if and only if:

$$\mathfrak{L}_{J^r\Xi}\lambda = 0, \tag{1.54}$$

for canonical lifts $\Xi = l(\xi)$ of all vector fields $\xi \in \mathcal{X}(M)$.

1.2 Variational completion of differential equations

This section presents the concept of canonical *variational completion* of a system of differential equations, which was first introduced in my joint paper with D. Krupka, [202]. An application to the so-called 4-dimensional Einstein-Gauss-Bonnet theory of gravity, presented in the Examples section, is a shortened version of my paper with M. Hohmann and C. Pfeifer, [98].

1.2.1 Introduction

The *inverse problem of the calculus of variations* consists in deciding whether a given system of differential equations is *variational*, i.e., if it arises, locally or globally, as the Euler-Lagrange system associated to a Lagrangian - and, in the negative case, transforming it into a variational one.

Typically, local variationality of a given system of equations is established by checking the so-called *Helmholtz conditions,* which are a PDE system to be satisfied by the coefficients of the original system; in the case the system is not variational, the most well-known method of transforming it into a variational one are the so-called *variational multipliers*, see, e.g., [35].

Another possibility of transforming an arbitrary PDE or ODE system into a variational one, is to simply add a correction term, [202]. This idea was motivated by the following example. Historically, the first variant of gravitational field equations proposed by Einstein was:

$$R_{ij} = 8\pi\kappa T_{ij}, \tag{1.55}$$

where: R_{ij} is the Ricci tensor of a 4-dimensional Lorentzian manifold (M, g), T_{ij} is the energy-momentum tensor and κ is a constant, [126]. This variant correctly predicted some physical facts, but was inconsistent with local energy-momentum conservation. Therefore, by a reasoning based on contracted Bianchi identities, Einstein added the "correction term" $-\frac{1}{2}Rg_{ij}$, which led to the nowadays famous:

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi\kappa T_{ij}. \tag{1.56}$$

On the other hand, the variational deduction of (1.56), due to Hilbert, relies on a heuristic argument: simplicity. Hilbert chose to construct the action for the left hand side using the simplest invariant scalar which can be constructed from the metric tensor and its derivatives alone, which is the scalar curvature R; the Euler-Lagrange expressions resulting from this simplest scalar coincide with the left hand side of (1.56).

Yet, knowing that contracted Bianchi identities can be understood, [126], as a direct result of the existence of a generally covariant *Lagrangian* for the left hand side of (1.56), the natural question that arises is: Does there exist a systematic algorithm that would determine the correction term $-\frac{1}{2}Rg_{ij}$ on variationality grounds? If so, such an algorithm should be able to correct in a meaningful way, basically any intuitively found system of differential equations, into a variational one.

The answer to the above question is affirmative. Any ordinary or partial differential system can be expressed as the vanishing of some source form ε along sections of an appropriate jet bundle. Further, to this source form and to each vertically star-shaped local chart domain on which ε is defined, one can naturally attach a Lagrangian λ_ε, called the *Vainberg-Tonti Lagrangian* of ε, [188], [114]; this Lagrangian has the property that the difference

$$\tau := \mathcal{E}(\lambda_\varepsilon) - \varepsilon \tag{1.57}$$

between its Euler-Lagrange form $\mathcal{E}(\lambda_\varepsilon)$ and ε gives a measure of the non-variationality of ε; to be more precise, the form τ is expressed in terms of the coefficients of the so-called *Helmholtz form of* ε - whose vanishing is equivalent to the Helmholtz conditions. In particular, if ε is locally variational, then $\varepsilon = \mathcal{E}(\lambda_\varepsilon)$.

We will call the correction term τ, the *canonical variational completion* of ε.

Apart from the above example, this method has already been proven to have some interesting applications:

1) *Energy-momentum tensors*, [202], [199]. Knowing a term of an energy-momentum tensor, one can recover both the corresponding Lagrangian and the full expression of the respective energy-momentum tensor, by canonical variational completion. In particular, symmetrization of canonical (Noether) energy-momentum tensors in the special relativistic limit can be understood this way.

2) *Extended gravity theories.* Applications to Einstein-Gauss-Bonnet gravity, [98] and Finsler gravity, [93], will be discussed in the next sections.

3) In *classical mechanics*, equations of *damped small oscillations* are known to be non-variational; in the case of linear damping, the correction term $\mathcal{E}(\lambda_\varepsilon) - \varepsilon$ is, up to a sign, the friction force, [202].

1.2.2 Source forms and the inverse problem of the calculus of variations

Local variationality. The Helmholtz conditions in field theory

Let, in the following, (Y, π, X) denote a fibered manifold with $\dim X = n$. We recall that a *source form* (or *dynamical form*) of order r is a $\pi^{r,0}$-horizontal, 1-contact $(n+1)$-form ε on J^rY; in the local contact basis (1.10) of $\Omega(J^rY)$, this gives:

$$\varepsilon = \varepsilon_\sigma \theta^\sigma \wedge d^n x, \qquad \varepsilon_\sigma = \varepsilon_\sigma(x^A, y^\mu, y^\mu{}_A, ..., y^\mu{}_{B_1...B_r}). \tag{1.58}$$

The set of source forms of order at most r over Y is closed under addition and under multiplication with functions $f \in \mathcal{F}(J^rY)$.

The most notable example of a source form is the *Euler-Lagrange form* $\mathcal{E}(\lambda)$ of a Lagrangian λ.

A source form $\varepsilon \in \Omega_{n+1}(J^rY)$ is called:

a) *locally variational* if corresponding to any fibered chart (V, ψ) of Y, there exists a Lagrangian $\lambda_V \in \Omega_{n,X}(J^sV)$ (of some order s) such that, on the respective domain, $\varepsilon = \mathcal{E}(\lambda_V)$;

b) *globally variational* if there exists a Lagrangian λ over the whole manifold Y such that $\varepsilon = \mathcal{E}(\lambda)$.

Local variationality of a source form $\varepsilon = \varepsilon_\sigma \theta^\sigma \wedge d^n x$ of order r is equivalent to a generalization of classical Helmholtz conditions, [114], [199]:

$$H_{\sigma\nu}{}^{B_1...B_k}(\varepsilon) = 0, \quad k = 0, ..., r, \tag{1.59}$$

where the local functions:

$$H_{\sigma\nu}{}^{B_1...B_k}(\varepsilon) = \frac{\partial \varepsilon_\sigma}{\partial y^\nu{}_{B_1...B_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma{}_{B_1...B_k}} - \tag{1.60}$$
$$- \sum_{l=k+1}^{r} (-1)^l \tbinom{l}{k} d_{A_{k+1}} d_{A_{k+2}} ... d_{A_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma{}_{B_1...B_k A_{k+1}...A_l}}$$

are the local coefficients of an $(n+2)$-form $H(\varepsilon) = \sum_{k=0}^{r} H_{\sigma\nu}{}^{B_1...B_k}(\varepsilon) \omega^\nu{}_{B_1...B_k} \wedge \omega^\sigma \wedge dx$ on $J^{2r}Y$, called the *Helmholtz form* of ε.

In the following, by "variationality", we will always mean local variationality.

The Vainberg-Tonti Lagrangian.

Given a source form ε defined on some fibered chart domain $V^r \subset J^rY$, one can attach to ε and to the respective chart a Lagrangian called the Vainberg-Tonti Lagrangian, with the following property: if the source form ε is locally variational, then the Vainberg-Tonti Lagrangian is a Lagrangian for ε. The results in this preliminary paragraph can be found in more detail in Sections 4.9 and 2.7 of the book by Krupka [114].

Consider an arbitrary source form $\varepsilon = \varepsilon_\sigma \theta^\sigma \wedge d^n x$, defined over a fibered chart domain $V^r \subset J^rY$ where $V^r = J^rV$ for some fibered chart domain $V \subset Y$. In the following, we assume that the set $\psi^r(V^r)$ is *vertically star-shaped* with center $\left(x^A, 0\right)$, i.e. if $(x^A, y^\sigma, y^\sigma{}_A, ..., y^\sigma{}_{A_1...A_r}) \in \psi^r(V^r)$, then the whole segment $(x^A, uy^\sigma, uy^\sigma{}_A, ..., uy^\sigma{}_{A_1...A_r})$, $u \in [0,1]$, remains in $\psi^r(V^r)$. Under this assumption[4], the correspondence

$$((x^A, y^\sigma, y^\sigma{}_A, ..., y^\sigma{}_{A_1...A_r}), u) \mapsto (x^A, uy^\sigma, uy^\sigma{}_A, ..., uy^\sigma{}_{A_1...A_r}) \tag{1.61}$$

is the coordinate expression of a well defined mapping:

$$\chi : V^r \times [0,1] \to V^r, \qquad \chi((J^r_x\gamma), u) =: \chi_u(J^r_x\gamma). \tag{1.62}$$

Further, for any $\rho \in \Omega_k(V^r)$, set:

$$I\rho := \int_0^1 \rho^{(0)}(u) du, \tag{1.63}$$

where $\rho^{(0)}(u) \in \Omega_{k-1}(V^r)$ is obtained from the decomposition:

$$\chi^*\rho = du \wedge \rho^{(0)}(u) + \rho'(u) \tag{1.64}$$

into a du-term and a term $\rho'(u)$ which does not contain du. The obtained mapping $I : \Omega_k(V^r) \to \Omega_{k-1}(V^r)$, called the *fibered homotopy operator*, is $\mathbb{R}$-linear and obeys:

$$\rho = Id\rho + dI\rho + (\pi^r)^* \rho_0, \tag{1.65}$$

where

$$\rho_0 := 0^*\rho \tag{1.66}$$

and 0 denotes the zero section $0 : \left(x^i\right) \mapsto \left(x^i, 0, 0, ..., 0\right)$ of V^r. The k-form ρ_0 is defined over $\pi(V) \subset X$.

The following properties will be used in the following sections:

$$Ih\rho = 0, \qquad Ip_k\rho = p_{k-1}I\rho, \qquad 1 \le k \le q. \tag{1.67}$$

Applying the above operator I to a source form $\varepsilon = \varepsilon_\sigma \theta^\sigma \wedge d^n x \in \Omega_{n+1}(V^r)$, the obtained n-form

$$\lambda_\varepsilon := I\varepsilon \tag{1.68}$$

[4]The construction can also be extended to cases when $\psi(V)$ is not vertically star-shaped, see, e.g., [98].

is a Lagrangian on V^r, called the *Vainberg-Tonti Lagrangian* attached to ε. In coordinates:

$$\lambda_\varepsilon = \mathcal{L}_\varepsilon d^n x, \qquad \mathcal{L}_\varepsilon = y^\sigma \int_0^1 \varepsilon_\sigma(x^A, uy^\mu, uy^\mu{}_A, ...uy^\mu{}_{A_1...A_r}) du. \tag{1.69}$$

If the source form ε admits a Lagrangian on V, then: $\mathcal{E}_{\lambda_\varepsilon} = \varepsilon$.

Remark. The Vainberg-Tonti Lagrangian λ_ε of a source form ε is of the same order as ε; in particular, for a second order source form, λ_ε is also of second order. That is, very often (e.g., in classical mechanics), λ_ε can be order-reduced, i.e., it is equivalent to a lower order Lagrangian.

1.2.3 Canonical variational completion of a source form

By *variational completion* of a given source form ε on Y, one could in principle understand any source form τ on Y with the property that $\varepsilon + \tau$ is locally variational. But, clearly, every source form has infinitely many variational completions, since *any* Lagrangian λ gives rise to a variational completion of ε by the rule $\tau := \mathcal{E}(\lambda) - \varepsilon$. Thus, the question is whether it is possible to choose this Lagrangian λ in a meaningful, "canonical" way.

An answer is given, as we will see below, by the Vainberg-Tonti Lagrangian. The key property that justifies this choice is the following.

The Euler-Lagrange form $\mathcal{E}(\lambda_\varepsilon) = \mathcal{E}_\nu \theta^\nu \wedge d^n x$ of the Vainberg-Tonti Lagrangian λ_ε is given, [117], by:

$$\mathcal{E}_\nu = \varepsilon_\nu - \int_0^1 u\{y^\sigma(H_{\nu\sigma} \circ \chi_u) + y^\sigma{}_B(H_{\nu\sigma}{}^B \circ \chi_u) + ... + y^\sigma{}_{B_1...B_r}(H_{\nu\sigma}{}^{B_1...B_r} \circ \chi_u)\} du, \tag{1.70}$$

where the coefficients $H_{\sigma\nu}{}^{B_1...B_k}$ are the Helmholtz coefficients (1.59).

From (1.59), it follows that the coefficients $H_{\sigma\nu}{}^{B_1...B_k}$ above have the meaning of "obstructions from variationality" of the source form ε. In particular, if the source form ε is variational, then $\mathcal{E}(\lambda_\varepsilon) = \varepsilon$.

Pick a vertically star-shaped fibered coordinate chart domain $V^r = J^r V \subset Y$ as above. Then, it makes sense:

Definition 7 *, [202]: The* ***canonical variational completion*** *of a source form $\varepsilon \in \Omega_{n+1}(J^r V)$, is the source form $\tau(\varepsilon)$ given by the difference between the Euler-Lagrange form of the Vainberg–Tonti Lagrangian of ε and ε itself:*

$$\tau(\varepsilon) = \mathcal{E}(\lambda_\varepsilon) - \varepsilon. \tag{1.71}$$

The local coefficients τ_ν of the canonical variational completion $\tau(\varepsilon) = \tau_\nu \theta^\nu \wedge d^n x$ can thus be directly expressed in terms of the Helmholtz form coefficients $H_{\nu\sigma}{}^{B_1...B_k}$:

$$\tau_\nu = -\int_0^1 u\{y^\sigma(H_{\nu\sigma} \circ \chi_u) + y^\sigma{}_B(H_{\nu\sigma}{}^B \circ \chi_u) + ... + y^\sigma{}_{B_1...B_r}(H_{\nu\sigma}{}^{B_1...B_r} \circ \chi_u)\} du.$$

Generally, the Vainberg-Tonti Lagrangian and, accordingly, the canonical variational completion of a source form of order r, are of order $2r$.

Remarks.

1. **Tensor equations:** In the case when $Y = T^p_q X$ is a bundle of tensors of type (p,q) over X and the functions ε_σ behave as components of a tensor (respectively, as components of a tensor density) of type (q,p) under arbitrary coordinate changes, then their Vainberg-Tonti Lagrangian function (1.69), whenever defined, is an invariant scalar (respectively, a scalar density). That is, if some globally defined tensor equations are locally variational, then their Vainberg-Tonti Lagrangian is natural; in particular, it is globally well defined.

2. **Applicability of the algorithm.** As constructed above, the Vainberg-Tonti Lagrangian is quite widely, but not universally applicable (or at least, not without modification). Limitations of its applicability come from the assumption of vertical star-shapedness with center $(x^A, 0, ..., 0)$ of the coordinate chart (V^r, ψ^r). An immediate counterexample in this sense are metric theories of gravity, where the dynamical variables are the metric components g_{ij} or the inverse metric components g^{ij}; in this case, the coordinate neighborhood V^r is *not* vertically star-shaped with center $\big(x^i, 0, 0..., 0\big)$ as neither $g_{ij} = 0$, nor $g^{ij} = 0$ can define a metric tensor. A simple fix that solves the problem in most such cases, is to consider the lower integration endpoint 0 as a *limit*, [98] (which will be done in all the examples below), or to replace 0 with a different value $a \in \bar{\mathbb{R}}$.

1.2.4 Examples

Einstein tensor obtained from canonical variational completion of Ricci tensor, [202].

In general relativity, gravity is encoded in a Lorentzian metric on a 4-dimensional manifold M interpreted as spacetime. Hence, let in the following (M, g) denote a 4-dimensional manifold equipped with local charts (U, ϕ), $\phi = (x^i)_{i=\overline{0,3}}$ and a metric of Lorentzian signature $(+,-,-,-)$. We denote by ∇ the Levi-Civita connection of g, by R_{ij} the Ricci tensor of ∇ and by $R = g^{ij}R_{ij}$, the scalar curvature. Indices of tensors will be lowered or raised by means of the metric g_{ij} and its inverse g^{ij}.

Einstein field equations (1.56) arise by varying with respect to the metric tensor the Lagrangian $\lambda = \lambda_g + \lambda_m$, where:

i) $\lambda_g = R\sqrt{|\det g|}d^4x$ is the Hilbert Lagrangian[5];

ii) the *matter Lagrangian* $\lambda_m = \mathbb{L}_m\sqrt{|\det g|}d^4x$, is given by an invariant scalar $\mathbb{L}_m = \mathbb{L}_m(g_{ij}, g_{ij,h}; y^I, {y^I}_j, ..., {y^I}_{j_1...j_r})$ depending on the metric tensor components and their derivatives up to order 1 and on the r-jet of some other field with coordinates y^I.

In the following, we will focus on the *vacuum* Einstein field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0. \tag{1.72}$$

[5] Here, we have omitted the constant $\alpha = -\dfrac{1}{16\pi\kappa}$ in front of λ_g (which is present in the paper [202]), as this constant, on the one hand, plays no role in determining the *vacuum* field equations and, on the other hand, it is not to be fixed by variationality reasonings, but by further considerations (imposing that the Newtonian limit of the obtained field equations - i.e., the Einstein field equations - coincides with the Poisson equation).

In this case, the dynamical variables are the metric tensor components g_{ij}, meaning that our configuration manifold (Y, π, M) is the *bundle of metrics*

$$(Met(M), \pi, M),$$

where $Met(M) \subset T_2^0 M$ is defined as the set of symmetric nondegenerate tensors of type (0,2) on M and π is the restriction to $Met(M)$ of the canonical projection of $T_2^0 M$. On the other hand, since both R_{ij} and R are of second order in g_{ij}, the space we actually have to work on is $J^2 Met(M)$.

We denote the local charts on $Met(M)$ by (V, ψ), with $\psi = (x^i, g_{jk})$ and the induced fibered chart on $J^2 Met(M)$, by (V^2, ψ^2), with $\psi^2 = (x^i, g_{jk}; g_{jk,i}; g_{jk,il})$. We will also use the notations $\theta^\sigma =: \theta_{jk}$, $\theta^\sigma{}_i =: \theta_{jk,i}$ for the local contact basis forms on $J^2 Met(M)$, more precisely:

$$\theta_{jk} = dg_{jk} - g_{jk,i} dx^i; \quad \theta_{jk,l} = dg_{jk,l} - g_{jk,li} dx^i.$$

The Riemann tensor components $R_j{}^i{}_{kl} = d_l \Gamma^i{}_{jk} - d_k \Gamma^i{}_{jl} + \Gamma^h{}_{jk} \Gamma^i{}_{hl} - \Gamma^h{}_{jl} \Gamma^i{}_{hk}$, the Ricci tensor components $R_{jk} = R_j{}^i{}_{ki}$ and Ricci scalar $R = g^{jk} R_{jk}$ thus become objects on $J^2 Met(M)$. We will refer to these objects as *formal* ones, meaning that they are constructed by means of the usual relations, but, in their expressions, g_{ij}, $g_{ij,k}$ and $g_{ij,kl}$ are the *coordinate functions* on $J^2 Met(M)$. Only when evaluated along *sections* $g \in \Gamma(Met(M))$, $g : (x^i) \mapsto g_{jk}(x^i)$, these become the usual geometric objects, defined on M; in other words, the base manifold quantities introduced in the first paragraphs of this section (e.g., in (1.72) are, in the new notations, $R_{jk} \circ J^2 g$ and $R \circ J^2 g$.

Fix a fibered chart on $J^2 Met(M)$ and consider the following source form:

$$\varepsilon := R^{ij} \sqrt{|\det g|} \theta_{ij} \wedge d^n x, \tag{1.73}$$

with components $\varepsilon^{ij} = \varepsilon^{ij}(g_{kl}; g_{kl,h}, g_{kl,hm})$ given by

$$\varepsilon^{ij} = R^{ij} \sqrt{|\det g|}.$$

The Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon d^n x$ is:

$$\mathcal{L}_\varepsilon = g_{ij} \int_0^1 \varepsilon^{ij}(u g_{kl}; u g_{kl,h}, u g_{kl,hm}) du,$$

where the lower integration endpoint $u = 0$ will be considered *as a* limit (since $g_{kl} = 0$ cannot be, properly speaking, the local components of any metric tensor).

Let us study the behavior of the integrand with respect to homotheties χ_u : $(x^i; g_{kl}; g_{kl,h}, g_{kl,hm}) \mapsto (x^i; u g_{kl}; u g_{kl,h}; u g_{kl,hm})$. These homotheties induce the transformation $g^{kl} \mapsto u^{-1} g^{kl}$ of the inverse metric tensor components. The Christoffel symbols

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{ih} (g_{hj,k} + g_{hk,j} - g_{jk,h})$$

are invariant with respect to χ_u and hence the curvature tensor components $R_j{}^i{}_{kl}$ are also invariant. The Ricci tensor R_{jk} is obtained just by a summation process from $R_j{}^i{}_{kl}$, which means that it is also insensitive to χ_u. That is,

$$R^{ij} \circ \chi_u = (g^{ih} g^{jl} R_{hl}) \circ \chi_u = u^{-2} R^{ij}.$$

It remains to compute the contribution of χ_u, $u \in [0,1]$, to the factor $\sqrt{|\det g|}$. Since each line of the matrix (g_{jk}) is multiplied by u, we get

$$\sqrt{|\det(g \circ \chi_u)|} = \sqrt{u^4\,|\det g|} = u^2\sqrt{|\det g|}.$$

Substituting into the expression of $\mathcal{L}_\varepsilon$, we get this way,

$$\mathcal{L}_\varepsilon = g_{ij}\int_0^1 u^0 R^{ij}\sqrt{|\det g|}du = g_{ij}R^{ij}\sqrt{|\det g|}\int_0^1 u^0 du = R\sqrt{|\det g|},$$

i.e., the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon d^n x$ is nothing but the Hilbert Lagrangian λ_g :

$$\lambda_\varepsilon = \lambda_g = R\sqrt{|\det g|}d^4x, \tag{1.74}$$

leading to the vacuum Einstein field equations (1.72).

Variational completion and 4-dimensional Einstein-Gauss-Bonnet gravity, [98].

The so-called *renormalized,* or *truncated 4-dimensional Einstein-Gauss-Bonnet gravity* is a theory which recently received a lot of attention - but also some criticism. One of the main criticisms, [9], is based on the fact that the resulting field equations cannot be obtained as the Euler–Lagrange equations from a generally covariant Lagrangian. In the following, we show by means of the canonical variational completion algorithm that, also in dimension $n \neq 4$, the renormalized truncated Gauss–Bonnet source form cannot be obtained from any Lagrangian *at all*; moreover, its canonical variational completion is still ill-defined in dimension 4.

Consider, in the following, $n = \dim M \geq 4$.

The original (non-truncated) Einstein-Gauss-Bonnet theory is based on the Lagrangian $\lambda = \lambda_g + \lambda_m$, where λ_m denotes the matter Lagrangian and the vacuum Lagrangian $\lambda_g = \mathcal{L}_g d^n x \in J^2Met(M)$ is given by:

$$\mathcal{L}_g = \left[\frac{M_{\mathrm{P}}^2}{2}R - \Lambda_0 + \frac{\alpha}{n-4}\mathcal{G}\right]\sqrt{|\det g|}\,, \tag{1.75}$$

where M_P, Λ_0 and α are real constants and $\mathcal{G}$ denotes the *Gauss-Bonnet scalar,* see, e.g., [50]:

$$\mathcal{G} = 6R^{ij}{}_{[ij}R^{kl}{}_{kl]} = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}\,. \tag{1.76}$$

Variation of λ with respect to the metric is known to lead to the field equations:

$$E_{ij} = M_{\mathrm{P}}^2 G_{ij} + \Lambda_0 g_{ij} - \frac{2\alpha}{n-4}\mathcal{G}_{ij} = T_{ij}\,, \tag{1.77}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ are the components of the Einstein tensor and the term:

$$\mathcal{G}_{ij} := 15 g_{i[j}R^{kl}{}_{kl}R^{hm}{}_{hm]} = \frac{1}{2}\mathcal{G}g_{ij} - 2R_{ihkl}R_j{}^{hkl} + 4R_{ikjl}R^{kl} + 4R_{ik}R_j{}^k - 2RR_{ij}, \tag{1.78}$$

originating from the Gauss-Bonnet scalar, is still of second order. Λ_0 is physically interpreted as the *cosmological constant.*

The truncated equations. Due to the $n-4$ denominator in the last term of (1.75), neither the Einstein-Gauss-Bonnet Lagrangian λ_g, nor the resulting field equations (1.77) make any sense for $n=4$. One idea to overcome this drawback was to truncate these equations, [83], as follows.

Noticing that

$$-\mathcal{G}_{ij} = (n-4)A_{ij} + W_{ij}\,, \tag{1.79}$$

where:

$$A_{ij} = \frac{n-3}{(n-2)^2}\left[\frac{2n}{n-1}RR_{ij} - 4\frac{n-2}{n-3}R^{kh}C_{ikjh} - 4R_i{}^kR_{jk} + 2R_{kh}R^{kh}g_{ij} - \frac{1}{2}\frac{n+2}{n-1}R^2g_{ij}\right] \tag{1.80}$$

and W_{ij} is a combination of the Weyl tensor components of (M,g), from which no factor $(n-4)$ can be extracted[6], the idea in [83] was to discard the W-term, i.e., to consider instead of (1.77), the truncated equations:

$$M_{\mathrm{P}}^2G_{ij} + \Lambda_0 g_{ij} + 2\alpha A_{ij} = \kappa T_{ij}\,, \tag{1.81}$$

where $\kappa \in \mathbb{R}$ is a constant.

But, as noted in [9], the left hand side of the above (more precisely, the term A_{ij}) cannot originate from the variation of any *generally covariant* Lagrangian. This is consistent, in the case $n=4$, with an old result by Lovelock, [133], stating that the only second order PDE system that can be obtained as the Euler-Lagrange system of a generally covariant Lagrangian on $Met(M)$, is given by the Einstein equations with a cosmological constant: $M_{\mathrm{P}}^2G_{ij} + \Lambda_0 g_{ij} = T_{ij}$.

In the following, we will show that the variationally completed equations for (1.81) are of fourth order and still diverge in the case $n=4$. We present here the discussion in the case $n \geq 4$; the case $n \leq 4$ was discussed, with similar conclusions, in the paper [98].

In order to correctly use g_{ij} as our dynamical variables in (1.81), we must first raise the indices in equation (1.81). The relevant source form, obtained by truncating the Euler-Lagrange form of (1.75) is $\tilde{\varepsilon} := \tilde{\varepsilon}^{ij}\theta_{ij} \wedge d^nx$ given by:

$$\tilde{\varepsilon}^{ij} := -\frac{1}{2}(M_{\mathrm{P}}^2G^{ij} + \Lambda_0 g^{ij} + 2\alpha A^{ij})\sqrt{|\det g|}; \tag{1.82}$$

the factor $-\dfrac{1}{2}$ arises from the definition of the energy-momentum tensor, see [126] or Section 1.3 below.

Under a rescaling $g_{ij} \to ug_{ij}$, the terms in the field equations (1.82) transform as $G_{ij} \to G_{ij}\,,\quad A_{ij} \to u^{-1}A_{ij}$ and $\sqrt{|\det g|} \to u^{\frac{n}{2}}\sqrt{|\det g|}$; taking into account $g^{ij} \to u^{-1}g^{ij}$, this gives:

$$G^{ij} \to u^{-2}G^{ij}\,,\quad A^{ij} \to u^{-3}A^{ij}. \tag{1.83}$$

For $n>4$, the resulting Vainberg-Tonti Lagrangian integral is finite:

$$\begin{aligned}
\mathcal{L}_{\tilde{\varepsilon}} &= -\frac{1}{2}g_{ij}\int_0^1 u^{n/2}\sqrt{|\det g|}\left(u^{-2}M_{\mathrm{P}}^2G^{ij} + u^{-1}\Lambda_0 g^{ij} + 2u^{-3}\alpha A^{ij}\right)du \\
&= -\frac{1}{2}\sqrt{|\det g|}g_{ij}\left(\frac{2M_{\mathrm{P}}^2}{n-2}G^{ij} + \frac{2\Lambda_0}{n}g^{ij} + \frac{4\alpha}{n-4}A^{ij}\right) \\
&= \sqrt{|\det g|}\left(\frac{M_{\mathrm{P}}^2}{2}R - \Lambda_0 - \frac{2\alpha}{n-4}A^i{}_i\right),
\end{aligned} \tag{1.84}$$

[6] The term W_{ij} does indeed, vanish in $n=4$ dimensions, but, for combinatorial reasons.

where we used $g_{ij}g^{ij} = n$ as well as

$$g_{ij}G^{ij} = g_{ij}\left(R^{ij} - \frac{1}{2}Rg^{ij}\right) = \left(1 - \frac{n}{2}\right)R. \tag{1.85}$$

The first two terms in the last line of (1.84) give us the Einstein-Hilbert Lagrangian density plus the (densitized) cosmological constant, as one would have expected.

Varying the obtained Vainberg-Tonti Lagrangian with respect to g_{ij} leads to the variationally completed field equations, [98], which, after lowering the indices, read:

$$\begin{aligned}\tilde{\mathcal{E}}_{ij} := M_{\mathrm{P}}^2 G_{ij} + \Lambda_0 g_{ij} + \frac{4\alpha(n-3)}{(n-1)(n-2)(n-4)}\Big[g_{ij}\left(\Box R - \frac{n}{4}R^2 + (n-1)R^{kl}R_{kl}\right) \\ - 2(n-1)\Box R_{ij} + (n-2)\nabla_{\partial_i}\nabla_{\partial_j}R + nRR_{ij} - 4(n-1)R^{kl}R_{ikjl}\Big] = T_{ij}\,,\end{aligned} \tag{1.86}$$

where $\Box = g^{ij}\nabla_{\partial_i}\nabla_{\partial_j}$ denotes the d'Alembert operator.

These equations are, as announced, of fourth order (and this will not change if we choose, e.g., to "densitize" the original source form by a different power of $|\det g|$). In particular, they cannot not coincide with the original equations (1.81), which means, using (1.70) that the considered source form is not locally variational[7]. Also, due to the presence of $(n-4)$ in the denominator of the last term, its canonical variational completion (which is the "closest" variational source form to the given one) does not make any sense for $n = 4$.

1.3 Energy-momentum tensor and energy-momentum balance law

1.3.1 Introduction

This section, which is based on our paper [201], tries to bring a bit more clarity to the long and intricate history of the topic of energy-momentum tensors in Lagrangian field theories. Essentially, it does two things:

1. It points out, for arbitrary Lagrangian field theories, a generalized "Hilbert-type" definition of the energy-momentum tensor based on Euler-Lagrange forms, which agrees along critical sections, with the one based on *corrected Noether currents* introduced by Gotay and Marsden, [84].

2. It finds, for these theories, an *energy-momentum balance*[8] *law*, that generalizes to *arbitrary* field theories based on natural Lagrangians, the covariant energy-momentum conservation law known in metric theories. Whereas point 1. above is not really - or not completely - new, but just done using a somewhat different framework (a similar expression of the energy-momentum tensor can be found, e.g., in [72]), to the best of our knowledge, the latter result was found for the first time in full generality, in [201].

[7] In [98], the statement on the non-variationality of equations (1.81) was justified using Corrollary 1, Section 4.11 of the book [114], which apparently (as pointed out by Bence Racsko in a private talk) has a gap in the proof. Yet, this does not affect our result, which can be justified directly using the Vainberg-Tonti Lagrangian.

[8] We preferred the term *balance law* to the one of *conservation law,* as this law does **not** imply, for arbitrary field theories, that the energy-momentum tensor has zero divergence (or not even zero "covariant divergence" - which would anyway make sense only provided that a notion of covariant derivative exists in the given theory), which would be required by a true conservation law.

As a first application, we find an explicitly covariant generalized energy-momentum conservation law for *general metric-tensor* (in particular, for *general metric-affine*) *theories of gravity*, with a much simpler expression compared to the known ones in the literature, e.g., [152], [132].

The main motivation for using a generalized Hilbert-type formula for the energy-momentum tensor, over a Noether-type one, is its computational simplicity. Yet, such a formula has at least two other advantages: it is based on a uniquely defined differential form (the Euler-Lagrange form), whereas Noether currents are not uniquely defined; also, it opens up the possibility of using results of the inverse problem of variational calculus, such as: the classification of first-order energy-momentum source forms, [116], or the notion of variational completion discussed above.

A brief history. Before passing to the technical details, let us present in brief the main episodes in the history of energy-momentum tensors (see also [74], [84]).

- In special relativity, whose spacetime manifold is $\mathbb{R}^4$ equipped with the Minkowski metric $\eta = diag(1, -1, -1, -1)$, the *canonical (Noether) energy-momentum tensor* is defined by means of the Noether currents due to the invariance of the Lagrangian to the group of *spacetime translations*. It is conserved on-shell and its time-time and time-space components give the correct energy and momentum densities of the described physical system. Still, as it fails to fulfil two basic requirements for physical applications - symmetry and gauge invariance - it requires an "improvement" procedure; the classical special-relativistic improvement recipe (Belinfante&Rosenfeld, 1940) is based on enlarging the group of translations to the whole Poincaré group.
- General relativity, based on a manifold M with a dynamical Lorentzian metric $g \in \Gamma(Met(M))$, came with a completely different toolkit. The *Hilbert, or metric energy-momentum tensor,* obtained as the variational derivative of the matter Lagrangian with respect to the metric, has the desired properties - symmetry, gauge invariance and vanishing covariant divergence on–shell. Hence, it does not require any improvement procedure. But, on the other hand, [84], it is not obvious at all that it gives the correct energy and momentum densities of the physical system under discussion.
- It thus appeared the idea of obtaining the energy-momentum tensor of *any* natural Lagrangian field theory as a kind of "improved Noether current" which, in the case of general relativity, coincides with the Hilbert one. This was done for first order Lagrangians by Gotay and Marsden, [84] (with some refinements brought by Forger and Rőmer[9], [74]) and extended to higher order ones by Fernández, García and Rodrigo, [72]. Here are the underlying ideas:
  - *General covariance of the Lagrangian:* On a general spacetime manifold M, translations (let alone the Poincaré group) make no sense. The natural choice in this case is the group $Diff(M)$ of diffeomorphisms of M; assuming that there exists a canonical lift of diffeomorphisms of M to the configuration manifold Y and the Lagrangian of the theory is invariant under lifts of arbitrary diffeomorphisms, then any 1-parameter subgroup of $Diff(M)$ gives rise to a Noether current.

[9] The main advance brought in [74] resides in the fact that the energy-momentum tensor is built from the matter Lagrangian λ_m only (while in [84], it is built from the total Lagrangian λ - and thus vanishes on-shell). Also, the energy-momentum tensor is regarded as a geometric object on a *jet bundle* of the configuration manifold, rather than on the spacetime manifold X - a standpoint which we will also adopt in the following.

– *Splitting of the variables into background and matter ones, and accordingly, of the Lagrangian into a background and a matter component:* The problem with the above algorithm based on the infinite-dimensional group $Diff(M)$ is that, when *all* the variables are subject to Euler-Lagrange equations, the corresponding "Noether currents" are always zero [74], [84]. The way out of this impasse consists, [74], [80], in dividing the variables of the theory into *background* ones (e.g., a metric and/or a connection, a tetrad etc.) and *dynamical* (or *matter*) ones and, accordingly, in splitting the total Lagrangian λ into a sum

$$\lambda = \lambda_b + \lambda_m,$$

where the *background Lagrangian* λ_b only depends on the background variables and their derivatives, while the *matter Lagrangian* λ_m may depend on all the variables of the theory. For the matter component λ_m taken separately, the background variables will no longer be subject to any Euler-Lagrange equations (these are only supposed to obey Euler-Lagrange equations for the *total* Lagrangian λ). This way, one can obtain nonzero Noether currents, leading to the energy-momentum tensor.

The original 1992 result by Gotay and Marsden, [84], states the following. Assume that, for a fibered manifold $Y \xrightarrow{\pi} M$, there exists a canonical lifting of vector fields $l : \mathcal{X}(M) \to \mathcal{X}(Y)$, $\xi \mapsto \Xi$ of index 1. Then, for any first order Lagrangian λ which is invariant under the lifts of arbitrary vector fields $\xi \in \mathcal{X}(M)$ and for any critical section $\gamma \in \Gamma(Y)$ for λ, there uniquely exists a (1,1)-tensor density $\mathcal{T}(\gamma) = \mathcal{T}^i_{\ j}(\gamma)\dfrac{\partial}{\partial x^i} \otimes dx^j$ on M such that, for all compact hypersurfaces $\Sigma \subset M$,

$$\int_\Sigma J^1\gamma^* \mathcal{J}^\Xi = \int_\Sigma \mathcal{T}^i_{\ j}(\gamma)\xi^j \mathbf{i}_{\partial_i} d^n x, \tag{1.87}$$

where $\mathcal{J}^\Xi$ denotes the Noether current (1.33), (1.43), associated with Ξ.

The "corrected Noether current map" $\mathcal{T}(\gamma)$ is gauge invariant and given, in metric theories (i.e., for $Y = Met(M)$), by a Hilbert-type formula. Moreover, relation (1.87) ensures that $\mathcal{T}(\gamma)$ is a "physically correct" energy-momentum tensor, in the following sense, [84]: if Σ is a Cauchy hypersurface and the vector field ξ is transversal to Σ, then the *energy* corresponding to the direction of evolution ξ, is $H_\xi = -\int_\Sigma J^1\gamma^* \mathcal{J}^\Xi$, i.e., it is given, again up to a sign, by relation (1.87).

The latter remark points out that one could hardly overestimate the importance of having the energy-momentum tensor related to Noether currents as in (1.87). But, instead of using this relation as a definition, we will obtain it as a *consequence* of a general "Hilbert-type" definition.

This is based on noticing that the known algorithm for relating the Hilbert and Noether-type energy-momentum tensors in the special-relativistic limit of general relativity, [127], can actually be extended to almost completely arbitrary (not necessarily metric) field theories, as follows. Assuming that the differential index of the theory *in the background variables* is 1, and the matter Lagrangian λ_m is generally covariant, in its Euler-Lagrange form $\mathcal{E}^{(b)}$ with respect to the background variables, we can isolate a divergence term - which thus couples to the Noether (boundary) term in the first variation formula and gives the correct energy-momentum tensor. The remaining part of $\mathcal{E}^{(b)}$ gives the *energy-momentum balance law,* which is obtained as an immediate consequence of the same first variation formula (1.33).

These results hold true regardless of the order of the matter Lagrangian, in any of the (background or matter) variables, or of the index of the lifting with respect to the matter variables. We will assume for simplicity, that Y is a natural bundle over M; yet, all the results remain valid if we assume, as in [84], that a canonical lift $Diff(M) \to Aut(Y)$ somehow exists.

1.3.2 Energy-momentum tensor and energy-momentum balance function

The setting.

- **Configuration manifold**: Assume, in the following, that the configuration manifold (Y, π, M) is a fibered product
$$Y = Y^{(b)} \times_M Y^{(m)},$$
over a spacetime manifold M, where both factors $Y^{(b)}$ and $Y^{(m)}$ are natural bundles over M. We denote the coordinates in an arbitrary fibered chart on Y by (x^i, y^σ, y^I); the local coordinates y^σ will be called *background variables* and y^I, *matter variables.* Accordingly, we will denote the elements of the local contact basis (1.10) on J^rY by:
$$\theta^\sigma = dy^\sigma - y^\sigma{}_i dx^i, \quad \theta^I = dy^I - y^I{}_i dx^i \quad etc..$$

- **Lagrangian.** Consider, on Y a Lagrangian of order r :
$$\lambda = \lambda_b + \lambda_m \in \Omega_{n,X}(J^rY), \tag{1.88}$$
where the *background Lagrangian* λ_b is projectable onto $J^rY^{(b)}$, i.e., it depends only on $x^i, y^\sigma, y^\sigma{}_i, ..., y^\sigma{}_{i_1...i_r}$, while the *matter Lagrangian*
$$\lambda_m = \mathcal{L}_m(x^i, y^\sigma, ..., y^\sigma{}_{i_1...i_r}, y^I, y^I{}_i, ..., y^I{}_{i_1...i_r})d^nx$$
may depend on all the coordinates on J^rY.

- **Further assumptions.** We suppose that:

A1. The natural lifting $M \to Y^{(b)}$ has index 1; in other words, the canonical lifting
$$l : \mathcal{X}(M) \to \mathcal{X}(Y), \qquad \Xi := l(\xi) \tag{1.89}$$
is of index 1 *in the background variables* y^σ. That is, in any fibered chart and for any $\xi = \xi^i\partial_i \in \mathcal{X}(M)$,
$$\Xi = \xi^i(x^j)\partial_i + \Xi^\sigma(x^j, y^\sigma)\partial_\sigma + \Xi^I(x^j, y^J)\partial_I; \tag{1.90}$$
where Ξ^σ are expressed as
$$\Xi^\sigma = C^\sigma_i\xi^i + C^{\sigma j}_i\xi^i_{,j} \tag{1.91}$$
for some $C^\sigma_i = C^\sigma_i\left(x^j, y^\mu\right), C^{\sigma j}_i = C^{\sigma j}_i\left(x^j, y^\mu\right)$. No restriction is imposed upon Ξ^I.

A2. The matter Lagrangian λ_m is natural (generally covariant), i.e.,
$$\mathfrak{L}_{J^r\Xi}\lambda_m = 0, \quad \forall\xi \in \mathcal{X}(M). \tag{1.92}$$

First variation formula revisited.

Under the above assumptions, the first variation formula (1.33) will have a peculiar form, which we will investigate in the following.

Assume that λ_m admits a Lepage equivalent $\rho_\lambda \in \Omega_n(J^sY)$ of order s (where $s \leq 2r-1$), i.e., the Euler-Lagrange form $\mathcal{E}(\lambda_m) = p_1 d\rho_\lambda$ is of order $s+1$ and the Noether current $J^\Xi = \mathbf{i}_{J^s\Xi}\rho_\lambda$ is of order s. The naturality condition (1.92), together with the first variation formula (1.33) imply:

$$J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}(\lambda_m) - J^s\gamma^* d\mathcal{J}^\Xi = 0, \tag{1.93}$$

for *any* section $\gamma := (\gamma^{(b)}, \gamma^{(m)}) \in \Gamma(Y)$, locally represented as $(x^i) \mapsto (x^i, y^\sigma(x^i), y^I(x^i))$.

Further, the Euler-Lagrange source form $\mathcal{E}(\lambda_m)$ uniquely splits into a background component $\mathcal{E}^{(b)} \in \Omega_n(J^{s+1}Y)$ and a matter one $\mathcal{E}^{(b)} \in \Omega_n(J^{s+1}Y)$:

$$\mathcal{E}(\lambda_m) = \mathcal{E}^{(b)} + \mathcal{E}^{(m)}, \tag{1.94}$$

such that $\mathcal{E}^{(b)}$ is $\pi_{Y^{(m)}}$-horizontal (i.e., in any fibered chart, it contains no θ^I-terms), whereas $\mathcal{E}^{(m)}$ is $\pi_{Y^{(b)}}$-horizontal (i.e., it contains, in any fibered chart, no θ^σ-terms); in coordinates,

$$\mathcal{E}^{(b)} = \frac{\delta\mathcal{L}_m}{\delta y^\sigma}\theta^\sigma \wedge d^n x, \qquad \mathcal{E}^{(m)} = \frac{\delta\mathcal{L}_m}{\delta y^I}\theta^I \wedge d^n x.$$

On-shell for the matter component[10] $\gamma^{(m)}$ of γ (i.e., for $y^I = y^I(x^j)$ subject to the Euler-Lagrange equations), the $\mathcal{E}^{(m)}$ part of (1.93) vanishes. So we remain with:

$$J^{s+1}\gamma^*(\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)}) - J^s\gamma^* d\mathcal{J}^\Xi \approx_{\gamma^{(m)}} 0, \tag{1.95}$$

where $\approx_{\gamma^{(m)}}$ denotes equality on-shell for $\approx_{\gamma^{(m)}}$. As in the following we will need *horizontal* forms, it will be more convenient to rewrite this relation using (1.12), as:

$$J^{s+2}\gamma^*(h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)}) - J^{s+1}\gamma^* hd\mathcal{J}^\Xi \approx_{\gamma^{(m)}} 0. \tag{1.96}$$

Definition of the energy-momentum tensor.

Under the assumption **A1** above, we prove below that the surviving Euler-Lagrange term $h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)}$ splits into a linear term in ξ and a divergence expression; the latter will provide the energy-momentum tensor.

Lemma 8 *If the canonical lifting $l : \mathcal{X}(M) \to \mathcal{X}(Y)$, $\xi \mapsto \Xi$, is of index 1 in the background variables y^σ, then there uniquely exist the $\mathcal{F}(M)$-linear mappings $\mathcal{B} : \mathcal{X}(M) \to \Omega_n(J^{s+2}Y)$ and $\mathcal{T} : \mathcal{X}(M) \to \Omega_{n-1}(J^{s+1}Y)$ with horizontal values, satisfying:*

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)} = \mathcal{B}(\xi) + hd(\mathcal{T}(\xi)), \qquad \forall \xi \in \mathcal{X}(M). \tag{1.97}$$

[10] For instance, in general relativity, if we take as the matter component $\gamma^{(m)}$, the electromagnetic 4-potential A (which is a section of $Y^{(m)} = T^*M$), on-shell for the matter component means that $F = dA$ is subject to the Maxwell equations.

Proof. We first define $\mathcal{B}$ and $\mathcal{T}$ locally. In an arbitrary fibered chart (V^{s+2}, ψ^{s+2}) on $J^{s+2}Y$, with $U = \pi^{s+2}(V^{s+2})$, $h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)}$ is expressed as:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)} = (\tilde{\Xi}^{\sigma}\mathcal{E}_{\sigma})d^n x, \tag{1.98}$$

where $\tilde{\Xi}^{\sigma} = \Xi^{\sigma} - y^{\sigma}{}_{i}\xi^{i}$, $\mathcal{E}_{\sigma} = \dfrac{\delta\mathcal{L}}{\delta y^{\sigma}}$. Substituting Ξ^{σ} from (1.91), we find:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)} = \{(C^{\sigma}{}_{i} - y^{\sigma}{}_{i})\mathcal{E}_{\sigma}\xi^{i} + C^{\sigma j}_{i}\mathcal{E}_{\sigma}\xi^{i}{}_{,j}\}d^n x = \{[(C^{\sigma}{}_{i} - y^{\sigma}{}_{i})\mathcal{E}_{\sigma} - d_j(C^{\sigma j}_{i}\mathcal{E}_{\sigma})]\xi^{i} + d_j(C^{\sigma j}_{i}\mathcal{E}_{\sigma}\xi^{i})\}d^n x. \tag{1.99}$$

Then, denoting, for any $\forall\xi \in \mathcal{X}(U)$:

$$\begin{aligned} \mathcal{B}(\xi) &= [(C^{\sigma}{}_{i} - y^{\sigma}{}_{i})\mathcal{E}_{\sigma} - d_j(C^{\sigma j}_{i}\mathcal{E}_{\sigma})]\xi^{i}\ d^n x, && (1.100)\\ \mathcal{T}(\xi) &: = (C^{\sigma j}_{i}\mathcal{E}_{\sigma}\xi^{i})\mathbf{i}_{\partial_j}d^n x, && (1.101) \end{aligned}$$

we obtain two linear mappings $\mathcal{B} : \mathcal{X}(U) \mapsto \Omega^{s+2}_{n}(Y)$ and $\mathcal{T} : \mathcal{X}(U) \mapsto \Omega^{s+1}_{n-1}(Y)$, having horizontal values and obeying (1.97).

Now, take two arbitrary, intersecting fibered chart domains $V^{s+2}, V^{s+2'} \subset J^{s+2}Y$, with $U = \pi^{s+2}(V^{s+2}), U' = \pi^{s+2}(V^{s+2'})$ and an arbitrary vector field $\xi \in \mathcal{X}(U \cap U')$. We denote by $\mathcal{T}$ and $\mathcal{T}'$ the mappings corresponding by (1.101) to the two domains. Taking into account the rules of transformation (1.18), (1.52), (1.17) of $\mathcal{E}_{\sigma}$, $C^{\sigma j}_{i}$ and $\mathbf{i}_{\partial_i}d^n x$, a brief calculation shows that:

$$\mathcal{T}(\xi) = (C^{\sigma j}_{i}\mathcal{E}_{\sigma}\xi^{i})\mathbf{i}_{\partial_j}d^n x = (C^{\sigma' j'}_{i'}\mathcal{E}_{\sigma'}\xi^{i'})\mathbf{i}_{\partial_{j'}}d^n x' = \mathcal{T}'(\xi) \tag{1.102}$$

i.e., the mapping $\mathcal{T}$ can be defined globally on $\mathcal{X}(M)$. As $h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)}$ is globally well defined, we find by (1.97) that $\mathcal{B}$ is also globally well defined.

Uniqueness of $\mathcal{B}$ and $\mathcal{T}$ can be established as follows. Assume that the mappings $\tilde{\mathcal{B}} : \mathcal{X}(U) \mapsto \Omega^{s+2}_{n}(Y)$, $\tilde{\mathcal{T}} : \mathcal{X}(U) \mapsto \Omega^{s+1}_{n-1}(Y)$ also obey the above requirements. Then, for any $\xi \in \mathcal{X}(M)$, we have:

$$0 = (\mathcal{B} - \tilde{\mathcal{B}})(\xi) + hd[(\mathcal{T} - \tilde{\mathcal{T}})(\xi)].$$

Since these mappings have horizontal values, they can be expressed as: $\mathcal{B}(\xi) = \mathcal{B}_i\xi^i d^n x$, $\tilde{\mathcal{B}}(\xi) = \tilde{\mathcal{B}}_i\xi^i d^n x$, $\mathcal{T}(\xi) = (\mathcal{T}^j_i\xi^i)\mathbf{i}_{\partial_j}d^n x$, $\tilde{\mathcal{T}}(\xi) = (\tilde{\mathcal{T}}^j_i\xi^i)\mathbf{i}_{\partial_j}d^n x$; substituting into the above equality, we find:

$$0 = [(\mathcal{B}_i - \tilde{\mathcal{B}}_i) + d_j(\mathcal{T}^j_i - \tilde{\mathcal{T}}^j{}_i)]\xi^i + (\mathcal{T}^j_i - \tilde{\mathcal{T}}^j{}_i)\xi^i_{,j}.$$

As this relation holds for *any* ξ, we obtain $\mathcal{T}^j_i - \tilde{\mathcal{T}}^j{}_i = 0$ and $\mathcal{B}_i - \tilde{\mathcal{B}}_i = 0$. Therefore, $\mathcal{B} = \tilde{\mathcal{B}}$ and $\mathcal{T} - \tilde{\mathcal{T}}$. ■

The above Lemma gives us the right to introduce

Definition 9 *The **energy-momentum tensor** of λ_m is the mapping:*

$$\mathcal{T} : \mathcal{X}(M) \to \Omega^{s+1}_{n-1}(Y), \quad \xi \mapsto \mathcal{T}(\xi), \tag{1.103}$$

uniquely defined by the splitting

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)} = \mathcal{B}(\xi) + hd(\mathcal{T}(\xi)), \tag{1.104}$$

where the mappings $\mathcal{T} : \mathcal{X}(M) \to \Omega^{s+1}_{n-1}(Y)$, $\xi \mapsto \mathcal{T}(\xi)$ and $\mathcal{B} : \mathcal{X}(M) \to \Omega^{s+2}_{n}(Y)$, $\xi \mapsto \mathcal{B}(\xi)$ are $\mathcal{F}(M)$-linear and have π^r-horizontal values.

In the following, we will call the mapping $\mathcal{B}$ defined by (1.104), the *balance function.*

Remarks.

1. **Coordinate expression of** $\mathcal{T}$: In coordinates, $\mathcal{T}$ is given by (1.101), i.e., $\mathcal{T} = \mathcal{T}_i^j dx^i \otimes \mathbf{i}_{\partial_j} d^n x$, with:
$$\mathcal{T}_i^j = C_i^{\sigma j} \frac{\delta \mathcal{L}_m}{\delta y^\sigma}, \tag{1.105}$$
where, y^σ denote the background variables and the coefficients $C_i^{\sigma j}$ are as in (1.91).

2. Relation (1.102) says that, with respect to fibered coordinate changes $\left(x^i, y^\sigma\right) \mapsto (x^{i'}, y^{\sigma'})$ on $J^{s+1}Y$, the functions $\mathcal{T}_i^j$ obey the rule:
$$\mathcal{T}_i^j = \frac{\partial x^j}{\partial x^{h'}} \frac{\partial x^{l'}}{\partial x^i} \det(\frac{\partial x^{k'}}{\partial x^k}) \mathcal{T}_{l'}^{h'}. \tag{1.106}$$

Remark. The local expression (1.105) of $\mathcal{T}$ coincides, up to a minus sign and to a pullback by sections of $J^{s+1}Y$, to the one found by Gotay and Marsden, [84] and extended by Fernandez, Garcia and Rodrigo [72] to higher order Lagrangians, as a result of a different, Noether-type construction. In the following, we will go a step further and reveal more of its potential in the next subsection, by getting a general energy-momentum balance law.

1.3.3 Properties of the energy-momentum tensor

The results below hold valid for *any* field theory obeying the assumptions **A1, A2** in the beginning of this subsection. Using Definition 9, we will prove:

Theorem 10 ***(Coordinate-free energy-momentum balance law):*** *For any piece $D \subset M$ and any $\xi \in \mathcal{X}(M)$ with support contained in D, there holds:*
$$\int_D J^{s+2}\gamma^* \mathcal{B}(\xi) \approx_{\gamma^{(m)}} 0, \tag{1.107}$$
where $\approx_{\gamma^{(m)}}$ means equality on-shell for the matter component $\gamma^{(m)}$ of the section $\gamma = (\gamma^{(b)}, \gamma^{(m)}) \in \Gamma(Y)$.

Proof. Consider an arbitrary vector field $\xi \in \mathrm{X}(M)$ and by denote $\Xi := l(\xi)$ its canonical lift to Y. Using the splitting (1.104) together with the integral first variation formula (1.36), we get:
$$0 \approx_{\gamma^{(m)}} \int_D J^{s+2}\gamma^* \mathcal{B}(\xi) + \int_{\partial D} J^{s+1}\gamma^* (\mathcal{T}(\xi) - \mathcal{J}^{\Xi}). \tag{1.108}$$
Equation (1.108) holds, in particular, for any vector field ξ *with support contained in* D; but, for such vector fields, the boundary term vanishes, which leads to the statement. ■

A direct consequence of the above is the main result of this section:

Theorem 11 *(i)* ***(Coordinate expression of the energy-momentum balance law):*** *Corresponding to any fibered chart on the configuration manifold $Y = Y^{(b)} \times_M Y^{(m)}$, there holds:*

$$\left(d_j \mathcal{T}^j_i - (C^\sigma{}_i - y^\sigma{}_i)\frac{\delta \mathcal{L}}{\delta y^\sigma}\right) \circ J^{s+2}\gamma \approx_{\gamma^{(m)}} 0, \quad i = 0, ..., n-1, \tag{1.109}$$

where $C^\sigma{}_i$ and $\mathcal{T}^j_i$ are given in (1.91) and (1.105).

(ii) ***(Energy-momentum tensor versus Noether currents):*** *For any compact domain $D \subset M$ and any vector field $\xi \in X(M)$ with support contained in D, there holds:*

$$\int_{\partial D} J^{s+1}\gamma^* \mathcal{T}(\xi) \approx_{\gamma^{(m)}} \int_{\partial D} J^{s+1}\gamma^* \mathcal{J}^{l(\xi)}, \tag{1.110}$$

where $l : \mathcal{X}(M) \to \mathcal{X}(Y)$ denotes the canonical lift.

Proof. *(i)* follows immediately from Theorem 11, taking into account the coordinate expression (1.100) of $\mathcal{B}$ and the arbitrariness of ξ. Then, choosing ξ with $\xi_{|\partial D} \neq 0$, and using *(i)*, together with the first variation formula (1.108), we find *(ii)*. ■

Property *(ii)* above tells us that $\mathcal{T}(\xi)$ coincides on-shell with the "improved Noether current" given by the general covariance of λ_m, i.e., it expresses the correct energy and momentum of the described physical system.

The next result concerns the behavior of $\mathcal{T}$ with respect to *gauge transformations* of the matter fields, which will be understood as strict fibered automorphisms that only affect these fields.

Theorem 12 *If a strict automorphism $\Phi \in Aut_s(Y^{(b)} \times_M Y^{(m)})$ is a symmetry of λ_m acting trivially on the background manifold $Y^{(b)}$, then:*

$$J^{s+1}\Phi^* \mathcal{T}(\xi) = \mathcal{T}(\xi), \qquad \forall \xi \in \mathcal{X}(M). \tag{1.111}$$

Proof. According to the hypothesis Φ acts trivially on M and on the fibers of $Y^{(b)}$, i.e., in any fibered chart, it is described as: $\Phi : \left(x^i, y^\sigma, y^I\right) \mapsto \left(x^i, y^\sigma, \tilde{y}^I(x^i, y^J)\right)$. Then, it is easily seen that in the expression $\mathcal{T}(\xi) = (C^{\sigma j}_i \mathcal{E}_\sigma \xi^i)\mathbf{i}_{\partial_j}(d^n x)$, all the appearing terms are Φ-invariant (the invariance of $\mathcal{E}_\sigma$ follows by noticing that $J^r\Phi^*\lambda_m = \lambda_m$ implies, by (4): $J^r\Phi^*\mathcal{E}^{(b)} = \mathcal{E}^{(b)}$; but, as $\mathcal{E}^{(b)} = \mathcal{E}_\sigma \theta^\sigma \wedge d^n x$ and θ^σ is Φ-invariant, one gets the statement), meaning that $\mathcal{T}(\xi)$ itself is invariant. ■

1.3.4 The case of metric and tensor backgrounds. Metric-affine gravity theories

The general case.

In the case when the background variables consist of a pseudo-Riemannian metric and, possibly, some other tensor quantity - which includes, in particular, all metric-affine theories of gravity - we will write the energy-momentum balance law (1.109) in a manifestly covariant form, using formal Levi-Civita covariant derivatives.

Denoting by $Met(M)$ the bundle of metrics, understood here for convenience as the set of all symmetric nondegenerate *contravariant* tensors of rank 2 on M, the background manifold becomes:

$$Y^{(b)} = Met(M) \times_M T^p_q(M)$$

and, accordingly, $Y = Met(M) \times_M T^p_q(M) \times_M Y^{(m)}$. We will use on $Y^{(b)}$ natural fibered coordinates $y^\sigma \in \{g^{jk}, y^{i_1...i_p}_{j_1...j_q}\}$, obtained by decomposing the involved tensors with respect to the natural local bases ∂_i, dx^i; also, by $dV_g = \sqrt{|\det g|} d^n x$, we will mean the (formal) invariant Riemannian volume form.

Assume that the matter Lagrangian λ_m is generally covariant, i.e.,

$$\lambda_m = \mathbb{L}_m dV_g,$$

where $\mathbb{L}_m = \mathbb{L}_m(x^i, y^I, g^{jk}, y^I{}_i, g^{jk}{}_i, ...y^I{}_{i_1...i_r}, g^{jk}{}_{i_1...i_r})$ is an invariant scalar; with the notations in the Section 1.1.5, we have: $\lambda_m = \mathcal{L}_m d^n x$, where:

$$\mathcal{L}_m = \mathbb{L}_m \sqrt{|\det g|}. \tag{1.112}$$

- The background component $\mathcal{E}^{(b)}$ of $\mathcal{E}(\lambda_m)$ can be also more conveniently expressed in terms of the invariant volume form, as:

$$\mathcal{E}^{(b)} = \mathfrak{T}_\sigma \theta^\sigma \wedge dV_g, \tag{1.113}$$

  where $\mathfrak{T}_\sigma \in \{\mathfrak{T}_{ij}, \mathfrak{T}^{j_1...j_q}_{i_1 i_2...i_p}\}$ are given by:

$$\mathfrak{T}_\sigma = \frac{\mathcal{E}_\sigma}{\sqrt{|\det g|}} = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta y^\sigma}. \tag{1.114}$$

- Accordingly, the energy-momentum tensor $\mathcal{T}$ will be expressed as:

$$\mathcal{T}(\xi) =: (T^j{}_i \xi^i) \mathbf{i}_{\partial_j} dV_g, \quad \forall \xi \in \mathcal{X}(M).$$

  where the coefficients

$$T^j{}_i = C^{\sigma j}_i \mathfrak{T}_\sigma = \frac{1}{\sqrt{|\det g|}} \mathcal{T}^j_i \tag{1.115}$$

  obey, with respect to arbitrary fibered coordinate changes, a tensor-type transformation rule: $T^{j'}{}_{i'} = \dfrac{\partial x^{j'}}{\partial x^j} \dfrac{\partial x^i}{\partial x^{i'}} T^j{}_i$; this rule as can be immediately checked using (1.106).

- In detail, the coefficients $C^{\sigma j}_i \in \{C^{(jk)h}_i, C^{(i_1...i_p)j}_{(j_1....j_q)i}\}$ (and $C^\sigma_i = 0$) are given by the tensor lifting rule (1.53), which leads to:

$$T^j{}_i = 2g^{jh} \mathfrak{T}_{hi} + (y^{j i_2...i_p}_{j_1...j_q} \mathfrak{T}^{j_1...j_q}_{i i_2...i_p} + ... - y^{i_1...i_p}_{j_1...j_{q-1} i} \mathfrak{T}^{j_1...j_{q-1} j}_{i_1 i_2...i_p}). \tag{1.116}$$

A direct calculation using (1.109) and the relations $d_j \sqrt{|\det g|} = \Gamma^i{}_{ji} \sqrt{|\det g|}$, $g^{jh}{}_i = -(\Gamma^{jh}{}_i + \Gamma^{hj}{}_i)$, (where $\Gamma^i{}_{jk}$ denote the formal Christoffel symbols of g) leads to the following result:

Theorem 13 ***(energy-momentum balance law in general metric-tensor theories):*** *If the background manifold is*

$$Y^{(b)} = Met(M) \times_M T^p_q(M),$$

then, for any natural matter Lagrangian $\lambda_m = \mathcal{L}_m d^n x \in \Omega^r_{n,X}(Y^{(b)} \times_M Y^{(m)})$ *and for any section* $\gamma : M \to Y$ *:*

$$(y^{\sigma}{}_{;i}\mathfrak{T}_{\sigma} + T^j{}_{i;j}) \circ J^{s+2}\gamma \approx_{\gamma^{(m)}} 0, \quad i = 1, ..., n, \tag{1.117}$$

where: $\mathfrak{T}_{\sigma} = \dfrac{1}{\sqrt{|\det g|}} \dfrac{\delta \mathcal{L}_m}{\delta y^{\sigma}}$, $y^{\sigma} = y^{i_1 \dots i_p}_{j_1 \dots j_q}$, *semicolons denote Levi-Civita covariant derivatives and* $\approx_{\gamma^{(m)}}$ *means equality on-shell for the matter field* $\gamma^{(m)}$.

Note. Actually, since $g^{jk}{}_{;i} = 0$, in the above relation, the $Met(M)$ part of the expression $y^{\sigma}{}_{;i}\mathfrak{T}_{\sigma}$ vanishes, i.e., only its $T^p_q M$-component will have a nonzero contribution: $y^{\sigma}{}_{;i}\mathfrak{T}_{\sigma} = y^{i_1 \dots i_p}_{j_1 \dots j_q;i} \mathfrak{T}^{j_1 \dots j_q}_{i_1 \dots i_p}$.

Let us investigate, in the following, the particular cases of metric and metric-affine theories.

Metric-affine theories.

In a metric-affine theory, the background variables are, *a priori*, a metric and an independent linear connection. Yet, once a metric is present, it comes automatically with its Levi-Civita connection; hence, any other connection D can be expressed in terms of *distortion tensors*, whose coefficients

$$N^i{}_{jk} = K^i{}_{jk} - \Gamma^i{}_{jk}, \tag{1.118}$$

give the difference between the coefficients $K^i{}_{jk}$ of D and the Christoffel symbols of the metric. Thus, metric-affine theories are metric-tensor theories, with background manifold $Y^{(b)} = Met(M) \times_M T^1_2(M)$. In natural fibered coordinates by, $\left(x^i, y^{\sigma}\right) =: (x^i, g^{ij}, N^i{}_{jk})$, we find the energy-momentum tensor components T^j_i by (1.116), as:

$$T^j_i = 2\mathfrak{T}^j{}_i + (\mathfrak{T}^{hl}{}_i N^j{}_{hl} - \mathfrak{T}^{jl}{}_m N^m_{il} - \mathfrak{T}^{hj}{}_m N^m_{hi}), \tag{1.119}$$

where:

$$\mathfrak{T}_{ij} = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}, \quad \mathfrak{T}^{jk}{}_i = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta N^i{}_{jk}} = \frac{\delta \mathbb{L}_m}{\delta N^j{}_{kh}}. \tag{1.120}$$

We note that, in metric-affine theories, $T_{ij} := T_{kj} T^k_i$ is generally non-symmetric.

Applying (1.117), we find:

Proposition 14 ***(Energy-momentum balance law in metric-affine theories):*** *In any metric-affine theory with generally covariant matter Lagrangian* $\lambda_m = \mathbb{L}_m \sqrt{|\det g|} d^n x$ *of order* r*, the energy-momentum tensor (1.119) obeys the balance law:*

$$(T^j{}_{i;j} + N^j{}_{kh;i} \frac{\delta \mathbb{L}_m}{\delta N^j{}_{kh}}) \circ J^{s+2}\gamma \approx_{\gamma^{(m)}} 0, \tag{1.121}$$

where semicolons denote Levi-Civita covariant derivatives.

The above relation holds for any matter Lagrangian function $\mathbb{L}_m$, of any order; also, the dimension of the spacetime manifold is irrelevant.

Particular case: purely metric theories. The case when the only background variable is a metric, i.e., $Y^{(b)} = Met(M)$, $y^\sigma = g^{jk}$ and the connection is the Levi-Civita one, can be obtained from the above with $N^i{}_{jk} = 0$. This immediately leads to the following result.

Proposition 15 *1. If the only background variable is a metric tensor g^{ij}, then the energy-momentum tensor $\mathcal{T}$ has the lower-index local components:*

$$T_{ij} = \frac{2}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}. \tag{1.122}$$

2. The energy-momentum balance law (1.117) becomes the usual covariant conservation law:

$$T^j{}_{i;j} \circ J^{s+2}\gamma \approx_{\gamma^{(m)}} 0. \tag{1.123}$$

Remark. Applying the same algorithm to the *Hilbert Lagrangian* $\lambda_g = RdV_g$, the corresponding "energy-momentum tensor" becomes the Einstein tensor $G = G^j{}_i \sqrt{|\det g|} dx^i \otimes \omega_j$ and the covariant conservation law (1.123) gives the contracted Bianchi identities $G^j{}_{i;j} \circ J^{s+2}\gamma = 0$.

1.4 A special property of Lepage equivalents of Lagrangians

This section reproduces with almost identically parts of my joint paper with S. Garoiu and B. Vasian, [198].

1.4.1 Introduction

As already discussed above, Lepage equivalents of a Lagrangian are a higher-order, field theoretical analogue of the Poincaré-Cartan form (1.45) known from mechanics and play the same role: they give rise to a geometric formulation of the calculus of variations. But, whereas in mechanics, the Poincaré-Cartan form is unique, in field theory, any given Lagrangian λ admits multiple Lepage equivalents ρ_λ, exhibiting different features. One of the most desirable such features is the so-called *closure property:*

$$\lambda \text{ is variationally trivial} \quad \Leftrightarrow \quad d\rho_\lambda = 0.$$

Once the closure property is satisfied, all Lagrangians producing the same Euler-Lagrange equations will be characterized by one and the same $d\rho_\lambda$. This property, which was initially motivated by the study of symmetries of the Euler-Lagrange form, see [33], [34], is a very promising one in at least two other directions:

- Geometric formulation of Hamiltonian field theory: given a Lagrangian form λ, a Hamiltonian form H_λ is constructed via the exterior derivative $d\rho_\lambda$ - and generally, it is not guaranteed that Lagrangians that produce the same Euler-Lagrange equations will also lead to the same Hamilton equations. This drawback is eliminated if the mapping $\lambda \mapsto \rho_\lambda$ is $\mathbb{R}$-linear and satisfies the closure property.

- Variational sequences (e.g., [114]), where it offers an elegant characterization of the kernel of the Euler-Lagrange mapping.

The closure property is notoriously obeyed in mechanics by the Poincaré-Cartan form Θ_λ, both in the first order case (1.45) and for higher order Lagrangians. But, in *field theory*, finding Lepage equivalents with the closure property has been for many years an open problem. Actually, to the best of our knowledge, mappings $\lambda \mapsto \rho_\lambda$ obeying it were only known, prior to the paper [198], in some very specific situations:

- First order Lagrangians. In this case, a globally defined Lepage equivalent with the desired feature, called the *fundamental Lepage equivalent* ρ_λ, was introduced by Krupka, [119] and rediscovered by Bethounes, [33]; for first order homogeneous Lagrangians, a similar notion was introduced by Urban and Brajercik, [194].

- Homogeneous Lagrangians with two independent variables; in this case, an extension of the fundamental form ρ_λ was constructed by Saunders and Crampin, [177].

In [198], we proposed a general procedure which solves this problem, at least, locally, for general Lagrangians λ, of any order $r \geq 1$. Our construction relies on a different idea than the first order construction in [119], as it uses as a raw material, the *principal Lepage equivalent* (1.44), which is much simpler; more specifically, it is 1-contact, whereas the fundamental Lepage equivalent ρ_λ has a higher degree of contactness.

The principal Lepage equivalent as it stands, does not generally obey the closure property, but we show that it can be tailored in such a way as to eliminate this drawback, as follows. To any Lagrangian λ over a given fibered chart domain, one can canonically attach the Vainberg-Tonti Lagrangian λ_{VT} of the Euler-Lagrange form of λ, see (1.68). The difference between λ and λ_{VT} is thus a trivial Lagrangian, which can be written, [114], up to pullback by the corresponding jet projections, as

$$\lambda = \lambda_{VT} + hd\alpha, \tag{1.124}$$

where h denotes the horizontalization operator and $d\alpha$ is uniquely determined, via a specific homotopy operator. Using the above decomposition, we prove that

$$\Phi_\lambda := \Theta_{\lambda_{VT}} + d\alpha, \tag{1.125}$$

where $\Theta_{\lambda_{VT}}$ is the principal Lepage equivalent of λ_{VT}, gives a Lepage equivalent of λ (which we call *canonical*), obeying the closure property. The construction is a local one; yet, we show that, for globally defined Lagrangians of order $r \leq 2$ on tensor bundles, having second order Euler-Lagrange equations - which represent most of the cases of interest for physical theories - Φ_λ is actually globally well defined.

A variant of the above construction, which is convenient in the case when λ is locally equivalent to a lower order Lagrangian λ', is to consider in (1.125), instead of the Vainberg-Tonti Lagrangian λ_{VT}, a reduced Lagrangian λ'. This leads, in general, to non-unique Lepage equivalents ϕ_λ, which we will call *reduced*; and, if we can ensure that λ' is truly of minimal order, the obtained reduced Lepage equivalent will still possess the closure property.

In particular, for reducible second order Lagrangians, any reduced Lepage equivalent will be of order 1.

1.4.2 The closure property

In the following, let (Y, π, X) denote an arbitrary fibered manifold. We recall that by $\Omega_k(J^rY)$ we mean the set of all k-forms defined on open subsets of J^rY; yet, whenever necessary, the precise domain of definition of these forms will be indicated explicitly.

A Lagrangian $\lambda \in \Omega_{n,X}(J^rY)$ is called *trivial,* or *null*, if its Euler-Lagrange form $\mathcal{E}_\lambda$ vanishes identically. It is known, e.g., [114], p. 123, that λ is trivial if and only if, for each fibered chart domain $V^r := J^rV$ in the domain of definition of λ, there exists an $(n-1)$-form $\alpha \in \Omega_{n-1}(V^{r-1})$ of order $r-1$ over V, such that:

$$\lambda = hd\alpha, \tag{1.126}$$

i.e., in coordinates, λ is given by a divergence expression (1.14).

A mapping $\rho : \Omega_{n,X}(J^rY) \to \Omega_n(J^sY)$ attaching to any Lagrangian $\lambda \in \Omega_{n,X}(J^rY)$, a Lepage equivalent ρ_λ of some order s, is said, [177], to have the *closure property,* if:

$$\lambda \text{ - } \mathit{trivial} \ \Rightarrow \ d\rho_\lambda = 0. \tag{1.127}$$

Remark. The converse implication: $d\rho_\lambda = 0 \Rightarrow \lambda$ - *trivial*, is true for *any* Lepage equivalent ρ_λ, since $d\rho_\lambda = 0$ implies $\mathcal{E}_\lambda = p_1 d\rho_\lambda = 0$; hence, whenever it holds, (1.127) is actually an equivalence.

A first consequence of the closure property is the following.

Proposition 16 *: If the mapping $\lambda \mapsto \rho_\lambda : \Omega_{n,X}(J^rY) \to \Omega_n(J^sY)$ is $\mathbb{R}$-linear and has the closure property, then, for any two dynamically equivalent Lagrangians $\lambda, \lambda' \in \Omega_{n,X}(J^rY)$:*

$$d\rho_\lambda = d\rho_{\lambda'}. \tag{1.128}$$

Proof. Assuming that the Lagrangians $\lambda, \lambda' \in \Omega_{n,X}(J^rY)$ are equivalent, it follows that the difference $\lambda - \lambda'$ is a trivial Lagrangian, hence $d\rho_{\lambda-\lambda'} = 0$, which, by linearity, implies (1.128). ■

The closure property is a very convenient one for physical applications, as, basically, equality (1.128) says that all Lagrangians describing the same physics will produce the same $d\rho_\lambda$.

Example: The fundamental (Krupka) Lepage equivalent for first order Lagrangians. For $\lambda \in \Omega_{n,X}(J^1Y)$, a globally defined, first order Lepage equivalent possessing the closure property is, [119], [170], [177]:

$$\tilde{\rho}_\lambda = \mathcal{L}d^nx + \sum_{k=1}^{\min\{m,n\}} \frac{1}{(k!)^2} \frac{\partial^k \mathcal{L}}{\partial y^{\sigma_1}_{A_1} ... \partial y^{\sigma_k}_{A_k}} \theta^{\sigma_1} \wedge ... \wedge \theta^{\sigma_k} \wedge \omega_{A_1 ... A_k}; \tag{1.129}$$

the degree of contactness of $\tilde{\rho}_\lambda$ is $\min\{m,n\}$.

1.4.3 Canonical Lepage equivalent

Definition and properties.

Fix a vertically star-shaped fibered coordinate chart (V, ψ) and an arbitrary Lagrangian $\lambda \in \Omega_{n,X}(V^r)$ of order $r \geq 2$ defined on $V^r := J^rV$. As, by definition, λ is a Lagrangian for its own Euler-Lagrange form $\mathcal{E}_\lambda = E_\sigma \theta^\sigma \wedge d^nx$, the Vainberg-Tonti Lagrangian (of order $\leq 2r$):

$$\lambda_{VT} := I\mathcal{E}_\lambda \in \Omega_{n,X}(V^{2r}), \tag{1.130}$$

where I denotes the fibered homotopy operator (1.63), is always equivalent to λ. The difference between λ and λ_{VT} is thus a trivial Lagrangian; more precisely, see Sec. 4.9 of [114], we can write:

$$\left(\pi^{2r,r}\right)^* \lambda = \lambda_{VT} + hd\alpha, \tag{1.131}$$

where:

$$\alpha := I\Theta_\lambda + \left(\pi^{2r-1}\right)^* \mu_0; \tag{1.132}$$

and μ_0 is an $(n-1)$-form on $\pi(V) \subset X$ such that

$$0^*\Theta_\lambda = d\mu_0 \tag{1.133}$$

(μ_0 is guaranteed to exist, as $0^*\Theta_\lambda$ is a form of maximal degree on X).

We will call the Lagrangian λ_{VT}, the Vainberg-Tonti Lagrangian *associated* to λ.

Let us make the following remarks.

1. As Θ_λ is 1-contact, we obtain by (1.67) that α is horizontal; moreover, since Θ_λ is of order $\leq 2r-1$, it follows that $\alpha \in \Omega_{n-1,X}(V^{2r-1})$, its coordinate expression is:

$$\alpha = \alpha^A \omega_A, \qquad \alpha^A = \alpha^A(x^C, y^\sigma{}_C, ..., y^\sigma{}_{C_1 \ldots C_{2r-1}}).$$

2. For a Lagrangian λ of order r, the Euler-Lagrange expressions (1.34) are of order $\leq 2r$, but their dependence on the variables $y^\sigma{}_{A_1 \ldots A_{2r}}$ is, in any fibered chart, at most affine. Hence, the associated Vainberg-Tonti Lagrangian λ_{VT} is also at most affine in $y^\sigma{}_{A_1 \ldots A_{2r}}$. Consequently, using Proposition 6, we find out that the order of its principal Lepage equivalent $\Theta_{\lambda_{VT}}$ does not exceed $4r-2$.

We are now able to prove the following result.

Theorem 17 ***(Canonical Lepage equivalent):*** *Let $\lambda \in \Omega_{n,X}(V^r)$ be an arbitrary Lagrangian of order r over a vertically star-shaped fibered chart domain V^r and $\lambda_{VT} = I\mathcal{E}_\lambda \in \Omega_{n,X}(V^{2r})$, its associated Vainberg-Tonti Lagrangian corresponding to the given chart. Then:*

(i) The differential form $\Phi_\lambda \in \Omega_n(V^{4r-2})$ given by:

$$\Phi_\lambda := \Theta_{\lambda_{VT}} + \left(\pi^{4r-2,2r-1}\right)^* d\alpha, \tag{1.134}$$

where α is given by (1.132)-(1.133), is a Lepage equivalent of λ;

(ii) If λ is a trivial Lagrangian, then $d\Phi_\lambda = 0$.

Proof. *(i)* Write λ as in (1.131). Then, since the horizontalization h is a linear mapping, we have, up to the corresponding jet projections:

$$h\Phi_\lambda = h\Theta_{\lambda_{VT}} + hd\alpha = \lambda_{VT} + hd\alpha = \lambda;$$

moreover, taking the exterior derivative of (1.131), we obtain: $d\Phi_\lambda = d\Theta_{\lambda_{VT}}$, therefore,

$$p_1 d\Phi_\lambda = p_1 d\Theta_{\lambda_{VT}} = \mathcal{E}_{\lambda_{VT}} = \mathcal{E}_\lambda,$$

which proves that Φ_λ is a Lepage equivalent of λ.

(ii) Assuming that λ is trivial, we have $\mathcal{E}_\lambda = 0$, which implies $\lambda_{VT} = 0$ and, accordingly, $\Theta_{\lambda_{VT}} = 0$; as a consequence, $\Phi_\lambda = \left(\pi^{4r-2,2r-1}\right)^* d\alpha$ is locally exact - therefore, closed. ■

We will call the differential form Φ_λ in (1.131)-(1.134), the *canonical* Lepage equivalent of λ.

Remarks.

1. *(Uniqueness of* Φ_λ*)*: Though the $(n-1)$-form μ_0 in (1.133) is not unique, in the expression of Φ_λ, it only appears through

$$d\alpha = dI\Theta_\lambda + \left(\pi^{2r-1}\right)^* d\mu_0 = dI\Theta_\lambda + \left(\pi^{2r-1}\right)^* 0^*\Theta_\lambda$$

which is uniquely defined.

2. *Linearity of* Φ: The mappings I, Θ and $\mathcal{E}$ involved in constructing Φ are all $\mathbb{R}$-linear ones, therefore,

$$\Phi : \Omega_{n,X}(V^r) \mapsto \Omega_n(V^{4r-2}), \quad \lambda \mapsto \Phi_\lambda$$

is also an $\mathbb{R}$-linear mapping. Together with the closure property, this ensures that, for equivalent Lagrangians λ_1, λ_2, we will have $d\Phi_{\lambda_1} = d\Phi_{\lambda_2}$.

Lagrangians admitting globally defined canonical Lepage equivalents. As the above construction heavily relies on quantities that are defined on a specified chart, such as the Vainberg-Tonti Lagrangian and the principal Lepage equivalent, a natural question is whether (or rather, when) could Φ_λ be globally defined. Though a complete answer to this question is out of the scope of this work, here is a result which covers a lot of the situations of interest for physical applications.

Theorem 18 *Assume that Y is a tensor bundle over X and $\lambda \in \Omega_{n,X}(J^rY)$ is a globally defined Lagrangian of order at most 2, having second order Lagrange equations. Then, the canonical Lepage equivalent (1.134) is globally well defined.*

Proof. Let us start by the following remark on the fibered homotopy operator I. In the particular case when (Y, π, X) has a vector bundle structure, the fiber rescalings $\chi_u = \chi(\cdot, u)$, $u \in \mathbb{R}$, given by (1.61) are nothing but the jet prolongations of the fiberwise scalar multiplication on Y, i.e., they make sense globally on J^rY. Accordingly, $\chi : J^rY \times \mathbb{R} \to J^rY$, $(J^r_x\gamma, u) \mapsto \chi_u(J^r_x\gamma)$ is a well defined, smooth mapping. Hence, for any globally defined form $\rho \in \Omega(J^rY)$, $\chi^*\rho$ is also globally defined on $J^rY \times \mathbb{R}$. Further, noticing that, in (1.64), we can actually write $\rho^{(0)} = \mathbf{i}_{\partial_u}(\chi^*\rho)$, we obtain

$$I\rho = \int_0^1 \mathbf{i}_{\partial_u}(\chi^*\rho)du,$$

where, in this case, all the involved operations make sense globally. Therefore, on vector bundles, $I\rho$ is globally defined.

Assume now that λ satisfies the above hypotheses; as λ is globally defined, its Euler-Lagrange form $\mathcal{E}_\lambda$ is also globally defined. Using the above remark, we get that $\lambda_{VT} = I\mathcal{E}_\lambda$ is globally defined

- and, according to our hypothesis, of second order. But, for second order Lagrangians, the principal Lepage equivalent is globally defined, which means that so is $\Theta_{\lambda_{VT}}$.

On the other hand, as the order of λ does not exceed two, Θ_λ is globally well defined. Applying again the above remark on the operator I, we finally get that $d\alpha = dI\Theta_\lambda + \left(\pi^{2r-1}\right)^* 0^*\Theta_\lambda$, is also globally defined. Summing up, we obtain that both terms of Φ_λ are globally defined, which completes the proof. ∎

The above result applies, for instance, to:

- all generally covariant, *first order* Lagrangians on tensor bundles;

- Lovelock gravity, Horndeski theories, metric-affine gravity theories with second order field equations.

The result below gives the difference between the canonical and the principal Lepage equivalent.

Proposition 19 *For a Lagrangian $\lambda = \lambda_{VT} + hd\alpha$ as in (1.131), there holds, up to the corresponding jet projections:*

$$\Phi_\lambda = \Theta_\lambda + (d\alpha - \Theta_{hd\alpha}). \tag{1.135}$$

Proof. From the linearity of Θ, we have: $\Theta_\lambda = \Theta_{\lambda_{VT}} + \Theta_{hd\alpha}$. Adding to both hand sides $d\alpha$ and taking into account that, up to jet projections, $\Phi_\lambda = \Theta_{\lambda_{VT}} + d\alpha$, this leads to (1.135). ∎

Remark. The term $d\alpha - \Theta_{hd\alpha}$ in (1.135) is 1-contact. Therefore, using (1.49), there exists a 1-contact form ν such that, up to the corresponding jet projections:

$$d\alpha - \Theta_{hd\alpha} = p_1 d\nu. \tag{1.136}$$

The precise coordinate expression of ν is calculated, for first order Lagrangians, in [198].

1.4.4 Reduced Lepage equivalents

In the following, we present an alternative construction, which is advantageous in the case when the Lagrangian λ can be order-reduced; for reducibility criteria, see, e.g., [85] [170], [176].

Consider an arbitrary open subset $W \subset Y$ and a Lagrangian $\lambda \in \Omega_{n,X}(W^r)$, where $W^r := J^r W$. Picking any equivalent Lagrangian $\lambda' \in \Omega_{n,X}(W^s)$ to λ of *minimal* order $s \leq r$ over W, we can again write

$$\lambda = (\pi^{r,s})^* \lambda' + hd\alpha, \tag{1.137}$$

for some $\alpha \in \Omega_{n-1}(W^{r-1})$.

In particular, for a trivial Lagrangian λ, minimal order Lagrangians equivalent to λ are π^r-projectable n-forms $\lambda' = f(x^A) d^n x$.

Proposition 20 *Let $\lambda \in \Omega_{n,X}(W^r)$ be an arbitrary Lagrangian and $\lambda' \in \Omega_{n,X}(W^s)$, a dynamically equivalent Lagrangian to λ, of minimal order $s \leq r$. Then:*

(i) The n-form

$$\phi_\lambda := \Theta_{\lambda'} + d\alpha, \tag{1.138}$$

where λ' and α are as in (1.137) and the equality must be understood up to the corresponding jet projections, is a Lepage equivalent of λ.

(ii) If λ is variationally trivial, then any ϕ_λ constructed as above is closed.

Proof. *(i)* The proof is similar to the one of Theorem 17. First, we note that, up to jet projections:

$$h\phi_\lambda = h\Theta_{\lambda'} + hd\alpha = \lambda' + hd\alpha = \lambda;$$

moreover, $d\phi_\lambda = d\Theta_{\lambda'}$ implies $p_1 d\phi_\lambda = p_1 d\Theta_{\lambda'} = \mathcal{E}_{\lambda'} = \mathcal{E}_\lambda$, which is a source form, that is, ϕ_λ is a Lepage equivalent of λ.

(ii) If λ is trivial, then $\lambda' = f(x^A)d^n x$, which gives: $\Theta_{\lambda'} = \lambda'$. But, as λ' is an n-form on X, $\dim X = n$, we find that: $d\phi_\lambda = d\Theta_{\lambda'} = d\lambda' = 0$. ∎

Definition 21 *We will call any Lepage equivalent built as in (1.137)-(1.138), a reduced Lepage equivalent of* λ.

Remarks.

1. The reduced Lagrangian λ' of λ (if it exists) is, generally, not unique. As a consequence, we may obtain multiple reduced Lepage equivalents ϕ_λ for the same Lagrangian. Even so, the multi-valued correspondence $\lambda \mapsto \phi_\lambda$ is $\mathbb{R}$-linear, in the following sense: for any $\lambda_1, \lambda_2 \in \Omega_{n,X}(W^r)$ and $a_1, a_2 \in \mathbb{R}$, if $\phi_{\lambda_1} \in \phi(\lambda_1)$ and $\phi_{\lambda_2} \in \phi(\lambda_2)$, then $a_1\phi_{\lambda_1} + a_2\phi_{\lambda_2}$ belongs to the image $\phi(a_1\lambda_1 + a_2\lambda_2)$.

2. The splitting (1.137) is, generally, only local - therefore, reduced Lepage equivalents are, in general, defined only locally.

In particular, for second order Lagrangians, we obtain:

Proposition 22 *Any reducible second order Lagrangian admits a local first order Lepage equivalent.*

Proof. If $\lambda \in \Omega_{n,X}(W^2)$ is reducible to a first order Lagrangian $\lambda' \in \Omega_{n,X}(W^1)$, the corresponding reduced Lepage equivalent ϕ_λ is of order 1, as both $\Theta_{\lambda'}$ and α are, in this case, of order 1. ∎

Example: the Hilbert Lagrangian. This is a very peculiar example, for which:

$$\Phi_{\lambda_g} = \Theta_{\lambda_g} = \phi_{\lambda_g}, \tag{1.139}$$

where ϕ_{λ_g} corresponds to the famous non-invariant, first order Lagrangian equivalent to λ_g, see, e.g., [126]. In particular, Φ_{λ_g} is of order 1.

To prove this statement, denote, again, by $Y = Met(M)$, the bundle of nondegenerate tensors of type (0,2) over a 4-dimensional manifold M and by $(x^i, g_{ij}; g_{ij,k}; g_{ij,kl})$, the coordinates in a fibered chart on J^2Y. By a quick direct computation similar to the one in Section 1.2, it follows that the Hilbert Lagrangian

$$\lambda_g := \mathcal{R} d^n x, \qquad \mathcal{R} = R\sqrt{|\det g|}$$

(having as its Euler-Lagrange form $\mathcal{E}(\lambda_g) = (R^{ij} - \frac{1}{2}Rg^{ij})\theta_{ij} \wedge d^4x = (R^{ij} - \frac{1}{2}Rg^{ij})dg_{ij} \wedge d^4x)$ coincides with its associated Vainberg-Tonti Lagrangian λ_{VT}, which leads to: $\Phi_{\lambda_g} = \Theta_{\lambda_g}$.

The second equality (1.139) is based on the following remark that Θ_{λ_g} can be locally decomposed, [199], as:

$$\Theta_{\lambda_g} = \Theta_{\lambda'_g} + d\alpha, \tag{1.140}$$

where

$$\lambda'_g = g^{jk}({\Gamma^i}_{jl}{\Gamma^l}_{ki} - {\Gamma^i}_{jk}{\Gamma^l}_{il})\sqrt{|\det g|}d^n x =: \mathcal{L}'_g d^n x \tag{1.141}$$

is the reduced non-invariant Lagrangian for λ_g and $\alpha = ({\Gamma^{ij}}_j - {\Gamma^{ji}}_j)\sqrt{|\det g|}\omega_i$. Therefore, Θ_{λ_g} is also a reduced Lepage equivalent.

The coordinate expression of Θ_{λ_g} in the natural basis $\left\{dx^i, dg_{jk}, dg_{jk,i}\right\}$ of $\Omega(J^1Y)$ is known, [120], as:

$$\begin{aligned}\Theta_{\lambda_g} \quad = \quad & g^{ip}({\Gamma^j}_{ip}{\Gamma^k}_{jk} - {\Gamma^j}_{ik}{\Gamma^k}_{jp})\sqrt{|\det g|}d^n x \\ & + \left(g^{jp}g^{iq} - g^{pq}g^{ij}\right)\sqrt{|\det g|}(dg_{pq,j} + {\Gamma^k}_{pq}dg_{jk}) \wedge \omega_i.\end{aligned}$$

Two more examples of canonical Lepage equivalents: the Klein-Gordon Lagrangian and the Lagrangian of classical electromagnetic field, are discussed in the paper [198]. In the first case, it turns out that $\Phi_\lambda = \Theta_\lambda$, while in the second one, the canonical and the principal Lepage equivalents are different.

Chapter 2

Geometry of Finsler spacetimes

This chapter presents the notion of Finsler spacetime as introduced in my joint paper with C. Pfeifer and M. Hohmann, [97], and points out its peculiarities - with a focus on the differences from the established and in-depth studied case of (smooth, positive definite) Finsler spaces. As the geometry-generating functions of Finsler spacetimes are typically smooth on a smaller set than their positive definite counterparts, these differences are quite often unexpected; hence, one has to proceed with maximum care when extending results from Finsler spaces to Finsler spacetimes. Yet, as we will point out in the following, physical field theories can still be safely built over Finsler spacetimes.

Throughout the chapter, we denote by M a connected, orientable $\mathcal{C}^\infty$-smooth manifold of dimension $n \geq 2$ and by (TM, π_{TM}, M), its tangent bundle; x^i will designate the coordinates of a point $x \in U \subset M$ in a local chart (U, φ) and $(x^i, \dot{x}^i)$, the naturally induced local coordinates of points $(x, \dot{x}) \in TU$, i.e., $\dot{x} = \dot{x}^i \partial_i\rfloor_x$ is the decomposition of the vector $\dot{x} \in T_xM$ in the natural local basis $\{\partial_i\rfloor_x\}$. Whenever there is no risk of confusion, we will omit the indices of the coordinates. Commas $_{,i}$ will mean partial differentiation with respect to the coordinates x^i and dots $_{\cdot i}$ partial differentiation with respect to the fiber coordinates $\dot{x}^i$. The notation

$$\overset{\circ}{TM} = TM \backslash \{0\}$$

will mean the tangent bundle of M without its zero section (the *slit tangent bundle*). Also, pseudo-Riemannian metrics will typically be denoted by a, whereas the notation g will be reserved for properly pseudo-Finslerian metric tensors.

2.1 Definitions and basic geometric objects

This section introduces Finsler spacetimes and the associated notions playing a core role in physical applications: light cones, cones of future-pointing timelike vectors, observer space; also, we briefly review the related geometric objects to be used in the sequel. A special attention is paid to *homogeneity*, which is the key concept ensuring a well-defined notion of arc length in pseudo-Finsler geometry. With the exception of Subsections 2.1.1, 2.1.4 and 2.1.5, which are just a quick review of known results to be used later, the results in this section are obtained in [97] as joint work with C. Pfeifer and M. Hohmann and represent refined versions of results in our older papers [96], [93], [94].

2.1.1 Finsler spaces

Finsler geometry is a generalization of Riemannian one, in the following sense. Whereas a Riemannian space is a manifold equipped with a smoothly varying family of scalar products, a Finsler space is, roughly speaking, a manifold equipped with a smoothly varying family of *norms* of tangent vectors; the origin of this idea goes back to Riemann [169], but it was only systematically investigated 60 years later by Finsler in his thesis [73]. Nowadays, (positive definite) Finsler geometry is an established and active area of study; here is just a very quick introduction.

Definition 23 *[26], A Finsler space is a pair (M, F), where M is a smooth n-dimensional manifold and the Finsler function $F : TM \to [0, \infty)$ has the following properties:*

1. *Regularity: F is smooth on $\overset{\circ}{TM}$ and continuous on TM.*
2. *Positive 1-homogeneity: $F(x, \alpha\dot{x}) = \alpha F(x, \dot{x})$, $\forall \alpha > 0$, $\forall (x, \dot{x}) \in TM$.*
3. *Strong convexity: At any point $(x, \dot{x}) \in \overset{\circ}{TM}$, the bilinear form $g_{(x,\dot{x})} : T_xM \times T_xM \to \mathbb{R}$, $(u, v) \mapsto g_{(x,\dot{x})}(u, v)$ given by:*

$$g_{(x,\dot{x})}(u, w) := \frac{1}{2}\frac{\partial^2 F^2}{\partial t \partial s}(x, \dot{x} + tu + ws) \mid_{t=s=0} \tag{2.1}$$

is positive definite.

Interpretation. In a Finsler space (M, F), each partial function $F_x = F(x, \cdot) : T_xM \to \mathbb{R}$, $x \in M$, which we will call in the following, a *Finsler norm*, is actually, a smooth "almost norm", as it is positive definite, obeys the triangle inequality, but is typically only *positively* homogeneous. Using F, the length of a regular curve $c : [a, b] \to M$ is defined as:

$$l(c) = \int_a^b F(c(t), \frac{dc}{dt}(t))dt. \tag{2.2}$$

The positive 1-homogeneity axiom ensures that $l(c)$ is invariant with respect to orientation-preserving reparametrizations of c.

The third axiom in Definition 23 introduces the *Finsler metric tensor*, which is a mapping $g : \overset{\circ}{TM} \to T_2^0M, \quad (x, \dot{x}) \mapsto g_{(x,\dot{x})}$. In coordinates, it is given by:

$$g_{(x,\dot{x})} = g_{ij}(x, \dot{x})\, dx^i \otimes dx^j, \tag{2.3}$$

where

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2 F^2}{\partial \dot{x}^i \partial \dot{x}^j}(x, \dot{x}); \tag{2.4}$$

the strong convexity requirement is equivalent to the fact that, at any point $(x, \dot{x}) \in \overset{\circ}{TM}$ and in one (and then, in any) local chart around it, the matrix $(g_{ij}(x, \dot{x}))$ is positive definite.

Particular cases.

1. *Riemannian spaces,* with:

$$F(x, \dot{x}) = \sqrt{a_x(\dot{x}, \dot{x})} = \sqrt{a_{ij}(x)\dot{x}^i\dot{x}^j}, \tag{2.5}$$

where a is a Riemannian metric on M. In this case, each Finsler norm F_x arises from a scalar product. In terms of the metric tensor (2.4), Riemannian manifolds are singled out in the class of Finsler manifolds by the fact that

$$g_{ij}(x,\dot{x}) = a_{ij}(x)$$

are, in any local chart, independent of $\dot{x}$.

2. (α, β)-*metrics*, [21], [173]. These are obtained by deforming a given Riemannian metric a on M with the help of a 1-form $b \in \Omega_1(M)$. From 1-homogeneity, one finds that these are always expressible as:

$$F = \alpha\Phi(\frac{\beta}{\alpha}), \tag{2.6}$$

where Φ is a smooth real function and for all $x \in M$, $\dot{x} \in T_xM$:

$$\alpha\left(x,\dot{x}\right) = \sqrt{a_x(\dot{x},\dot{x})}, \qquad \beta\left(x,\dot{x}\right) = b_x\left(\dot{x}\right). \tag{2.7}$$

This class includes as its simplest subclasses:

(i) Randers metrics, see, e.g., [26]*:*

$$F(x,\dot{x}) = \sqrt{a_x(\dot{x},\dot{x})} + b_x\left(\dot{x}\right) \tag{2.8}$$

where, in order to ensure the positive definiteness of (g_{ij}), one must impose the condition $a^{-1}(b,b) \in (0,1)$ (where $a^{-1}(b,b) := a^{ij}b_ib_j$).

(ii) Kropina metrics, see, e.g., [223]*:*

$$F\left(x,\dot{x}\right) = \frac{a_x\left(\dot{x},\dot{x}\right)}{b_x\left(\dot{x}\right)}. \tag{2.9}$$

Kropina metrics are among the Finsler metrics most used in applications; yet, to be honest, they do not completely fit into the above definition, since F cannot be defined on the whole TM. They are, actually, a first example of so-called *conic Finsler metrics*, to be discussed in the next subsection.

The 2-homogeneous Finsler function L. The whole geometry of (M,F) can be just as well characterized in terms of the square

$$L = F^2; \tag{2.10}$$

actually, when passing Lorentzian signature, the 2-homogeneous function L will turn out to be even more convenient, as it will physically interpreted as relativistic *interval.* Using the 2-homogeneity of L and (2.4), one finds, in any local chart:

$$L\left(x,\dot{x}\right) = g_{(x,\dot{x})}\left(\dot{x},\dot{x}\right) = g_{ij}\left(x,\dot{x}\right)\dot{x}^i\dot{x}^j. \tag{2.11}$$

2.1.2 Finsler spacetimes; future directed timelike cones, observer space

A first remark that comes to one's mind is that, even for the simplest nontrivial examples of Finsler spaces, which are Randers spaces, when replacing the positive definite Riemannian metric a with a Lorentzian one, the obtained Finsler function will *not* be smooth for $\dot{x} \in T_xM$ such that $a_x(\dot{x}, \dot{x}) = 0$. Moreover, for another very simple example, which is the Kropina metric, we have seen that the requirement that F should be defined on the entire TM is too strong even in the positive definite case.

Actually, for a lot of applications - including gravity theories - having with F (or L) defined and smooth on a wisely chosen *conic subbundle* of TM is just enough, which will be done below.

A *conic subbundle* of TM (see, e.g., Bejancu&Farran, [30], or Javaloyes&Sanchez, [101]), is a non-empty open submanifold $\mathcal{Q} \subset TM\backslash\{0\}$, with the following properties:

- $\pi_{TM}(\mathcal{Q}) = M$;
- *conic property:* if $(x, \dot{x}) \in \mathcal{Q}$, then, for any $\lambda > 0 : (x, \lambda\dot{x}) \in \mathcal{Q}$.

Any conic subbundle is thus a fibered manifold over M, having as fibers $\mathcal{Q}_x := \mathcal{Q} \cap T_xM$, conic subsets of T_xM, $x \in M$.

The above notion allows one to introduce the notion of pseudo-Finsler space. The definition below is due to Bejancu&Farran, [30].

Definition 24 *A* ***pseudo-Finsler space*** *is a pair* (M, L), *where* M *is a smooth manifold,* $\mathcal{A} \subset \overset{\circ}{TM}$ *is a conic subbundle and* $L : \mathcal{A} \to \mathbb{R}$ *is a smooth function obeying the following conditions:*

1. *Positive 2-homogeneity*: $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall\alpha > 0$, $\forall(x, \dot{x}) \in \mathcal{A}$.
2. *Nondegeneracy:* At any $(x, \dot{x}) \in \mathcal{A}$ and in one (and then, in any) local chart around $(x, \dot{x})$, the Hessian:
$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial\dot{x}^i\partial\dot{x}^j} \tag{2.12}$$
is nondegenerate.

The conic subbundle $\mathcal{A}$, where L is defined, smooth and with nondegenerate Hessian, is called the set of *admissible vectors*. In the following, unless elsewhere specified, we will always consider as $\mathcal{A}$, the maximal set with these properties.

The nondegeneracy condition implies that the matrix (g_{ij}) has constant signature on each connected component of $\mathcal{A}$ - but it might very well fail to have the same signature on all of $\mathcal{A}$.

In a pseudo-Finsler space, the *Finslerian pseudo-norm* $F : \mathcal{A} \to \mathbb{R}_+$, which defines a notion of arc length for curves similarly to (2.2), is given by
$$F := \sqrt{|L|}. \tag{2.13}$$
In turn, F defines L up to a sign:
$$L = \epsilon F^2, \quad \epsilon = sign(L). \tag{2.14}$$

Examples:

- *Finsler spaces* are characterized by the fact that $\mathcal{A} = \overset{\circ}{TM}$ and (g_{ij}) is everywhere positive definite. The case when (g_{ij}) is positive definite on $\mathcal{A}$, but $\mathcal{A}$ is strictly contained in $\overset{\circ}{TM}$, is known under the name of *conic Finsler spaces,* [101].

- *Lorentz-Finsler spaces* are characterized by $q = n - 1$. As we will see below, in Lorentzian signature, the admissible set $\mathcal{A}$ is most often strictly contained in $\overset{\circ}{TM}$.

Finsler spacetimes are a more nuanced version of Lorentz-Finsler spaces; though there is yet no general consensus in literature over their precise definition, most recent versions, e.g., [101], [96], [53], [88], tend to converge to the following: roughly speaking, a Finsler spacetime is a pseudo-Finsler space (M, L) such that $L > 0$ and g has Lorentzian signature on a "large enough" conic subbundle $\mathcal{T}$ of $\overset{\circ}{TM}$; the precise conditions to be imposed to $\mathcal{T}$ are, in principle, meant to ensure the existence of a well defined causal structure.

Definition 25 ***(Finsler spacetimes,*** *[97]): A Finsler spacetime is a 4-dimensional, connected pseudo-Finsler space* (M, L), *with admissible set* $\mathcal{A} \subset \overset{\circ}{TM}$, *obeying the extra condition:*

There exists a connected conic subbundle $\mathcal{T} \subset \mathcal{A}$ *with connected fibers* $\mathcal{T}_x = \mathcal{T} \cap T_xM$, $x \in M$, *such that, on each* $\mathcal{T}_x$:

- $L > 0$;

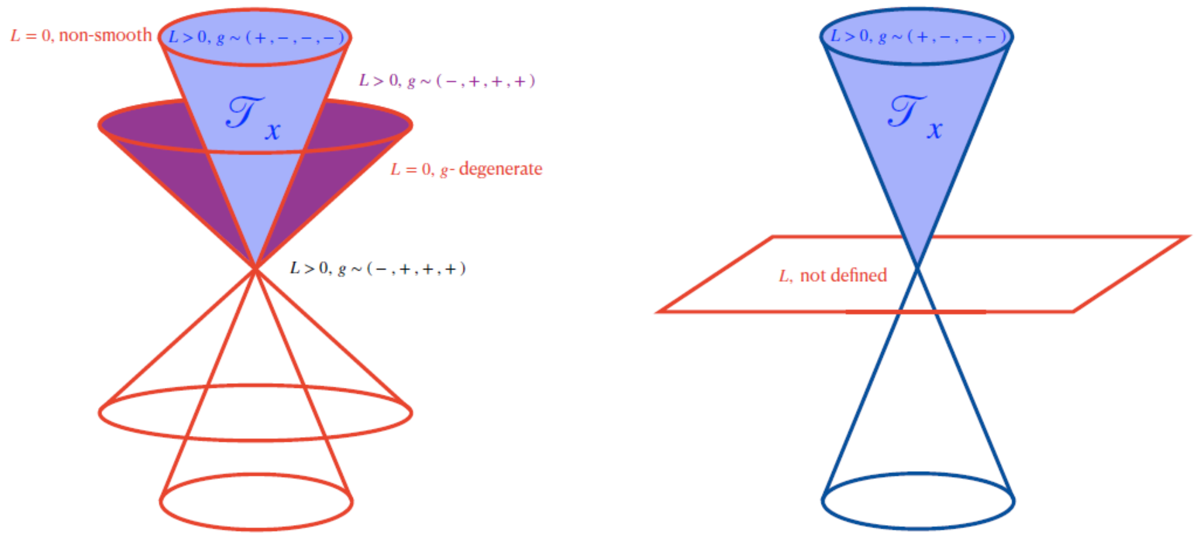
- *g has Lorentzian signature* $(+, -, -, -)$;

- *L can be continuously extended as* 0 *to the boundary* $\partial\mathcal{T}_x$.

The set $\mathcal{T}$ *is called the set of* ***future-pointing timelike vectors.***

Convention. In the following, though we will not specify this explicitly, we will always consider that L is continuously prolonged as 0 on $\partial\mathcal{T}$; in particular, $L(0) = 0$.

The above definition allows for situations like the ones below, where, in red, we have depicted the set $TM\backslash\mathcal{A}$ of non-admissible vectors and $\mathcal{T}_x := \mathcal{T} \cap T_xM$:



Future-pointing timelike cones in a Finsler spacetime

Comparison to other definitions in the literature. Our definition presented above is a slightly more relaxed one than the one of improper Finsler spacetimes in [32], which is recovered for $\mathcal{A} := \mathcal{T}$; having $\mathcal{T}$ just contained in $\mathcal{A}$ allows one to talk about L also outside $\mathcal{T}$, which is useful, e.g., in situations like in the first picture above, corresponding to the physical situation of birefringence - and is characterized in terms of 4-th root metrics (see below for a discussion).

The existence and uniqueness of geodesics with given initial conditions $(x, \dot{x}) \in \bar{\mathcal{T}}$, which was explicitly required in our older definition in [93], follows from the axioms $1. - 3.$ above, see [32], and thus the definition presented here also covers the Finsler spacetimes discussed in [125], [88].

In principle it would be possible to include particular directions in $\mathcal{T}$ that are not in $\mathcal{A}$, but just in $\overline{\mathcal{A}}$, as in [52], [53]. Yet, when deriving the necessary geometric objects for field theory (connections, curvature), one needs smoothness, hence, it will be more convenient to assume that $\mathcal{T} \subset \mathcal{A}$, in order to avoid unnecessary complications in variational procedures involving $\mathcal{T}$.

Timelike vectors and the observer space. Assume, in the following, that (M, L) is a Finsler spacetime; in particular, $\dim M = 4$. An important conic subbundle in a pseudo-Finsler space is the set of *non-null admissible vectors:*

$$\mathcal{A}_0 := \mathcal{A} \setminus L^{-1}\{0\}. \tag{2.15}$$

This is the set where we can divide by L in order to adjust the homogeneity degree of geometric objects in $\dot{x}$.

For the application of Finsler spacetimes in physics, besides the sets of admissible (respectively, non-null admissible) directions $\mathcal{A}$ and $\mathcal{A}_0$, the following subsets of TM play an important role:

1. The conic subbundle $\mathcal{T}$ of future pointing timelike vectors.

2. The *observer space*, or set of *unit* future pointing timelike vectors:

$$\mathcal{O} := \{(x, \dot{x}) \in \mathcal{T} \mid L(x, \dot{x}) = 1\} : \tag{2.16}$$

3. The conic set $L^{-1}(0)$ has the meaning of set of *null* or *lightlike* vectors. By continuously extending L as zero to the boundary $\partial\mathcal{T}$, as specified above, we always have the inclusion

$$\partial\mathcal{T} \subset L^{-1}(0). \tag{2.17}$$

It is important to notice that the null set $L^{-1}(0)$ might not be contained in $\mathcal{A}$, but just in $\overline{\mathcal{A}}$.

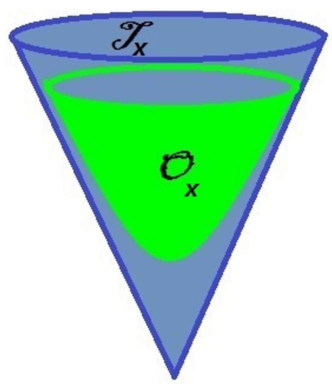
Relations between $\mathcal{O}, \mathcal{T}, \mathcal{A}_0$ and $\mathcal{A}$: From the above definitions, we find the inclusions:

$$\mathcal{O} \subset \mathcal{T} \subset \mathcal{A}_0 \subset \mathcal{A}.$$

Moreover, due to the homogeneity of L, we have at any $x \in M$:

$$\mathcal{T}_x = (0, \infty) \cdot \mathcal{O}_x, \tag{2.18}$$

where $\mathcal{O}_x = \mathcal{O} \cap T_xM$ is the observer space *at the point x*.



Observer space

An immediate result, yet, with quite deep implications, is the following, [97].

Proposition 26 *At each point $x \in M$ of a Finsler spacetime (M, L) :*
(i) The future timelike cone $\mathcal{T}_x$ is an entire connected component of $L^{-1}((0,\infty)) \cap T_xM$.
*(ii) The observer space $\mathcal{O}_x$ is a connected component of the **indicatrix** $I_x := L^{-1}(1) \cap T_xM$.*

Proof. *(i)* holds by virtue of the null boundary condition $\partial\mathcal{T}_x \subset L^{-1}(0) \cap T_xM$ and *(ii)* follows immediately from the connectedness and maximality of $\mathcal{T}_x$. ■

As a consequence of the maximal connectedness of $\mathcal{O}_x$, a result by Beem, [29] ensures that $\mathcal{O}_x$ is a strictly convex hypersurface of T_xM and moreover, the set

$$\mathcal{S}_x := \mathcal{T}_x \cap L^{-1}([1,\infty)) = [1,\infty) \cdot \mathcal{O}_x$$

is also convex. Based on this, we can state, [97]:

Proposition 27 *In a Finsler spacetime as defined above, all future timelike cones $\mathcal{T}_x$, $x \in M$, are convex.*

Proof. Fix $x \in M$ and consider two arbitrary vectors $u, v \in \mathcal{T}_x$. In order to show that the segment $\{w := (1-\alpha)u + \alpha v \mid \alpha \in [0,1]\}$ lies in $\mathcal{T}_x$, we rescale it by $\beta \geq \max(L(u)^{-1/2}, L(v)^{-1/2})$; this way, the endpoints $\beta u, \beta v$ lie in $\mathcal{S}_x$ and, by the convexity of $\mathcal{S}_x$, we find that $\beta w \in \mathcal{S}_x \subset \mathcal{T}_x$. The statement then follows from the conicity of $\mathcal{T}_x$. ∎

The convexity of the cones $\mathcal{T}_x$ allows one to prove (see Section 2.4 below) a Finslerian version of *reverse Cauchy-Schwarz inequality* and a *reverse triangle inequality* - which are important in discussing causal relations.

2.1.3 Examples of Finsler spacetimes

Here are some classes that are allowed by Definition 25.

1. Lorentzian spacetimes: If $a : M \to T_2^0 M$, $x \mapsto a_x = a_{ij}(x) dx^i \otimes dx^j$ is a Lorentzian metric on M, then

$$L(x, \dot{x}) = a_x(\dot{x}, \dot{x}) = a_{ij}(x)\dot{x}^i \dot{x}^j \tag{2.19}$$

defines a Finsler spacetime function that is smooth on the entire TM.

In a Lorentzian space (M, a), a tangent vector $\dot{x} \in T_x M$ is called *timelike,* if $a_x(\dot{x}, \dot{x}) > 0$, *lightlike if* $a_x(\dot{x}, \dot{x}) = 0$, *causal* if $a_x(\dot{x}, \dot{x}) \geq 0$, respectively, *spacelike*, if $a_x(\dot{x}, \dot{x}) < 0$. The future-pointing timelike set $\mathcal{T}_x$ is typically chosen by specifying a *time orientation*, which is an everywhere timelike vector field $\mathbf{t}$; more precisely, one declares as $\mathcal{T}_x$, the connected component of the set $\{\dot{x} \in T_x M \mid a_x(\dot{x}, \dot{x}) > 0\}$ containing the vector $\mathbf{t}_x$.

2. Randers spacetimes: $L = \epsilon F^2$, with $\epsilon = sign(F)$, where:

$$F(x, \dot{x}) = \sqrt{|a_x(\dot{x}, \dot{x})|} + b_x(\dot{x}).$$

Here, a is a Lorentzian metric and $b = b_i dx^i$ is a 1-form on M. In our paper [93] (using a more restrictive definition of Finsler spacetimes), we proved that, if $a^{-1}(b, b) \in (0, 1)$, then L defines a Finsler spacetime structure; that is, it will also satisfy the more relaxed axioms presented above.

In physics, Randers spaces are employed to study the motion of an electrically charged particle in an electromagnetic field, the propagation of light in static spacetimes [221], Lorentz violating field theories from the standard model extension, [111], [178], [179] and Finsler gravitational waves [89]. Recently also spinors have been constructed on Randers geometries, [190].

3. Bogoslovsky/Kropina type:

$$L(x, \dot{x}) = \epsilon(|a_x(\dot{x}, \dot{x})|)^{1-q} |b_x(\dot{x})|^{2q} \tag{2.20}$$

[36], [113], where $q \in \mathbb{R}$, a is a Lorentzian metric as above, $b = b_i dx^i$ is a 1-form on M and $\epsilon \in \{-1, 1\}$ is a sign, e.g., $\epsilon = sign(a_x(\dot{x}, \dot{x}))$; the conditions upon the 1-form b, such that F defines a spacetime structure depend on the value of q, see our paper [93] for a discussion.

In physics, these have been used in approaches to quantum field theories and modifications of general relativity, known under the name of *very special/very general relativity*; their main feature is that each $L(x, \cdot) : T_x M \to \mathbb{R}$ is only invariant under a specific subgroup of the Lorentz group, [61], [67], [81], [77], [78].

4. Kundt spacetimes, [162]:

$$L(x,\dot{x}) = a_x(\dot{x},\dot{x})s^{-p}(k+m\ s)^{p+1}\,, \tag{2.21}$$

where $k,p,m \in \mathbb{R}$ are arbitrary constants, a, b are as above and $s := \dfrac{(b_x(\dot{x}))^2}{|a_x(\dot{x},\dot{x})|}$ The causal properties of this Finsler Lagrangian are discussed in Appendix B of the paper [76].

5. Polynomial (m-th root) type*:*

$$L\left(x,\dot{x}\right) = \epsilon|a_{i_1\cdots i_m}(x)\dot{x}^{i_1}\dots\dot{x}^{i_m}|^{\frac{2}{m}}, \tag{2.22}$$

where $\epsilon = sign\left(a_{i_1\cdots i_m}(x)\dot{x}^{i_1}\dots\dot{x}^{i_m}\right)$. In particular, for $m=4$, one obtains *quartic metrics,* which appear in physics, for example, in the description of propagation in birefringent media, in the context of premetric electrodynamics, and the minimal standard model extension, [158], [171], [178], [86].

To check the axioms in Definition 25 for m-th root metrics, pick any conic subbundle $\mathcal{T} \subset \mathcal{A}$ such that $L > 0$ on $\mathcal{T}$. On such a subbundle, one can define the polynomial function in $\dot{x}$:

$$H := L^{m/2} = F^m. \tag{2.23}$$

The Hessian of $(H_{\cdot i\cdot j})$ is related to g, [160], [39], by the rule $H_{\cdot i\cdot j} = mF^{m-2}[g_{ij} + (m-2)F_{\cdot i}F_{\cdot j}]$ which easily leads (see the Appendix of our paper [140] for a proof) to the implication:

$$(H_{\cdot i\cdot j})\ \text{- Lorentzian at}\ (x,\dot{x}) \Rightarrow \quad (g_{ij}\left(x,\dot{x}\right))\ \text{is Lorentzian.} \tag{2.24}$$

That is: if $(H_{\cdot i\cdot j})$ is Lorentzian on $\mathcal{T}$ and $\mathcal{T}$ obeys the extra conditions: connectedness of fibers, $L|_{\partial\mathcal{T}} = 0$, then the considered pseudo-Finsler space (M,L) is a Finsler spacetime.

6. Anisotropic conformal transformations of Lorentzian metrics a:

$$L\left(x,\dot{x}\right) = e^{2\sigma(x,\dot{x})}a_x\left(\dot{x},\dot{x}\right), \tag{2.25}$$

which have been studied in the context of an extension of the Ehlers-Pirani-Schild axiomatic to Finsler geometry [184], [185]. The light cones of L are the same as those of a.

7. General first order perturbations:

$$L = a_x\left(\dot{x},\dot{x}\right) + 2\varepsilon h(x,\dot{x}), \tag{2.26}$$

(where $\varepsilon^2 \simeq 0$) of pseudo-Riemannian metrics a; in physics, these are often used in the study of the physical phenomenology of Planck scale modified dispersion relations [4], [131], [166]. In particular, if the 2-homogeneous function h is smooth on $\overset{\circ}{TM}$, then L is smooth on $\overset{\circ}{TM}$.

Examples 2-4 above belong to the more general class of (α,β)*-metric spacetimes.*

2.1.4 Typical Finslerian geometric objects

Here, we briefly review the typical Finslerian objects to be used in the following sections; on pseudo-Finsler spaces (M, L) and, in particular, on Finsler spacetimes, these are obtained similarly to the corresponding objects in positive definite Finsler spaces, see, e.g., [26, 60, 46], just taking care that we have to restrict them to $\mathcal{A}$ or, if necessary, to $\mathcal{A}_0 = \mathcal{A}\backslash L^{-1}(0)$.

As further, in Chapter 3, we will explicitly need the *coordinate expressions* of these geometric objects, we will present, for brevity, directly these expressions; for their coordinate-free definitions, we mainly refer to the book by Bucătaru and Miron, [46].

Hilbert form and Finsler metric tensor. On a pseudo-Finsler space (M, L) the *Hilbert form* $\omega : \mathcal{A}_0 \to T_1^0 M$ and the *Finslerian metric tensor* $g : \mathcal{A} \to T_2^0 M$ are expressed, in every manifold induced local coordinate chart, as

$$\omega_{(x,\dot{x})} := F_{\cdot i}(x, \dot{x})dx^i \,, \qquad F_{\cdot i} = \epsilon \frac{g_{ij}\dot{x}^j}{F}, \tag{2.27}$$

$$g_{(x,\dot{x})} := g_{ij}(x, \dot{x})dx^i \otimes dx^j \,, \qquad g_{ij} := \frac{1}{2} L_{\cdot i \cdot j}, \tag{2.28}$$

where $\epsilon = sign(L)$ and $F = \sqrt{|L|}$. We note that the Hilbert form ω is only defined on $\mathcal{A}_0 = \mathcal{A}\backslash L^{-1}(0)$, as it involves derivatives of $F = \sqrt{|L|}$ - which are not defined at points where $L = 0$.

Arc length, geodesics and canonical nonlinear connection. A curve $c : [a, b] \to M$ is called *admissible* if all its tangent vectors are in $\mathcal{A}$. The arc length of a regular admissible curve $c : t \in [a, b] \mapsto c(t)$ on M is calculated as

$$l(c) = \int_a^b F(c(t), \dot{c}(t))dt, \tag{2.29}$$

where $\dot{c}(t) = \dfrac{dc}{dt}(t)$. If, moreover, $\dot{c}(t)$ is nowhere lightlike, i.e., $(c(t), \dot{c}(t)) \in \mathcal{A}_0$ for all t, then $l(c)$ can also be expressed in terms of the Hilbert form as:

$$l(c) = \int_a^b C^*\omega = \int_{ImC} \omega, \tag{2.30}$$

where $C : [a, b] \to TM$, $t \mapsto (c(t), \dot{c}(t))$ denotes the *natural lift* of c to TM.

Proposition 28 *, see, e.g., [46]: Critical points of the length functional* (2.29)*, called **Finsler geodesics**, are characterized, in the arc length parametrization $c : s \mapsto \left(x^i(s)\right)$, by the equations:*

$$\ddot{x}^i(s) + 2G^i(x(s), \dot{x}(s)) = 0, \tag{2.31}$$

where $\dot{x}^i(s) = \dfrac{dx^i}{ds}(s)$; the geodesic coefficients are well defined at all points $(x, \dot{x}) \in \mathcal{A}$ and given, in any coordinate chart, by

$$2G^i(x, \dot{x}) = \frac{1}{2} g^{ih}(L_{\cdot h,j}\dot{x}^j - L_{,h})\,. \tag{2.32}$$

A *nonlinear connection* will be understood as a connection on the fibered manifold $\mathcal{A}$ in the sense of [79, pp. 30-32], i.e., it is characterized by the existence of a vector subbundle $H\mathcal{A}$, called the *horizontal subbundle*, of the tangent bundle $T\mathcal{A}$, such that:

$$T\mathcal{A} = H\mathcal{A} \oplus V\mathcal{A},$$

where $V\mathcal{A} = \ker d(\pi_{TM}|_{\mathcal{A}})$ is the *vertical subbundle* of $T\mathcal{A}$, locally generated by the vectors $\dot{\partial}_i := \dfrac{\partial}{\partial \dot{x}^i}$. Such a splitting gives rise to a *local adapted basis* $(\delta_i, \dot{\partial}_i)$ of $T\mathcal{A}$, where the vectors:

$$\delta_i := \frac{\partial}{\partial x^i} - G^j{}_i \frac{\partial}{\partial \dot{x}^j} \tag{2.33}$$

locally span $H\mathcal{A}$ and, accordingly, to the dual basis $(dx^i, \delta\dot{x}^i := d\dot{x}^i + G^i{}_j dx^j)$; the locally defined functions $G^j{}_i = G^j{}_i(x, \dot{x})$ are called the local coefficients of the connection.

For pseudo-Finsler spaces, a canonical choice is the *Cartan nonlinear connection* N, given by:

$$G^i{}_j := G^i{}_{\cdot j}\,, \tag{2.34}$$

Arc-length parametrized geodesics of the Finsler spacetime (M, L) are autoparallel curves of the canonical nonlinear connection. This is equivalent to the fact that an admissible curve $c : [a, b] \to M$, $s \mapsto c(s)$ parametrized by arc length is a geodesic if and only if its natural lift C has everywhere horizontal velocity vector $\dot{C}(s) = \dot{x}^i(s)\delta_i$.

We denote by $\mathfrak{h}$ and $\mathfrak{v}$ the horizontal and, accordingly, the vertical projector determined by the canonical nonlinear connection; that is, for any vector $X \in T\mathcal{A}$, locally written as $X = X^i\delta_i + Y^i\dot{\partial}_i$ we will have:

$$\mathfrak{h}X = X^i\delta_i, \qquad \mathfrak{v}X = Y^i\dot{\partial}_i. \tag{2.35}$$

Any admissible vector field $V = V^i\partial_i \in \Gamma(\mathcal{A})$ can then be lifted to TM, either into a horizontal vector field as $V \mapsto V^h := (V^i \circ \pi_{TM|\mathcal{A}})\delta_i$ or into a vertical one, as: $V \mapsto V^v := (V^i \circ \pi_{TM|\mathcal{A}})\dot{\partial}_i$.

The *dynamical covariant derivative,* [45], [46]*,* determined by the canonical nonlinear connection can be regarded as an $\mathbb{R}$-linear map[1] $\nabla : \mathcal{X}(\mathcal{A}) \to \mathcal{X}(\mathcal{A})$*,* acting on horizontal, respectively, vertical vector fields as:

$$\nabla(X^i\delta_i) = (\dot{x}^j\delta_j X^i + G^i{}_j X^j)\delta_i, \qquad \nabla(Y^i\dot{\partial}_i) = (\dot{x}^j\delta_j Y^i + G^i{}_j Y^j)\dot{\partial}_i. \tag{2.36}$$

Nonlinear curvature tensor and Finsler Ricci scalar. The curvature tensor $\mathbf{R} = R^i{}_{jk} dx^j \wedge dx^k \otimes \dot{\partial}_i$ of the canonical nonlinear connection of (M, L) is a tensor on TM, with local components $R^i{}_{jk} := \delta\dot{x}^i([\delta_j, \delta_k])$ given by:

$$R^i{}_{jk} = \delta_k G^i{}_j - \delta_j G^i{}_k. \tag{2.37}$$

Geodesic deviations are characterized in terms of the canonical nonlinear connection as:

$$(\nabla\nabla V^h)_{|(c,\dot{c})} = \mathbf{R}(\dot{c}^h, V^h), \tag{2.38}$$

[1] In [46], ∇ was introduced as an operator acting on vertical vector fields only; yet, it can be also thought of as acting on horizontal ones.

where $V \in \mathcal{X}(M)$ is the corresponding deviation vector field and the symbol $_{|(c,\dot{c})}$ means that the dynamical covariant derivative ∇ is calculated along the natural lift $C = (c, \frac{dc}{ds})$ to TM of a geodesic $c : [a, b] \to M$.

The trace of the geodesic deviation operator $V \mapsto \mathbf{R}\left(\dot{c}^h, V^h\right)$ gives the so-called *Finsler-Ricci scalar*[2] R_0; on a Finsler spacetime, it makes sense on $\mathcal{A}_0$ and is given by

$$R_0 = \frac{1}{L} R^i{}_{ik} \dot{x}^k. \tag{2.39}$$

Besides the canonical nonlinear connection, it is possible to additionally introduce on $\mathcal{A}$ several *linear* connections (e.g., Chern-Rund, Berwald, Cartan, see [46]), which preserve the distributions generated by the canonical nonlinear connection N. In this work we will pick, for simplicity, one of these linear connections as an auxiliary tool to ensure that all obtained objects are well defined tensors. Our particular choice of the linear connection is, yet, unessential, as both the results in this chapter and our construction in Chapter 3 are independent of the typical Finslerian linear connections that one may use.

Chern-Rund linear connection. The Chern-Rund linear covariant derivative on a Finsler spacetime (M, L), defined on $\mathcal{A} \subset TM$, is locally given by the relations

$$\mathrm{D}_{\delta_k} \delta_j = \Gamma^i{}_{jk} \delta_i, \quad \mathrm{D}_{\delta_k} \dot{\partial}_j = \Gamma^i{}_{jk} \dot{\partial}_i, \quad \mathrm{D}_{\dot{\partial}_k} \delta_j = \mathrm{D}_{\dot{\partial}_k} \dot{\partial}_j = 0\,, \tag{2.40}$$

where

$$\Gamma^i{}_{jk} := \frac{1}{2} g^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}). \tag{2.41}$$

We denote by $_{|i}$ D-covariant differentiation with respect to δ_i.

In any local chart, there holds $\dot{x}^i \Gamma^k{}_{ji} = G^k{}_j$; as a consequence, the dynamical covariant derivative ∇ determined by the nonlinear canonical connection N can be conveniently expressed in terms of D-covariant derivatives as:

$$\nabla = \dot{x}^i \mathrm{D}_{\delta_i} = \nabla_{\dot{x}^i \delta_i}. \tag{2.42}$$

Relation (2.42) remains valid also when using other typical Finslerian connections (Berwald, Cartan) and is intuitively interpreted as follows. Since tangent vectors to natural lifts of geodesics c of (M, L) are horizontal, i.e., of the form $\dot{x}^i \delta_i$, ∇ actually measures the rate of change of tensors under parallel transport along (lifted) geodesics of M.

Cartan tensor and Landsberg tensor. The *Cartan tensor* $C : \mathcal{A} \to T_3^0 M$ is a measure of how "non-Riemannian" a (pseudo-)Finsler structure is; it is given in coordinates by:

$$C_{(x,\dot{x})} = C_{ijk}\left(x, \dot{x}\right) dx^i \otimes dx^j \otimes dx^k, \qquad C_{ijk} = \frac{1}{2} g_{ij \cdot k}. \tag{2.43}$$

Indeed, $C = 0$ is equivalent to the fact that g_{ij} depends on x only, i.e., it is pseudo-Riemannian.

The Landsberg tensor can be defined as $P = \nabla C$ - which would thus give a mapping from $\mathcal{A}$ to $T_3^0 M$. It is more customary, yet, to introduce it with one index raised, i.e., as a mapping $P : \mathcal{A} \to T_2^1 M$, $P = P^i{}_{jk} dx^j \otimes dx^k \otimes \partial_i$; in coordinates:

$$P^i{}_{jk} = g^{mi} \nabla C_{mjk} = G^i{}_{\cdot j \cdot k} - \Gamma^i{}_{jk}\,, \tag{2.44}$$

[2] This is equal to minus the Finsler-Ricci scalar denoted by *Ric* in [26].

Its trace $tr(P) = P_i dx^i$ has the components

$$P_i = P^j{}_{ij} = \nabla C_i, \tag{2.45}$$

where $C_i := g^{jk} C_{ijk}$ are the local coefficients of the trace of the Cartan tensor.

The Finsler spacetime (M, L) is called *of Landsberg type* if $P = 0$ and *weakly Landsberg*, if $tr(P) = 0$.

Finally, the following identities will also be useful when we consider action integrals and calculus of variations on Finsler spacetimes:

$$\delta_i L = L_{|i} = 0, \quad g_{ij|k} = 0, \quad \dot{x}^i_{|j} = 0, \tag{2.46}$$

$$\nabla L = 0, \qquad \nabla g_{ij} = 0, \qquad \nabla \dot{x}^i = 0, \tag{2.47}$$

$$P^i{}_{jk} \dot{x}^k = 0, \quad P_i \dot{x}^i = 0. \tag{2.48}$$

They can all be proven by using the homogeneity properties of the tensors involved and the definition of the canonical nonlinear connection in terms of the Finsler Lagrangian.

2.1.5 Lorentzian, flat and Berwald spacetimes

Lorentzian spaces. Pseudo-Riemannian (in particular, Lorentzian) manifolds (M, a) correspond to *quadratic* Finsler functions

$$L(x, \dot{x}) = a_{ij}(x) \dot{x}^i \dot{x}^j; \tag{2.49}$$

in this case, the canonical nonlinear connection coefficients and its curvature are expressed as:

$$G^i{}_j = \gamma^i{}_{jk}(x) \dot{x}^k, \quad R^i{}_{jk} = r_j{}^i{}_{kl} \dot{x}^l, \tag{2.50}$$

where we have denoted by lowercase letters the geometric objects specific to pseudo-Riemannian geometry.

Caveat: The Finsler-Ricci scalar $R_0 = L^{-1} R^i{}_{ik} \dot{x}^k$ (which explicitly depends on $\dot{x}$) does *not* coincide with the usual Riemannian Ricci scalar $r = g^{ij} r_i{}^k{}_{jk}$; the relation between these two scalars is:

$$g^{ij} (L R_0)_{\cdot ij} = -2r. \tag{2.51}$$

Flat Finsler spacetimes. We will call a Finsler spacetime (M, L) *flat*, if around any $(x, \dot{x}) \in \mathcal{A}$, there exists a local chart in which $L = L(\dot{x})$ depends on $\dot{x}$ only. Picking such a coordinate chart, we get:

$$G^i = 0,$$

which entails $G^i{}_j = 0$, $R^i{}_{jk} = 0$, $R = 0$. Also, $\Gamma^i{}_{jk} = 0$.

In the literature on positive definite Finsler spaces, flat spaces are most often called *locally Minkowski spaces.* We will, still, prefer to avoid this terminology, as, especially when passing to Lorentzian signature, it can lead to confusions with the Minkowski metric $\eta = diag(1, -1, ..., -1)$ on $\mathbb{R}^n$ or to its associated pseudo-Finsler function $F : \mathbb{R}^n \to \mathbb{R}$, $F(\dot{x}) = \sqrt{|\eta_{ij} \dot{x}^i \dot{x}^j|}$, which is just a very particular case. The term *flat* used here is justified by analogy with the Riemannian case ([128], p. 119), as, in this case, the canonical nonlinear connection has identically vanishing curvature.

Berwald spacetimes. Pseudo-Finsler spaces of Berwald type include as subclasses pseudo-Riemannian ones and flat ones. They are defined by the fact that its geodesic spray coefficients are, in any local chart, quadratic in $\dot{x}$:

$$G^i(x,\dot{x}) = \frac{1}{2}G^i{}_{jk}(x)\dot{x}^j\dot{x}^k\,. \tag{2.52}$$

The second $\dot{x}$-derivatives $G^i_{\cdot j\cdot k} = G^i{}_{jk}$ coincide, in this case, with the Chern-Rund connection coefficients:

$$\Gamma^i{}_{jk}(x,\dot{x}) = G^i{}_{jk}(x), \tag{2.53}$$

which, thus, depend on x only. In particular, the Chern-Rund connection of a Berwald space can be projected onto a well defined connection on M, called simply the *affine connection* of (M,L).

The above equalities imply $P^i{}_{jk} = 0$, i.e., all Berwald spaces are Landsberg spaces.

2.1.6 Homogeneity of geometric objects on $\overset{\circ}{TM}$

This subsection is devoted to positively homogeneous geometric objects defined on conic subbundles $\mathcal{Q} \subset \overset{\circ}{TM}$. The results, proven by us in [97], are fairly straightforward extensions of the results by Bucataru and Miron, [46] and Szilasi, [183], referring to objects defined on the whole slit tangent bundle $\overset{\circ}{TM}$. Yet, as these results will be essential in the following, we chose to present them in quite some detail.

We recall that homogeneity is a key concept in pseudo-Finslerian geometry, as the positive homogeneity of L is precisely the property ensuring that the arc length (2.29) of a curve is invariant under orientation-preserving reparametrizations. But, the positive homogeneity of L in $\dot{x}$ entails the positive homogeneity of some degree of all typical Finslerian geometric objects.

Let, for the rest of this section, M denote an arbitrary n-dimensional manifold, with no additional structure assumed.

Definition 29 (Fiber homotheties) *The mappings*

$$\chi_\alpha : \overset{\circ}{TM} \to \overset{\circ}{TM}, \qquad \chi_\alpha(x,\dot{x}) = (x,\alpha\dot{x})\,, \qquad \alpha > 0, \tag{2.54}$$

are called fiber homotheties on $\overset{\circ}{TM}$.

Fiber homotheties form a 1-parameter group of strict automorphisms of the fibered manifold $(\overset{\circ}{TM}, \pi_{TM}, M)$, isomorphic to $(\mathbb{R}^*_+, \cdot)$ and generated by the *Liouville vector field*

$$\mathbb{C} = \dot{x}^i\dot{\partial}_i. \tag{2.55}$$

The corresponding group action is given by the mapping:

$$\chi : \overset{\circ}{TM} \times \mathbb{R}^*_+ \to \overset{\circ}{TM}, \quad \chi\left((x,\dot{x}),\alpha\right) = \chi_\alpha(x,\dot{x}). \tag{2.56}$$

Definition 30 (Homogeneous tensor fields on TM) , *[97] Let T be a tensor field over a conic subbundle $\mathcal{Q} \subset \overset{\circ}{T}M$. T is called positively homogeneous of degree $k \in \mathbb{R}$ in $\dot{x}$, or simply, k-homogeneous, if, for all $\alpha > 0$, its pullback by the restriction $\chi_\alpha : \mathcal{Q} \to \mathcal{Q}$ satisfies*

$$\chi_\alpha^* T = \alpha^k T. \tag{2.57}$$

In particular, 0-homogeneity is synonimous with invariance under the fiber rescalings χ_α, $\alpha > 0$, i.e., with invariance under the flow of $\mathbb{C}$.

Note: In [46], k-homogeneity of *vector fields* is defined differently (it is, in our terms $(k+1)$-homogeneity). The reason for our convention on the homogeneity degree is that it allows a unitary treatment of tensors of any rank.

The result below extends to tensors of arbitrary type a result proven in the book by Szilasi [183, Lemma 4.2.9] for scalar functions and vector fields.

Theorem 31 *A tensor field T over $\mathcal{Q}$ is positively homogeneous of degree $k \in \mathbb{R}$ if and only if*

$$\mathfrak{L}_{\mathbb{C}} T = kT \tag{2.58}$$

Proof. In the following, it will be more convenient to reinterpret the multiplicative 1-parameter group $\{\chi_\alpha\}$ as the additive group $(\mathbb{R}, +)$, by setting $t := \log(\alpha) \in \mathbb{R}$ and

$$\phi_t(x, \dot{x}) := (x, e^t \dot{x}) = \chi_\alpha(x, \dot{x}).$$

for all $(x, \dot{x}) \in \overset{\circ}{T}M$.

$\to$: Assume, first, that T is k-homogeneous, i.e.: $\phi_t^* T = e^{kt} T$. Then,

$$\mathfrak{L}_{\mathbb{C}} T = \left. \frac{d}{dt}(\phi_t^* T) \right|_{t=0} = \left. \frac{d}{dt}(e^{kt} T) \right|_{t=0} = kT.$$

$\leftarrow$: Conversely, assume (2.58) holds. Differentiating the identity $\phi_t^* \phi_\varepsilon^* T = \phi_{t+\varepsilon}^* T$ with respect to ε at $\varepsilon = 0$, one finds, for all t:

$$\phi_t^* \mathfrak{L}_{\mathbb{C}} T = \frac{d}{dt}(\phi_t^* T). \tag{2.59}$$

Using (2.58), this leads to the differential equation $\frac{d}{dt}(\phi_t^* T) = k \phi_t^* T$ in the unknown $\phi_t^* T$. Integrating this equation with the initial condition $\phi_0^* T = T$, we find $\phi_t^* T = e^{kt} T$, which, reverting to the old notation, is precisely $\chi_\alpha^* T = \alpha^k T$. ■

Examples: Using (2.58), we find that:

1. The Liouville vector field $\mathbb{C}$ is 0-homogeneous, since $\mathfrak{L}_{\mathbb{C}} \mathbb{C} = [\mathbb{C}, \mathbb{C}] = 0$.
2. The vertical local basis vectors $\dot{\partial}_i$ are (-1)-homogeneous, as $[\mathbb{C}, \dot{\partial}_i] = -\dot{\partial}_i$.

Definition 32 (Homogeneous nonlinear connection) *A nonlinear connection $T\mathcal{Q} = H\mathcal{Q} \oplus V\mathcal{Q}$ on the conic subbundle $\mathcal{Q} \subset \overset{\circ}{TM}$, is called homogeneous, if fiber homotheties preserve the horizontal subbundle, i.e., $(\chi_\alpha)_* X \in H\mathcal{Q}$ for all $\alpha \in \mathbb{R}$ and all $X \in H\mathcal{Q}$.*

The above condition is a nontrivial one; in coordinates, homogeneity of a nonlinear connection is characterized (see, e.g., the book by Bucataru and Miron, [46]) by that fact its local coefficients (2.33) are 1-homogeneous functions in $\dot{x}$. A standard example of a homogeneous nonlinear connection is the canonical nonlinear connection (2.34) of a Finsler space.

Homogeneous anisotropic tensors. Homogeneous d-tensors.

Almost all Finsler geometric objects discussed in the previous section are *anisotropic tensor fields,* which thus deserve a special mentioning. These can be mapped into specific tensor fields on the tangent bundle, called *distinguished tensor fields*, or *d-tensor fields*; for the latter, homogeneity can be discussed in a natural manner.

Definition 33 *, [101]: An anisotropic tensor field on the conic subbundle $\mathcal{Q} \subset \overset{\circ}{TM}$ is a section of the pullback bundle $\pi^*_{TM|\mathcal{Q}}(\mathcal{T}^p_q M)$, i.e. , a smooth mapping:*

$$T : \mathcal{Q} \to \mathcal{T}^p_q M, \ \ (x, \dot{x}) \mapsto T_{(x,\dot{x})};$$

i.e., for any $(x, \dot{x}) \in \mathcal{Q}$, $T_{(x,\dot{x})}$ is a tensor on M, based at $x = \pi_{TM}(x, \dot{x})$.

Consequently, an anisotropic tensor field will be locally expressed as: $T_{(x,\dot{x})} = T^{i_1...i_p}_{j_1...j_q}(x, \dot{x}) (\partial_{i_1} \otimes ... \otimes \partial_{i_p} \otimes dx^{j_1} \otimes ... \otimes dx^{j_q})|_x$.

On the other hand, in the presence of a nonlinear connection on TM, with horizontal and, respectively, vertical projectors $\mathfrak{h}, \mathfrak{v}$ as in (2.35), the following definition makes sense.

Definition 34 *, [46]: A d-tensor field on a conic subbundle $\mathcal{Q} \subset \overset{\circ}{TM}$ (regarded as a manifold) is a tensor field $T \in \mathcal{T}^p_q(\mathcal{Q})$, obeying the condition:*

$$T(\omega_1, ..., \omega_p, V_1, ..., V_q) = T(\varepsilon_1\omega_1, ...\varepsilon_p\omega_p, \varepsilon_{p+1}V_1, ..., \varepsilon_{p+q}V_q),$$

for an arbitrarily fixed choice of the projectors $\varepsilon_1, .., \varepsilon_{p+q} \in \{\mathfrak{h}, \mathfrak{v}\}$.

For instance, if V is an arbitrary vector field on $\mathcal{Q}$, its horizontal and vertical components $\mathfrak{h}V$ and $\mathfrak{v}V$, taken separately, are d-tensor fields (of type $(1, 0)$), as each of them acts on a single specified component $\mathfrak{h}\omega$ or $\mathfrak{v}\omega$ of a 1-form $\omega \in \Omega_1(\mathcal{Q})$, whereas their sum is typically, not a d-tensor field.

With respect to the horizontal/vertical adapted local bases of $T\mathcal{Q}$ and $T^*\mathcal{Q}$, a d-tensor field T will be expressed as:

$$T_{(x,\dot{x})} = T^{i_1...i_p}_{j_1...j_q}(x, \dot{x}) (\delta_{i_1} \otimes ... \otimes \dot{\partial}_{i_p} \otimes dx^{j_1} \otimes ... \otimes \delta\dot{x}^{j_q})|_{(x,\dot{x})}. \qquad (2.60)$$

Local characterization of homogeneous d-tensors. In particular, in the presence of a *homogeneous* nonlinear connection, the local basis elements δ_i are 0-homogeneous. Taking into account that the natural vertical basis vectors $\dot{\partial}_i$ are (-1)-homogeneous, the degree of homogeneity (if any) of a d-tensor can be easily established in local coordinates, by summing up the $\dot{x}$-homogeneity degrees of the contributing factors $T^{i_1...i_p}_{j_1...j_q}$, $\delta_{i_1}, ..., \delta\dot{x}^{j_q}$ to each term of (2.60).

Relation between anisotropic tensors and d-tensors. Anisotropic tensor fields can be mapped into multiple d-tensor fields on $\mathcal{Q} \subset TM$, using either horizontal or vertical lifts determined by the nonlinear connection. When doing this, one must take into account that using horizontal lifts $\partial_i \mapsto \delta_i$, one obtains a d-tensor of different degree of homogeneity, compared to the one obtained via a vertical lift $\partial_i \mapsto \dot{\partial}_i$, due to the -1-homogeneity of $\dot{\partial}_i$.

Example: 0-homogeneous geometric objects on Finsler spacetimes. If, in particular, M is equipped with a Finsler spacetime function L, then, the following geometric objects are 0-homogeneous d-tensor fields:

- the Finslerian metric tensor $g = g_{ij} dx^i \otimes dx^j$;
- the curvature tensor $R = R^i{}_{jk} dx^j \otimes dx^k \otimes \dot{\partial}_i$ of the canonical linear connection (0-homogeneity follows as $R^i{}_{jk}$ are 1-homogeneous and $\dot{\partial}_i$ are (-1)-homogeneous).
- The Hilbert form $\omega = l_i dx^i$ and its dual vector field:

$$\ell = l^i \delta_i, \tag{2.61}$$

  where $l^i = \dfrac{\dot{x}^j}{\sqrt{|L|}}$(which, we recall, are only defined on $\mathcal{A}_0 = \mathcal{A} \backslash L^{-1}(0)$).

Also, on a Finsler spacetime, an important feature of both the Chern-Rund connection D (more generally, of any of the typical Finslerian connections in the literature) and of the dynamical covariant derivative ∇ on $T\mathcal{A}$, is that the covariant derivative of a d-tensor field is always a d-tensor field, [46]. Moreover, if $T \in T^p_q(\mathcal{Q})$ is k-homogeneous, then: $D_{\delta_i} T$ is k-homogeneous, $\nabla T = \dot{x}^i D_{\delta_i} T$ is $(k+1)$-homogeneous and $D_{\dot{\partial}_i} T = \dot{\partial}_i T$ is $(k-1)$-homogeneous.

2.2 The positively projectivized tangent bundle PTM^+

2.2.1 Introduction

The positively projectivized tangent bundle PTM^+ (also called in the literature, the *projective sphere bundle* over M, see, e.g., [26], [27]), is essential for a mathematically well defined calculus of variations on Finsler spacetimes.

Thus, in the first part of this subsection, we try to systematize and complete the common knowledge on projectivized bundles over general manifolds. The particular case when M is equipped with a Finsler spacetime structure is then studied in its second part. Since, passing to Finsler spacetimes, some of the facts that became folklore in the standard (smooth, positive definite) Finsler case, will inevitably suffer changes, we will proceed, in this case, with attention to details. Here are some key facts to be stressed:

- The construction of PTM^+ is a natural (functorial) one, referring to the whole category of differentiable manifolds. That is, it is completely independent of any Finsler, or pseudo-Finsler structure that may exist on the manifold.

- In particular case of smooth, positive definite Finsler spaces, PTM^+ is globally diffeomorphic to the *(unit) sphere bundle,* or *indicatrix bundle* $SM = \{(x, \dot{x}) \in TM \mid L(x, \dot{x}) = 1\}$. Yet, such a diffeomorphism does *not* hold anymore in Lorentzian signature; this is seen as PTM^+ has compact, connected fibers diffeomorphic to round spheres $\mathbb{S}^{n-1}$, whereas the Finslerian unit spheres $L = 1$ are non-compact and, generally, disconnected. This is why, in order to avoid any possible confusions, we preferred here the term of *positively projectivized tangent bundle,* rather than *projective sphere bundle.*

  What can still be established, in Finsler spacetimes, is a diffeomorphism between the *observer space* $\mathcal{O}$ and the subset $\mathcal{T}^+ \subset PTM^+$ consisting of *future-pointing (or observer) directions* at all points of M; a nice consequence is that integration on timelike domains in PTM^+ is computationally equivalent to integration over subsets of the observer space $\mathcal{O}$.

- In a Finsler spacetime, the set of non-null admissible directions (which includes the set of observer directions $\mathcal{T}^+$) in PTM^+ has a natural *contact structure*, given by the Hilbert form ω; this allows one to characterize geodesics as integral curves of the Reeb vector field, or to introduce a canonical volume form - in a similar way to what is done, e.g., on *observer spaces* in Lorentzian geometry, see, e.g., [175].

This section is based on our paper [97].

2.2.2 Definition and structure over general manifolds

Throughout this subsection, M will denote an arbitrary smooth, orientable manifold of dimension $n \geq 2$, with no pseudo-Finsler structure assumed.

Definition 35 (positively projectivized tangent bundle, projective sphere bundle) *, [26], [27] Let M be a connected, orientable smooth manifold of dimension n. The positively projectivized tangent bundle is the quotient space*

$$PTM^+ := \overset{\circ}{TM}_{/\sim} \tag{2.62}$$

where $\sim$ is the equivalence relation on $\overset{\circ}{TM}$ given by:

$$(x, \dot{x}) \ \sim \ (x, u) \Leftrightarrow \ u = \alpha \dot{x} \ \ \textit{for} \ \ \textit{some} \ \ \alpha > 0\,. \tag{2.63}$$

In other words we identify the half-line $\{(x, \alpha\dot{x}) \mid \alpha > 0\}$ as a single point. We denote by

$$\pi^+ : \overset{\circ}{TM} \to PTM^+, (x, \dot{x}) \mapsto [(x, \dot{x})] \tag{2.64}$$

the canonical projection.

Here are some properties of PTM^+, to be used in the following. As a clear and systematic formulation seems to be missing in the literature, we briefly sketch their proofs.

- PTM^+ is a smooth manifold of dimension $2n - 1$, [27]. A smooth atlas $\left\{(\pi^+(V_i^{\pm}), \psi^{\pm})\right\}$ on PTM^+, which will be occasionally used in the following, is constructed as follows. Start with an atlas $\{(U, \varphi)\}\,, \varphi = (x^i)$ on M and, for each local chart domain U and each $i = 0, \ldots, n-1$, consider the open sets: $V_i^+ = \left\{(x,\dot{x}) \in TU \mid \dot{x}^i > 0\right\}$, $V_i^- = \left\{(x,\dot{x}) \in TU \mid \dot{x}^i < 0\right\}$. Then, for each $[(x,\dot{x})] \in \pi^+(V_i^+)$, define the coordinate maps $\psi^+ := (x^i, u^\alpha)$ (respectively, $\psi^- = (x^i, u^\alpha)$) as:

$$(x^i, u^\alpha) = (x^0, ..., x^{n-1}, \frac{\dot{x}^0}{\dot{x}^i}, ..., \frac{\dot{x}^{i-1}}{\dot{x}^i}, \frac{\dot{x}^{i+1}}{\dot{x}^i}, ..., \frac{\dot{x}^{n-1}}{\dot{x}^i}). \tag{2.65}$$

Yet, a more practical choice will be *homogeneous local coordinates*, to be presented in the next paragraph.

- PTM^+ is the *orientable double cover* (see *[128],* Ch. 15) of the usual projectivized tangent bundle PTM; in particular, PTM^+ is orientable.

- $(\overset{\circ}{TM}, \pi^+, PTM^+, \mathbb{R}_+^*)$ is a principal bundle, with typical fiber $(\mathbb{R}_+^*, \cdot)$, [93]; this is seen as the mapping $\chi : \overset{\circ}{TM} \times \mathbb{R}_+^* \to \overset{\circ}{TM}$, $((x,\dot{x}), \alpha) \mapsto (x, \alpha\dot{x})$, defined in (2.56), is a smooth action which preserves the fibers $(\pi^+)^{-1}([x,\dot{x}]) = \left\{(x, \alpha\dot{x}) \mid \alpha \in \mathbb{R}_+^*\right\}$ and acts freely and transitively on them.

  The Liouville vector field $\mathbb{C}$ is tangent to the fibers of π^+ - i.e., it is π^+-*vertical*:

$$(\pi^+)_*\mathbb{C} = 0; \tag{2.66}$$

  actually, as these fibers are 1-dimensional, $\mathbb{C}$ generates the tangent spaces to the fibers.

- The triple (PTM^+, π_M, M), where $\pi_M : PTM^+ \to M$, $[(x,\dot{x})] \mapsto x$, is a natural bundle, with fibers diffeomorphic to the Euclidean sphere $\mathbb{S}^{n-1}$ (hence the name of *projective sphere bundle* used, e.g., in [26]*).* The fact that the fibers PT_xM^+ are diffeomorphic to $\mathbb{S}^{n-1}$ follows easily, as they are orientation coverings of the projective tangent spaces $PT_xM \simeq P\mathbb{R}^n$; but, the orientation covering of the projective space $P\mathbb{R}^n$ is the round sphere $\mathbb{S}^{n-1}$.

  Naturality is seen as the correspondence $\mathfrak{F} : \mathcal{M}_n \to \mathcal{FB}$, attaching to any manifold $M \in \mathcal{M}_n$, its positively projectivized tangent bundle PTM^+ and to any diffeomorphism $f : M \to M'$, the fibered morphism $\mathfrak{F}(f) : PTM^+ \mapsto PTM'^+$, $[(x,\dot{x})] \mapsto [(f(x), df_x(\dot{x}))]$ (which is well defined by virtue of the linearity of df_x), is a covariant functor.

From PTM^+ to $\overset{\circ}{TM}$ and back. Homogeneous local coordinates.

Homogeneous local coordinates (see, e.g., [27] and also, [60] for PTM) of a point (that is, of an equivalence class $[(x,\dot{x})]$) of PTM^+ are defined as the coordinates $(x^i, \dot{x}^i)$ in the corresponding chart on TM of an arbitrarily chosen representative $(x,\dot{x})$ of the class $[(x,\dot{x})]$. Thus, homogeneous coordinates are only unique up to multiplication by a positive scalar of the $\dot{x}$-coordinates.

In these coordinates, local computations on PTM^+ will become identical to those on TM, just, with due care that the involved expressions in $\left(x^i, \dot{x}^i\right)$ - which formally correspond to geometric objects on TM - should really define objects on PTM^+. A necessary (but not always sufficient) condition is that these formally defined geometric objects on TM should be positively 0-homogeneous in $\dot{x}$, i.e., invariant under the flow of $\mathbb{C}$. Here we list the most frequently encountered examples:

- **Functions.** A function $f : \overset{\circ}{TM} \to \mathbb{R}$, $f = f(x,\dot{x})$ can be identified with a function f^+ on PTM^+ such that $f = f^+ \circ \pi^+$, if and only if it is positively 0-homogeneous in $\dot{x}$; in this case, $f^+ : PTM^+ \to \mathbb{R}$ is defined by:

$$f^+[(x,\dot{x})] = f\,(x,\dot{x})\,.$$

- **Vector fields.** For a vector field $X = X^i\partial_i + \tilde{X}^i\dot{\partial}_i \in \mathcal{X}(\overset{\circ}{TM})$, the projection

$$X^+ := (\pi^+)_* X$$

is a well defined vector field on PTM^+ if and only if X is positively 0-homogeneous in $\dot{x}$, i.e., $\mathfrak{L}_{\mathbb{C}} X = 0$.

We note, [97], that the correspondence $X \mapsto X^+$ is surjective, but not injective, since the π^+-verticality of $\mathbb{C}$ implies that all vector fields of the form $X + f\mathbb{C} \in \mathcal{X}(\overset{\circ}{TM})$ will descend by π^+_* onto the same $X^+ \in \mathcal{X}(PTM^+)$.

- **Differential forms.** For differential forms ρ defined on conic subbundles of $\overset{\circ}{TM}$, 0-homogeneity is necessary, but not sufficient in order to be identified with differential forms on PTM^+. More precisely, given ρ, there exists a (unique) differential form $\rho^+ \in \Omega(PTM^+)$ such that $\rho = (\pi^+)^*\rho^+$, if and only if ρ is 0-homogeneous in $\dot{x}$ and π^+-horizontal, i.e.[3]:

$$\mathfrak{L}_{\mathbb{C}}\rho = 0, \quad \mathbf{i}_{\mathbb{C}}\rho = 0. \tag{2.67}$$

Remarks, see also [60]:

1. The projection π^+ is represented in homogeneous coordinate as the identity $\pi : (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i)$. Therefore, the geometric objects f, X, ρ etc. on TM and their correspondents f^+, X^+, ρ^+ etc. on PTM^+ (provided that they exist) have identical expressions in local homogeneous coordinates.

2. Exterior differentiation of forms $\rho^+ \in \Omega(PTM^+)$ (and, in particular, differentiation of functions) can be carried out, in homogeneous coordinates, identically to exterior differentiation of the corresponding form $\rho \in \Omega(\overset{\circ}{TM})$, since:

$$d\rho = d\left((\pi^+)^*\rho^+\right) = \left(\pi^+\right)^* d\rho^+.$$

The function f^+ is differentiable at $[(x, \dot{x})]$ if and only if $f = f^+ \circ \pi^+$ is differentiable at one representative $(x, \dot{x})$.

2.2.3 The positively projectivized tangent bundle of Finsler spacetimes

Assume, in the following, that M is equipped with a Finsler spacetime function L. This way, one can speak about the conic subbundles $\mathcal{A}, \mathcal{A}_0, \mathcal{T} \subset \overset{\circ}{TM}$ and the observer space $\mathcal{O}$, see Section 2.1.2. We will denote by a plus sign, e.g., $\mathcal{T}^+ = \pi^+(\mathcal{T})$, $\mathcal{A}^+ = \pi^+(\mathcal{A})$ etc., their images through $\pi^+ : \overset{\circ}{TM} \to PTM^+$. Also, we will always use local homogeneous coordinates on PTM^+.

Canonical nonlinear connection and Chern-Rund connection. The canonical nonlinear connection N on $\mathcal{A}$ (see eq. (2.34)) is naturally transplanted to $\mathcal{A}^+$, as follows. For any vector $X^+ \in T\mathcal{A}^+$, there exists a positively 0-homogeneous vector $X \in T\mathcal{A}$ - which is unique up to a multiple of $\mathbb{C}$ - such that $(\pi^+)_* X = X^+$. The vector field X is then decomposed using N,

[3] This result, proven in [93], is just the coordinate-free restatement of a result in [60].

as $X = \mathfrak{h}X + \mathfrak{v}X$, where $\mathfrak{h}X = X^i\delta_i$ and $\mathfrak{v}X = \dot{X}^i\dot{\partial}_i$; both these components are positively 0-homogeneous, due to the homogeneity of N, hence they descend back, by $(\pi^+)_*$ onto well defined vectors $\mathfrak{h}X^+, \mathfrak{v}X^+ \in T\mathcal{A}^+$; in homogeneous local coordinates,

$$\mathfrak{h}X^+ = X^i\delta_i, \quad \mathfrak{v}X^+ = \dot{X}^i\dot{\partial}_i.$$

The possible multiple of $\mathbb{C}$ appearing in the procedure will be projected back to PTM^+ into the zero vector, hence the components $\mathfrak{h}X^+, \mathfrak{v}X^+$ are uniquely defined by X^+. This gives rise to a splitting $X^+ = \mathfrak{h}X^+ + \mathfrak{v}X^+$, i.e., to a connection N^+ on $\mathcal{A}^+ := \pi^+(\mathcal{A})$:

$$T\mathcal{A}^+ = H\mathcal{A}^+ \oplus V\mathcal{A}^+. \tag{2.68}$$

Similarly, the Chern-Rund connection D gives rise to a linear connection D^+ on $\mathcal{A}^+$, given in homogeneous coordinates, by the same local expression of covariant derivatives as D.

Contact structure and volume form for the set of non-null admissible directions $\mathcal{A}_0^+ = \pi^+(\mathcal{A}_0)$. The Hilbert form $\omega = F_{\cdot i}dx^i$, defined on $\mathcal{A}_0$ obeys the conditions:

$$\mathbf{i}_{\mathbb{C}}\omega = 0, \quad \mathfrak{L}_{\mathbb{C}}\omega = d\mathbf{i}_{\mathbb{C}}\omega + \mathbf{i}_{\mathbb{C}}d\omega = 0,$$

which allow us to identify it with a differential form ω^+ on $\mathcal{A}_0^+ \subset PTM^+$, such that $(\pi^+)^*\omega^+ = \omega$. In homogeneous coordinates, this is:

$$\omega^+ = F_{\cdot i}dx^i. \tag{2.69}$$

Using the exterior derivative:

$$d\omega^+ = \frac{1}{F}(\epsilon g_{ij} - F_{\cdot i}F_{\cdot j})\delta\dot{x}^j \wedge dx^i, \tag{2.70}$$

where the sign ϵ is defined by the equality $L = \epsilon F^2$ (see Section 2.1.2), a similar calculation to the one in the positive definite case (e.g., [60]), shows that, for $\dim M = 4$, the 7-form:

$$\omega^+ \wedge \ d\omega^+ \wedge d\omega^+ \wedge d\omega^+ \ = 3!\frac{\det g}{L^2}\mathbf{i}_{\mathbb{C}}(d^4x \wedge d^4\dot{x}) = 3!\frac{\det g}{L^2}\mathrm{Vol}_0, \tag{2.71}$$

where

$$\mathrm{Vol}_0 := \mathbf{i}_{\mathbb{C}}(d^4x \wedge d^4\dot{x}), \tag{2.72}$$

is always nonzero. In other words, the Hilbert form ω^+ defines a *contact structure* on $\mathcal{A}_0^+ \subset PTM^+$.

Warning: In the above, Vol_0 should be understood just as a shorthand notation for $\mathbf{i}_{\mathbb{C}}(d^4x \wedge d^4\dot{x})$. Since it is 4-homogeneous, it does not represent any differential form on PTM^+. The one which *does* represent a well defined form on PTM^+ is (2.71); the notation Vol_0 is just meant will serve later, in Chapter 3, for a convenient bookkeeping, which explicitly separates quantities depending on L from those that do not depend on L.

In contact geometry, the *Reeb vector field* $\ell^+ \in \mathcal{X}(\mathcal{A}_0^+)$ corresponding to the contact structure ω^+ is uniquely defined by the conditions

$$\mathbf{i}_{\ell^+}(\omega^+) = 1, \quad \mathbf{i}_{\ell^+}d\omega^+ = 0. \tag{2.73}$$

In our case, a quick coordinate computation identifies ℓ^+ as:

$$\ell^+ = l^i \delta_i, \quad l^i = \frac{\dot{x}^i}{F}. \tag{2.74}$$

We note that the Reeb vector field ℓ^+ on PTM^+ is just the image through π^+ of the vector field $\ell = l^i \delta_i \in \mathcal{X}(\mathcal{A}_0)$, defined in Section 2.61.

The importance of the Reeb vector field is given by the following result, which extends to Lorentzian signature a result in [66].

Proposition 36 *Let $c : [a, b] \to M$, $s \mapsto x(s)$ be a non-lightlike admissible curve parametrized by arc length and $C : [a, b] \to \mathcal{A}_0^+$, $s \mapsto [(x(s), \dot{x}(s))]$, its canonical lift. Then, C is an integral curve of ℓ^+ if and only if c is an arc-length parametrized geodesic of (M, L).*

Proof. In homogeneous coordinates, $\dot{C} = \dot{x}^i(s)\delta_i + \delta_s \dot{x}^i(s)\dot{\partial}_i$, where $\delta_s \dot{x}^i(s) := \ddot{x}^i(s) + 2G^i(x^j(s), \dot{x}^j(s))$; that is, C is an integral curve of ℓ^+ is and only if:

$$\dot{x}^i(s) = l^i, \ \delta_s \dot{x}^i(s) = 0.$$

The first condition above is trivially satisfied by any curve parametrized by arc length, since $F(x(s), \dot{x}(s)) = 1$, whereas the second one means that c is an arc-length parametrized geodesic of (M, L), see (2.31). ■

The contact structure ω^+ enables us to identify a canonical volume form on $\mathcal{A}_0^+$. Taking into account that $\dim M = 4$ (i.e., $\dim PTM^+ = 7$), the following definition makes sense.

Definition 37 (Canonical volume form) *Let (M, L) be a Finsler spacetime, $\mathcal{A}_0^+ \subset PTM^+$, the set of its admissible, non-null directions and ω^+, the Hilbert form on $\mathcal{A}_0^+$. The 7-form:*

$$d\Sigma^+ := \frac{\epsilon}{3!}\omega^+ \wedge (d\omega^+)^3 = \frac{|\det g|}{L^2}\mathrm{Vol}_0 \tag{2.75}$$

(with $\epsilon = sign(\det g)$ and Vol_0 as in (2.72)) is called the canonical volume form on $\mathcal{A}_0^+$.

Note that, on $\mathcal{A}_0^+$, g is nondegenerate, so, $d\Sigma^+$ is well defined.

The divergence of a vector field $X \in \mathcal{X}(\mathcal{A}_0^+)$, with respect to this volume form is defined as usually by $(\operatorname{div} X)d\Sigma^+ = \mathfrak{L}_X d\Sigma^+$; in coordinates, this gives, for horizontal and vertical vector fields, $X^H = X^i \delta_i$ and $X^V = Y^i \dot{\partial}_i$:

$$\operatorname{div}(X^H) = (X^i{}_{|i} - P_i X^i), \tag{2.76}$$

$$\operatorname{div}(X^V) = (Y^i{}_{\cdot i} + 2C_i Y^i - \frac{4}{L} Y^i \dot{x}_i)\,, \tag{2.77}$$

where P_i are the components of the trace of the Landsberg tensor (2.44) and C_i, those of the trace of the Cartan tensor (2.43). For any $f : \mathcal{A}_0^+ \to \mathbb{R}$, the above equations imply

$$\nabla f = \operatorname{div}(f\ell^+) = \operatorname{div}(f l^i \delta_i). \tag{2.78}$$

Integration on PTM^+ and integration on observer space

As already seen above, in Finsler spacetimes, there is no global diffeomorphism between the indicatrix bundle $L^{-1}(1)$ and PTM^+. Yet, we can establish a diffeomorphism between the observer space $\mathcal{O} \subset L^{-1}(1)$ and the set of future pointing timelike directions $\mathcal{T}^+ := \pi^+(\mathcal{T}) \subset PTM^+$; as a consequence, integration of differential forms on the observer space of Finsler spacetimes can be understood as integration of differential forms on (subsets of) PTM^+.

Proposition 38 *1. The restriction $\pi^+ : \mathcal{O} \to \mathcal{T}^+$ of the projection $\pi^+ : \overset{\circ}{TM} \to PTM^+$ is a diffeomorphism.*

2. Pick any compactly supported 7-form ρ^+ on $\mathcal{T}^+$ and set: $\rho = (\pi^+)^* \rho^+$. Then:

$$\int_{\mathcal{T}^+} \rho^+ = \int_{\mathcal{O}} \rho. \tag{2.79}$$

Proof. 1. *Injectivity*: Assume $\pi^+(x, \dot{x}) = \pi^+(x', \dot{x}')$ for some $(x, \dot{x}), (x', \dot{x}') \in \mathcal{O}$. It follows that $[(x, \dot{x})] = [(x', \dot{x}')]$, i.e., $x = x'$ and there exists an $\alpha > 0$ such that $\dot{x}' = \alpha\dot{x}$. Applying L to both hand sides, we find $L(x, \dot{x}') = \alpha^2 L(x, \dot{x})$; but, on $\mathcal{O}$, $L = 1$, which means that $\alpha^2 = 1$. Since $\alpha > 0$, it follows that $(x, \dot{x}') = (x, \dot{x})$.

Surjectivity: Pick an arbitrary $[(x, \dot{x})] \in \mathcal{T}^+$ and an arbitrary representative $(x, \dot{x}) \in \mathcal{T}$. From the conicity of $\mathcal{T}$, we find that the vector $(x, \alpha\dot{x})$, with $\alpha := L(x, \dot{x})^{-1/2}$, also belongs to $\mathcal{T}$ and, in addition, $L(x, \alpha\dot{x}) = 1$. In other words, we have found a representative $(x, \alpha\dot{x}) \in \mathcal{O}$ of the class $[(x, \dot{x})]$. But, as $\pi^+(x, \dot{x}) = \pi^+(x, \alpha\dot{x}) = [(x, \dot{x})]$, it follows that $[(x, \dot{x})] \in \pi^+(\mathcal{O})$.

The smoothness of π^+ and of its inverse are immediate.

2. follows from $\rho = (\pi^+)^* \rho^+$ and point 1. ■

In particular, the above result shows that the set of observer directions $\mathcal{O}^+$ and the set of future-pointing timelike directions $\mathcal{T}^+$ are the same:

$$\mathcal{O}^+ = \mathcal{T}^+. \tag{2.80}$$

Integration on pieces of $\mathcal{A}_0^+$. A similar result holds for *pieces* $D^+ \subset \mathcal{A}_0^+$ (i.e., to compact 7-dimensional submanifolds with boundary of $\mathcal{A}_0^+$); in this case, the compact support condition on ρ can be dropped, i.e., for any differential form ρ^+ on PTM^+, there holds, [93]:

$$\int_{D^+} \rho^+ = \int_{D} \rho, \tag{2.81}$$

where $\rho = (\pi^+)^* \rho^+ \in \Omega(\overset{\circ}{TM})$ and $D := (\pi^+)^{-1}(D^+) \cap L^{-1}(1)$.

The following Lemma, proven in our paper [93], allows us to evaluate and manipulate action integrals in Chapter 3; on positive definite Finsler spaces, a similar relation to the first equation (2.82) below was proven by Chen and Shen, [59].

Lemma 39 *: Let (M, L) be a Finsler spacetime, $f : \mathcal{A}_0^+ \to \mathbb{R}$ be a smooth function defined on the set of non-lightlike admissible directions $\mathcal{A}_0^+$ of L and $X = (Lg^{ij} f_{\cdot i})\dot{\partial}_j$. Then, the following identities hold in homogeneous local coordinates:*

$$[g^{ij}(Lf)_{\cdot i \cdot j} - 8f]d\Sigma^+ = d(\mathbf{i}_X d\Sigma^+)\,, \tag{2.82}$$

$$[L^{-1}g^{ij}(L^2 f)_{\cdot i \cdot j} - 24f]d\Sigma^+ = d(\mathbf{i}_X d\Sigma^+) \tag{2.83}$$

$$(g^{ij} - 4L^{-1}\dot{x}^i\dot{x}^j)(Lf)_{\cdot i \cdot j} d\Sigma^+ = d(\mathbf{i}_X d\Sigma^+)\,. \tag{2.84}$$

Proof. For the first equation, we expand

$$g^{ij}(Lf)_{\cdot i \cdot j} d\Sigma^+ = g^{ij}(2g_{ij}f + 2L_{\cdot i}f_{\cdot j} + Lf_{\cdot i \cdot j})d\Sigma^+\,.$$

As f is defined on a subset of PTM^+, it must be given by a 0-homogeneous expression in $\dot{x}$, i.e.:

$$g^{ij}L_{\cdot i}f_{\cdot j} = 2\dot{x}^i f_{\cdot i} = 0; \tag{2.85}$$

in addition, $g^{ij}g_{ij} = \dim M = 4$, which leads to: $g^{ij}(Lf)_{\cdot i \cdot j}d\Sigma^+ = 8f d\Sigma^+ + Lg^{ij}f_{\cdot i \cdot j}d\Sigma^+$. The last step is to show that the last term in this sum is of the form $d(\mathbf{i}_X d\Sigma^+) = \operatorname{div} X d\Sigma^+$. This can be seen from:

$$Lg^{ij}f_{\cdot i \cdot j}d\Sigma^+ = g^{ij}f_{\cdot i \cdot j}\frac{|\det g|}{L}\mathbf{i}_{\mathbb{C}}\mathrm{Vol}_0 = (g^{ij}f_{\cdot i}\frac{|\det g|}{L})_{\cdot j}\mathbf{i}_{\mathbb{C}}\mathrm{Vol}_0 = \operatorname{div} X d\Sigma^+ = d(i_X d\Sigma^+),$$

where we have used $(g^{ij}\,|\det g|)_{\cdot j} = 0$ and (2.85).

Equation (2.83) can be proven by expanding

$$L^{-1}g^{ij}(L^2 f)_{\cdot i \cdot j} = L^{-1}g^{ij}(2g_{ij}(Lf) + 2L_{\cdot i}(Lf)_{\cdot j} + L(Lf)_{\cdot i \cdot j})$$

and using $L_{\cdot i} = 2g_{ik}\dot{x}^k$ together with the 2-homogeneity of Lf to write: $2g^{ij}L_{\cdot i}(Lf)_{\cdot j} = 4\dot{x}^j(Lf)_{\cdot j} = 8Lf$. The desired result then follows as a consequence of the first equation. The third equation (2.84) is then obtained from (2.82) and the 2-homogeneity of Lf. ■

2.3 Finsler spacetimes, Finsler spaces, Lorentzian manifolds: a brief comparison

2.3.1 Introduction

Finsler spacetimes differ from Finsler spaces in sometimes unexpected ways. The reason is that, besides the change of signature, there is one more detail that comes into play: the existence, in each tangent space, of *non-admissible* directions along which L is either non-smooth, or has degenerate $\dot{x}$-Hessian. This calls for extreme care when trying to extend to Lorentzian signature, results that hold true in Finsler spaces. Yet, on the other hand, Finsler spacetimes seem "manageable" enough for physical applications, while offering much more generality than Lorentzian ones.

This section, which combines results from my paper [200] and from my joint paper [76] with A. Fuster, S. Heefer and C. Pfeifer, tries to capture, on the one hand, some of the differences between Finsler and pseudo-Finsler spaces and, on the other hand, to show some results that *do* extend from Lorentzian spaces to Lorentz-Finsler ones.

Finsler spacetimes versus Finsler spaces. As already seen above in Section 2.1.4, computational results such as: geodesic equations, the canonical nonlinear connection, geodesic deviation equations, all have the same expressions as in (positive definite, $\overset{\circ}{TM}$-smooth) Finsler spaces, with the only difference that the geodesic coefficients G^i and geometric objects arising thereof can only be defined on a conic subbundle of $\overset{\circ}{TM}$.

But, not much else seems to extend from Finsler spaces, to Finsler spacetimes. Here are just some results that hold valid in Finsler geometry, but do *not* extend to Finsler spacetimes.

1. *Szábó Metrizability Theorem for Berwald spaces.* This result, [181], states that any Berwald-type Finsler structure (M, L) is Riemann metrizable, i.e., there exists a Riemannian metric on M which shares the same parametrized geodesics with L.

   This result does not extend to Finsler spacetimes, as explicitly shown in [76]. Yet, the question on the extendability of this result still remains open in the case of *smooth* (or regular) Finsler spacetime metrics, see Subsection 2.3.2 below.

2. *Averaged Riemannian metrics.* For smooth, positive definite Finsler structures (M, F), one can construct a so-called *averaged Riemannian metric* a on M, whose components $a_{ij} = a_{ij}(x)$ are obtained, see, e.g., [62] by integrating the Finslerian metric components $g_{ij} = g_{ij}(x, \dot{x})$ (alternatively, the products $\dot{x}_i \dot{x}_j$ or some linear combination of g_{ij} and $\dot{x}_i x_j$), over the indicatrix $I_x = F^{-1}(1) \cap T_xM$. Averaged Riemannian metrics have an essential property: if any affine connection on M is compatible with F, then it must be metrical with respect to a. The technique is used in proving Szabó's Theorem, [196], and extending results from Riemannian geometry to Finslerian spaces, e.g., [197], [137].

   Sadly, in Finsler spacetimes, the above construction makes no sense. Even in the case when the functions to be integrated exist and are smooth for all $\dot{x} \in S_x$ (which is, as seen above, rather the exception, than the rule, in the case of Finsler spacetimes), we have a bigger problem: the indicatrix I_x is non-compact - and, even its most "domesticated" connected component, which is the observer space $\mathcal{O}_x$, is still non-compact. Thus, the procedure typically leads to infinite integrals. Up to now, there is no fully-featured extension of this technique to Lorentzian signature[4]. Yet, as we will show below in Subsection 2.3.5, an old and simple technique of reducing Finslerian problems to their Riemannian counterparts, which is the use of *associated* (or *osculating*) pseudo-Riemannian metrics can still be very helpful, depending on the problem to be solved.

3. *Liouville-type classification theorem for conformal symmetries.* A diffeomorphism $f : M \to M$ is deemed a *conformal symmetry* of a pseudo-Finsler structure (M, L) if the Finsler functions L and $\tilde{L} = L \circ df$ are conformally related, i.e., if there exists a function $\sigma : M \to \mathbb{R}$, $\sigma = \sigma(x)$, such that:
   $$\tilde{L}(x, \dot{x}) = e^{2\alpha(x)} L(x, \dot{x}), \tag{2.86}$$
   at all admissible $(x, \dot{x}) \in \overset{\circ}{TM}$; the latter equality is obviously equivalent to conformal relation $\tilde{g}_{(x,\dot{x})} = e^{2\sigma(x)} g_{(x,\dot{x})}$ between the corresponding metric tensors, i.e., (2.86) extends the pseudo-Riemannian notion of conformal equivalence.

[4] A notion of averaging was proposed recently in [189]; yet, the affine connection of a Berwald space does not coincide with the Levi-Civita of the averaged metric.

In Riemannian geometry, the existence of a 1-parameter group of conformal symmetries can provide valuable information, which can go up to full classification results, [3], [48], [49], [123], on the metric structure. It is thus a natural question to try to classify conformal transformations, also in the Finslerian case.

Conformal groups of pseudo-Finsler metrics have, yet, a much more complicated structure than both pseudo-Riemannian and Finslerian conformal groups. To prove this statement, we show in Subsection 2.3.3 that there exist entire classes of examples of flat pseudo-Finsler spaces whose conformal symmetries depend on *arbitrary* functions. Comparatively, in dimension $n \geq 3$, conformal symmetries of a pseudo-Euclidean space can only be similarities, inversions and compositions thereof, [87], while the only conformal symmetries of a non-Euclidean flat Finsler space are similarities, [136].

4. *Deicke's Theorem.* This theorem states that, in a ($\overset{\circ}{TM}$-smooth, positive definite) Finsler space, if the trace of the Cartan tensor identically vanishes, then the full Cartan tensor is identically zero, i.e., the space is actually Riemannian. But this is known ([135], p. 154-156) not to extend to pseudo-Finsler spaces[5]. A concrete counterexample is given by the so-called Berwald-Moór metric on $\mathbb{R}^n$, see eq. 2.96 below - which, in the particular case $n = 4$, gives an example of Finsler spacetime structure.

Extending to Finsler spacetimes results from Lorentzian geometry. Despite the "oddities" of Finslerian spacetime structures presented above, from the point of view of physical applications, the situation is by far less discouraging than it might seem. Actually, Finsler spacetimes share with Lorentzian ones some essential features that make them desirable for physical applications: a well defined notion of proper time (i.e., of arc length), a well-defined causal structure (see, e.g., [142]), geodesic equations that can be interpreted as trajectories of freely falling particles.

But there are definitely more results from semi-Riemannian/Lorentzian geometry that can actually be extended to pseudo-Finsler, respectively, to Lorentz-Finsler geometry; actually, a full, in-depth exploration of this topic is still to be done. We will just present here, in Subsections 2.3.4 and 2.3.5, a few examples regarding projective and conformal transformations, proven in our paper [200].

1. One of the basic results for general relativity, which is *Weyl's Theorem* (see, e.g., [105]). This states that two metrics on a connected manifold of dimension greater than 1 that are both conformally and projectively related, can only differ by a multiplicative constant.
2. A result regarding the causal character of essential conformal vector fields.
3. Two results regarding the existence of zeros of Killing vector fields on Lorentzian manifolds.

2.3.2 On the non-metrizability of Berwald-Finsler spacetimes

This section presents in brief the results in [76].

In the following, assume (M, L) is a Berwald-type pseudo-Finsler space. It means that the Chern-Rund connection on TM descends into a well defined affine connection on M; more precisely, the coefficients ${\Gamma^i}_{jk}$ coincide with the derivatives ${G^i}_{\cdot j \cdot k}$ and depend on x only, see Section 2.1.5.

[5] The result was deduced in [203], but later I realized that this was actually, a rediscovery.

A natural question is then whether L is *affinely equivalent* to a Lorentzian metric a on M, i.e., whether there exists some pseudo-Riemannian metric tensor on M admitting $\Gamma^i{}_{jk}$ as its Christoffel symbols (which is the same as saying that a and L have the same parametrized geodesics).

For properly Finslerian (i.e., positive definite, smooth on $\overset{\circ}{TM}$) Berwald spaces, the following result is known in the literature:

Theorem 40 ***(Szábó's Metrizability Theorem,*** *[181]): Let (M, F) be a Finsler space of Berwald type. Then, there exists a Riemannian metric a on M such that the affine connection of the Berwald space is the Levi-Civita connection of a.*

The metric a above is explicitly constructed by averaging the Finsler metric g over the indicatrix at each point $x \in M$. Yet, as already noted above, this is not applicable in Lorentzian signature.

A necessary condition for Lorentz metrizability. We denote by

$$R_i{}^m{}_{jk} = \delta_k \Gamma^m{}_{ij} - \delta_j \Gamma^m{}_{ik} + \Gamma^s{}_{ij}\Gamma^m{}_{sk} - \Gamma^s{}_{ik}\Gamma^m{}_{sj}, \tag{2.87}$$

the horizontal components[6] of the curvature of the Chern-Rund connection of (M, L), and by $R_{ij} := R_i{}^k{}_{jk}$ the horizontal Chern-Rund Ricci tensor components:

$$R_{ij} = \delta_m \Gamma^m{}_{ij} - \delta_j \Gamma^m{}_{im} + \Gamma^s{}_{ij}\Gamma^m{}_{sm} - \Gamma^s{}_{im}\Gamma^m{}_{sj}. \tag{2.88}$$

A necessary condition for the connection defined by $\Gamma^i{}_{jk}$ to be the Levi-Civita connection of a pseudo-Riemannian metric is that the Ricci tensor (2.88) is symmetric. Yet, this is generally not the case, as shown from the example below.

An explicit example of a Berwald spacetime (M, L) **with non-symmetric Chern-Rund Ricci tensor**[7] is the following, [76]. Consider, on $M = \mathbb{R}^4$ equipped with global coordinates $(x^i)_{i=\overline{0,3}}$, the following Kundt-type metric:

$$L(x, \dot{x}) = a_x(\dot{x}, \dot{x}) s^{-p} (k + m\ s)^{p+1}, \tag{2.89}$$

where $k, m, p \in \mathbb{R}$ are constants, $s(x, \dot{x}) = \dfrac{(b_x(\dot{x}))^2}{a_x(\dot{x}, \dot{x})}$ and:

$$a := 2dx^0 \otimes dx^1 + x^1\, \phi(x^2, x^3)\, dx^0 \otimes dx^0 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3,\ \ b := dx^0, \tag{2.90}$$

for some arbitrary smooth function $\phi = \phi(x^2, x^3)$. Calculations using computer algebra show that, for $k \neq 0$, $p \neq 1$, the function L defines a Berwald spacetime structure on $\mathbb{R}^4$, whose Chern-Rund Ricci tensor satisfies:

$$R_{02} - R_{20} = \frac{2p}{p-1}\partial_2\phi, \qquad R_{03} - R_{30} = \frac{2p}{p-1}\partial_3\phi, \tag{2.91}$$

i.e., it is not symmetric for non-constant ϕ. As a consequence, this particular Berwald spacetime cannot be affinely equivalent to any pseudo-Riemannian metric. Hence, Szabó's Theorem does not extend to general Finsler spacetimes.

[6] Here, a different sign convention for the components $R_i{}^m{}_{jk}$ is used, compared to [76].

[7] Full credit for finding this precise example must go to my coauthors.

What makes R_{ij}, yet, be non-symmetric? Actually, the culprit for the non-symmetry of R_{ij} is the existence of *non-admissible directions* for L, rather than of the change of signature. This can be seen from the two results below.

Lemma 41 *In any pseudo-Finsler space (M, L) and in any local chart, the antisymmetric part of the Chern-Rund Ricci tensor (2.88) is expressed as:*

$$R_{ij} - R_{ji} = R_{m\ ji}^{\ k}\dot{x}^m C_k, \tag{2.92}$$

where C_k are the components of the trace of the Cartan tensor (2.43).

Proof. Fix an arbitrary local chart on TM, which intersects the set of admissible vectors of L. From (2.88), we find:

$$R_{ij} - R_{ji} = \delta_i \Gamma^m_{\ jm} - \delta_j \Gamma^m_{\ im}.$$

Then, denoting:

$$f := \ln \sqrt{|\det(g_{ij})|}, \tag{2.93}$$

we find that, in the given chart:

$$\delta_j f = \Gamma^m_{\ jm}, \quad \dot{\partial}_j f = C_j. \tag{2.94}$$

In particular, this gives:

$$R_{ij} - R_{ji} = \delta_i \delta_j f - \delta_j \delta_i f = [\delta_i, \delta_j] f = R^k_{\ ij} \dot{\partial}_k f,$$

Further, using the identity $\dot{x}^m R_{m\ ij}^{\ k} = R^k_{\ ij}$ (which follows immediately from the local expressions (2.37) and (2.87) and the second equality (2.94), we get the required equality (2.92). Moreover, as both hand sides of (2.92) are tensor components, it follows that the relation does not depend on the choice of the local chart - though the expression f does. ■

Relation (2.92) holds in any pseudo-Finsler space and, in particular, in any Finsler spacetime. The peculiarity of Berwald-type spaces is that, in this case, its left hand side does not depend on $\dot{x}$. This fact will be used in the following.

Theorem 42 ***(Symmetry of the Chern-Rund Ricci tensor for smooth Finsler spacetime metrics):*** *If (M, L) is a Berwald spacetime with $\mathcal{A} = \overset{\circ}{TM}$, then:*

$$R_{ij} = R_{ji}.$$

Proof. Fix an arbitrary point $x \in M$ and a chart (TU, ϕ), $\phi = \left(x^i, \dot{x}^i\right)$ on TM, naturally induced by some arbitrary chart with domain $U \subset M$, $x \in U$. This way, expression (2.93) locally defines a 0-homogeneous function $f : TU \to \mathbb{R}$ which is, according to the hypothesis $\mathcal{A} = \overset{\circ}{TM}$, smooth on the entire $TU \backslash \{0\}$.

The 0-homogeneity of f allows us to naturally identify it with a smooth function f^+ on the subset $TU^+ = \pi^+(TU) \subset PTM^+$ having in homogeneous coordinates, the same expression as f (see Section 2.2.2). Thus, the partial function $f^+_x(\cdot) = f^+(x, \cdot)$ is defined and smooth on the fiber of $\pi^+(TU)$ at x, i.e., on the compact set $PT_xM^+ \simeq \mathbb{S}^{n-1}$. As a consequence, it must admit at least a local extremum, say at $[(x, v)]$; using homogeneous coordinates, this implies that the partial derivatives $\dot{\partial}_k f$ vanish at (x, v), i.e.: $C_k(x, v) = 0$.

But then, evaluating (2.92) at (x, v), we find:

$$R_{ij}(x) - R_{ji}(x) = R_{m\ ji}^{\ k}(x) v^m C_k(x, v) = 0;$$

since the left hand side does not depend on v, it yields, in the given local coordinates: $R_{ij}(x) = R_{ji}(x)$. Moreover, from the tensorial character of (2.92), we find that the equality does not depend on the choice of local coordinates around x. The conclusion now follows from the arbitrariness of x. ∎

As we have seen in Section 2.1.3, for many (actually, most) interesting classes of Finsler spacetimes, the condition $\mathcal{A} = \overset{\circ}{T}M$ is *not* fulfilled. In this case, Berwald-Finsler spacetimes are not necessarily Lorentz metrizable.

2.3.3 How large is the set of conformal symmetries of a pseudo-Finsler space?

This subsection is a part of our paper [200]. The results below, if not otherwise stated, refer to the larger class of n-dimensional pseudo-Finsler spaces.

Definition 43 *Let (M, L) be a pseudo-Finsler space. A diffeomorphism $\phi : M \to M$ is called a **conformal symmetry** of (M, L), if there exists a smooth function $\sigma : M \to \mathbb{R}$, $x \mapsto \sigma(x)$, such that:*

$$L \circ d\phi = e^{\sigma} L. \tag{2.95}$$

In particular:
*(i) if $\sigma = const.$, then ϕ is called a **similarity;***
*(ii) if $\sigma = 0$, then ϕ is called an **isometry** of (M, L).*

Conformal symmetries of flat pseudo-Finsler functions on $\mathbb{R}^n$, $n \geq 3$, do *not* admit a Liouville-type classification, unlike their pseudo-Euclidean counterparts. Actually, as follows from the counterexamples below, there exist whole *classes* of pseudo-Finsler functions with larger (even infinite-dimensional) groups of conformal symmetries.

For $\dim M = 4$, a first such example is actually known for long in the literature, see [157]. But, this example can be immediately extended to any dimension, as presented below.

Example 1: Consider, on $M = \mathbb{R}^n$, $n > 1$:

$$\mathcal{A} = \left\{ (x^i, \dot{x}^i)_{i=\overline{0,n-1}} \mid \dot{x}^0 \dot{x}^1 .... \dot{x}^{n-1} \neq 0 \right\} \subset T\mathbb{R}^n \backslash \{0\}$$

and the n-dimensional *Berwald-Moor pseudo-Finsler function* ([135], pp. 155-156) on $\mathcal{A}$:

$$L(x, \dot{x}) = \epsilon \left| \dot{x}^0 \dot{x}^1 .... \dot{x}^{n-1} \right|^{\frac{2}{n}}, \tag{2.96}$$

where $\epsilon := sign(\dot{x}^0 \dot{x}^1 .... \dot{x}^{n-1})$. This is a flat (locally Minkowski) pseudo-Finsler metric with the property that *any* diffeomorphism of the form

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad x = \left(x^0, x^1, ..., x^{n-1} \right) \mapsto (f^0(x^0), f^1(x^1), ..., f^{n-1}(x^{n-1})), \tag{2.97}$$

with positive Jacobian determinant $J(x) := \frac{df^0}{dx^0}\frac{df^1}{dx^1}....\frac{df^{n-1}}{dx^{n-1}}$, is a conformal symmetry of L. The statement follows immediately, as:

$$L(x, df_x(\dot{x})) = J(x)^{\frac{2}{n}} L(x, \dot{x}), \quad \forall (x, \dot{x}) \in \mathcal{A}. \tag{2.98}$$

The conformal symmetry f depends on n arbitrary functions $f^0, ..., f^{n-1}$.

Remark. The Berwald-Moor metric is known, see [11], to be of Lorentzian signature $(+,-,-,-)$ on the conic convex set $\mathcal{T} = \left\{(x^i, \dot{x}^i)_{i=\overline{0,n-1}} \mid \dot{x}^i > 0,\ \forall i = 0, ..., n-1\right\}$. As the fibers of $\mathcal{T}$ are connected and $L_{|\partial\mathcal{T}} = 0$, in dimension 4, it defines a spacetime structure according to Definition 25.

Further, using the above example, we can build a whole class of flat pseudo-Finsler metrics on $\mathbb{R}^n$, $n \geq 2$, which admit conformal symmetries that are not similarities.

Example 2: *Weighted product Finsler functions.* Consider $M = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and a pseudo-Finsler metric function $L : \mathcal{A} \to \mathbb{R}$ of the form:

$$L = L_1^{\alpha} L_2^{1-\alpha}, \tag{2.99}$$

where $L_1 : \mathcal{A}_1 \to \mathbb{R}$, and $L_2 : \mathcal{A}_2 \to \mathbb{R}$ (with $\mathcal{A}_1 \subset T\mathbb{R}^k$, $\mathcal{A}_2 \subset T\mathbb{R}^{n-k}$) are smooth, 2-homogeneous functions and $\alpha \in (0, 1)$. The admissible set $\mathcal{A}$ of L is a subset of the Cartesian product $\mathcal{A}_1 \times \mathcal{A}_2$.

Assume that $f_1 : \mathbb{R}^k \to \mathbb{R}^k$, $\left(x^0, ..., x^{k-1}\right) \mapsto (\tilde{x}^0, ..., \tilde{x}^{k-1})$ is a conformal transformation for L_1, with non-constant factor $\sigma = \sigma(x)$. Then, the transformation

$$f : \mathbb{R}^n \to \mathbb{R}^n,\ f := (f_1, id_{\mathbb{R}^{n-k}}) \tag{2.100}$$

is a conformal symmetry of L with non-constant conformal factor $\alpha\sigma(x)$. The function L_1 can then be chosen. e.g., as the k-dimensional Berwald-Moor metric, or as the *k-dimensional Minkowski metric* $L_1(\dot{x}^0, ..., \dot{x}^{k-1}) = \left(\dot{x}^0\right)^2 - \left(\dot{x}^1\right)^2 - .... - (\dot{x}^n)^2$ (in which case, f_1 can be, e.g., an inversion), whereas the factor L_2 in (2.99) can be absolutely arbitrarily chosen, subject to the only condition that there exists a conic subset $\mathcal{A} \subset \mathcal{A}_1 \times \mathcal{A}_2$ on which L is smooth and has nondegenerate $\dot{x}$-Hessian.

2.3.4 A pseudo-Finslerian extension of Weyl's Theorem

The Weyl Theorem basically states that, in the class of Lorentzian spacetimes, knowing geodesics as point sets, i.e., the *projective structure*, and the *conformal structure* (which, as noted in the paper by Ehlers, Pirani&Schild [69], fixes: the light cones at each point, a notion of orthogonality and lightlike geodesics), uniquely determine the spacetime metric up to a constant rescaling. In the following, we will show that this property remains valid when passing to the much larger class of Finslerian spacetimes.

Before extending the Weyl Theorem to pseudo-Finsler spaces, let us briefly discuss Finslerian projective and, respectively, of conformal equivalence.

Assume (M, L) is a pseudo-Finsler space, with admissible set $\mathcal{A} \subset \overset{\circ}{TM}$. A diffeomorphism $f : M \to M$ between is called a *projective map* if geodesics of L coincide, up to re-parametrization, with geodesics of

$$\tilde{L} := L \circ df.$$

In a completely similar manner to the positive definite case ([7], pp. 110-111), it follows that the mapping f is projective if and only if there exists a 1-homogeneous scalar function $P : A \to \mathbb{R}$ such that, in any local chart,

$$2\tilde{G}^i(x,\dot{x}) = 2G^i(x,\dot{x}) + P(x,\dot{x})\,\dot{x}^i, \quad \forall\,(x,\dot{x}) \in \mathcal{A}. \tag{2.101}$$

On the other hand, if two pseudo-Finsler functions $L, \tilde{L} : \mathcal{A} \to \mathbb{R}$ over M are conformally related, i.e.,

$$\tilde{L}(x,\dot{x}) = e^{\sigma(x)}L(x,\dot{x}), \qquad \forall(x,\dot{x}) \in \mathcal{A}, \tag{2.102}$$

for some smooth function $\sigma = \sigma(x) : M \to \mathbb{R}$, then, the following properties follow easily:

1. L and $\tilde{L}$ share the same light cones at each point, as $L = 0 \Leftrightarrow \tilde{L} = 0$.
2. Fix a point $x \in M$. Then, for any admissible admissible vector $\dot{x} \in \mathcal{A}_x$ and any $v \in T_xM$, the conditions $L_{\cdot i}(x,\dot{x})v^i = 0$ and $\tilde{L}_{\cdot i}(x,\dot{x})v^i = 0$ (which can be interpreted, see [146], p. 127) as L, respectively $\tilde{L}$-orthogonality of $\dot{x}$ and v), are equivalent.
3. Null geodesics of two conformally related pseudo-Finsler metrics coincide up to parametrization, [200]. This is seen as follows. Along null geodesics, there holds $L = 0$; on the other hand, since L and $\tilde{L}$ are conformally related, we have (2.103), which leads to $2\tilde{G}^i = 2G^i + \sigma_{,k}y^ky^i =: 2G^i + Py^i$, for $P := \sigma_{,k}y^k$. The latter equality means exactly that (lightlike) geodesics of L and $\tilde{L}$ coincide up to reparametrization.

Note. The properties 1. and 3. above are also preserved, in the Finslerian realm, by the more general *anisotropically conformal* equivalence, obtained by allowing the conformal factor σ in (2.102) to depend smoothly on $\dot{x}$, as shown by Javaloyes and Soares, [104]. Yet, L-orthogonality is *not* preserved by anisotropically conformal transformations. Thus, the preservation of all the three features: light cones, lightlike geodesics, orthogonality - seems to remain a privilege of the usual, "isotropic" conformal relation.

The result below, proven by us in [200], is an extension to pseudo-Finsler spaces of Weyl's Theorem. This is, also, one of the results that *do* extend from (smooth, positive definite) Finsler spaces to general pseudo-Finsler spaces. For the standard Finsler case, we refer to Cheng [20] and Szilasi [182]; yet, as the cited results explicitly used the positive definiteness of the Finslerian metric tensor, a different technique was necessary for proving it in the indefinite case.

Theorem 44 *If a conformal symmetry of a connected pseudo-Finsler space (M, L) is also a projective map, then it is a similarity.*

Proof. Assume that the diffeomorphism $f : M \to M$ is a conformal symmetry of L, i.e., $\tilde{L} := L \circ df = e^{\sigma}f$ for some function $\sigma = \sigma(x) : M \to \mathbb{R}$. Then, at any $(x,\dot{x}) \in A$ and in any local chart around $(x,\dot{x})$, a direct calculation using (2.32) shows that:

$$2\tilde{G}^i = 2G^i + \frac{1}{2}g^{ih}\left(\sigma_{,k}\dot{x}^kL_{\cdot h} - \sigma_{,h}L\right) = 2G^i + \sigma_{,k}\dot{x}^k\dot{x}^i - \frac{1}{2}g^{ih}\sigma_{,h}L, \tag{2.103}$$

where in the last equality we have used: $L_{\cdot h} = 2\dot{x}_h$. If, moreover, f is a projective map, then (2.101) holds, meaning that,

$$\sigma_{,k}\dot{x}^k\dot{x}^i - \frac{1}{2}g^{ik}\sigma_{,k}L = P\dot{x}^i. \tag{2.104}$$

Now, fix an arbitrary $x \in M$ and an arbitrary open region of $\mathcal{A}_x = \mathcal{A} \cap T_xM$ where $L \neq 0$; on such a region, one can introduce the *angular metric tensor* components $h_{ij} = h_{ij}(x, \dot{x})$, as, [26]:

$$h_{ij} = g_{ij} - \frac{\dot{x}_i \dot{x}_j}{L};$$

these functions obey the identity:

$$h_{ij}\dot{x}^j = 0. \tag{2.105}$$

Contracting (2.104) with h_{ij} and using (2.105), it remains: $h_{ij}g^{ik}\sigma_{,k}L = 0$. Taking into account that $h_{ij}g^{ik} = \delta^k_j - \dfrac{\dot{x}^k \dot{x}_j}{L}$, this becomes:

$$L\sigma_{,j} - \sigma_{,k}\dot{x}^k \dot{x}_j = 0. \tag{2.106}$$

Differentiating with respect to $\dot{x}^i$ and taking into account that $L_{\cdot i} = 2\dot{x}_i$ and $\dot{x}_{\cdot j} = g_{ij}$, we get:

$$2\dot{x}_i \sigma_{,j} - \sigma_{,i}\dot{x}_j - \sigma_{,k}\dot{x}^k g_{ij} = 0.$$

Now, contract both hand sides of the above equality with $h^{ij} := g^{ik}g^{jl}h_{kl}$. Using again (2.105), we get rid of the first and of the second term. Further, noticing that $h^{ij}g_{ij} = n - 1$, we obtain: $(n-1)\,\sigma_{,k}\dot{x}^k = 0$. But, by hypothesis, $n = \dim M \geq 2$, therefore:

$$\sigma_{,h}\dot{x}^h = 0,$$

which, by differentiation with respect to $\dot{x}^k$, gives that: $\sigma_{,k}(x) = 0$. As the point x was arbitrarily chosen (and M is connected), we get $\sigma(x) = const.$, q.e.d. ■

2.3.5 Some results on conformal and Killing vector fields

In the following, (M, L) will denote a pseudo-Finsler space of arbitrary dimension.

A arbitrary vector field $\xi \in \mathcal{X}(M)$ is, by definition, a section of the natural bundle (TM, π_{TM}, M); the natural lift of its local 1-parameter group of diffeomorphisms $\phi_\varepsilon : M \to M$, $\varepsilon \in I$, is the 1-parameter group $\{d\phi_\varepsilon\}_{\varepsilon \in I}$ of fibered automorphisms of TM generated by the *complete lift* $\xi^{\mathbf{c}} \in \mathcal{X}(TM)$ (see [46]):

$$\xi^{\mathbf{c}} = \xi^i \partial_i + (\xi^i{}_{,j}\dot{x}^j)\dot{\partial}_i, \tag{2.107}$$

The flow $\{d\phi_\varepsilon\}$ acts on ξ, L and g as follows, see Section 1.1.5:

1. The vector field ξ is deformed by the rule: $\xi_\varepsilon = d\phi_\varepsilon \circ \xi \circ \phi_\varepsilon^{-1}$. But, ξ is invariant under its own flow, which means $\xi_\varepsilon = \xi$, equivalently:

$$d\phi_\varepsilon \circ \xi = \xi \circ \phi_\varepsilon. \tag{2.108}$$

2. The Finslerian metric tensor $g : \mathcal{A} \to T^0_2 M$ is transformed into the mapping:

$$g_\varepsilon := T^0_2\phi_\varepsilon \circ g \circ d\phi_\varepsilon : \mathcal{A} \to T^0_2 M \tag{2.109}$$

where $T^0_2\phi_\varepsilon : T^0_2 M \to T^0_2 M$ is the natural lift of ϕ_ε. A quick check in coordinates shows that g_ε is nothing but the Finslerian metric tensor corresponding to the deformed pseudo-Finsler function:

$$L_\varepsilon = L \circ d\phi_\varepsilon. \tag{2.110}$$

Assume, in the following, that $\xi \in \mathcal{X}(M)$ is everywhere admissible or zero, i.e., $\xi \in \Gamma(\mathcal{A}) \cup \{0\}$. Also, we will assume (if necessary, by restricting $\mathcal{A}$) that $d\phi_\varepsilon(\mathcal{A}) \subset \mathcal{A}$.

Definition 45 *An (either admissible, or zero) vector field $\xi \in \mathcal{X}(M)$ on a pseudo-Finsler space (M, L) is called a **conformal vector field,** if its flow consists of conformal symmetries of L.*

Conformality of ξ is equivalent to

$$L \circ d\phi_\varepsilon = e^{\sigma_\varepsilon} L, \forall \varepsilon \in I, \tag{2.111}$$

for some functions $\sigma_\varepsilon : M \to M$, $\varepsilon \in I$, respectively, to:

$$\mathfrak{L}_{\xi^c} L = \mu L, \tag{2.112}$$

where $\mu = \dfrac{d\sigma_\varepsilon}{d\varepsilon}|_{\varepsilon=0}$.

Particular case: By a *Killing vector field*, one understands an admissible or zero vector field $\xi \in \mathcal{X}(M)$ whose flow consists of isometries of L. In particular, Killing vector fields obey:

$$\mathfrak{L}_{\xi^c} L = 0. \tag{2.113}$$

The set of Killing vector fields of a pseudo-Finsler space of dimension n is known, see e.g., Pfeifer, [160], to form a Lie algebra of dimension at most $\dfrac{n\,(n+1)}{2}$.

Also, we have proven in [200] that the existence of a conformal or a Killing vector field gives rise to a conserved quantity along lightlike geodesics.

Here is a notion that will prove very useful in the following.

Associated (osculating) pseudo-Riemannian metrics are defined, see. e.g., [10], [180], as pseudo-Riemannian metrics on M obtained as:

$$g^\xi := g \circ \xi : M \to T_2^0 M, \tag{2.114}$$

for some admissible (hence, nonzero) vector field $\xi \in \Gamma(\mathcal{A})$. As ξ is assumed to be everywhere admissible, g^ξ is a well defined, smooth pseudo-Riemannian metric on M.

The associated metric g^ξ is thus a section of the bundle $T_2^0 M$ over M; as a consequence, the flow $\{\phi_\varepsilon\}_{\varepsilon \in I}$ of ξ (more precisely, its natural lift $T_2^0 \phi_\varepsilon$ to $T_2^0 M$) deforms it into the section:

$$\phi_\varepsilon^*(g^\xi) := T_2^0 \phi_\varepsilon \circ g^\xi \circ \phi_\varepsilon : M \to T_2^0 M. \tag{2.115}$$

That is: ξ is a conformal vector field for g^ξ if and only if $\phi_\varepsilon^*(g^\xi) = e^{\sigma_\varepsilon} g$ for some functions $\sigma_\varepsilon : M \to M$, $\varepsilon \in I$. In this case, $\phi_\varepsilon^*(g^\xi)$ is also nondegenerate.

Using these remarks, we can now prove the following Lemma, [200].

Lemma 46 *If $\xi : M \to \mathcal{A}$ is a conformal vector field for a pseudo-Finsler metric structure (M, L), with 1-parameter group $\{\phi_\varepsilon\}_{\varepsilon \in I}$, then:*

(i) ξ is also a conformal vector field for the pseudo-Riemannian metric g^ξ.

(ii) The conformal factor relating g^ξ and $\phi_\varepsilon^*(g^\xi)$ is the same as the one relating L and $L \circ d\phi_\varepsilon$.

Proof. *(i)* Assume that ξ is a conformal vector field for L, i.e., $L \circ d\phi_\varepsilon = e^{\sigma_\varepsilon} L$, $\forall \varepsilon \in I$. In terms of g, this is translated, using (2.109)-(2.110), into:

$$T_2^0\phi_\varepsilon \circ g \circ d\phi_\varepsilon = e^{\sigma_\varepsilon} g.$$

Composing this equality to the right by ξ and using (2.115), (2.108), this leads to:

$$e^{\sigma_\varepsilon} g^\xi = T_2^0\phi_\varepsilon \circ g \circ (d\phi_\varepsilon \circ \xi) \overset{(2.108)}{=} T_2^0\phi_\varepsilon \circ g \circ \xi \circ \phi_\varepsilon = T_2^0\phi_\varepsilon \circ g^\xi \circ \phi_\varepsilon \overset{(2.115)}{=} \phi_\varepsilon^*(g^\xi),$$

which means that ξ is a conformal vector field for g^ξ.

(ii) The statement follows immediately from the above relation. ■

In the following, we will present three results in Lorentzian geometry that can be extended, using the above Lemma, to Finsler spacetimes.

A property of essential conformal vector fields, [200].

A conformal vector field for a pseudo-Finsler metric L is called *essential* if it is not a Killing vector field for any conformally related metric to L. Here is a property (known in pseudo-Riemannian geometry from [123]) which can be extended to pseudo-Finsler spaces.

Proposition 47 *In a pseudo-Finsler space (M, L), any essential conformal vector field must be lightlike, i.e., $L \circ \xi = 0$, at least at a point.*

Proof. Let $\xi : M \to \mathcal{A}$ denote a conformal vector field and assume that ξ is nowhere lightlike. Since ξ is a conformal vector field for L, we have, for some $\mu : M \to \mathbb{R}$:

$$(\mathfrak{L}_{\xi^c} L)(x, \dot{x}) = \mu L(x, \dot{x}), \quad \forall (x, \dot{x}) \in \mathcal{A}. \tag{2.116}$$

We construct the real-valued function on M:

$$\alpha := g^\xi(\xi, \xi) = L \circ \xi.$$

Under our assumption that ξ is nowhere lightlike, α cannot have any zeros. Now, take:

$$\tilde{L}(x, \dot{x}) := \frac{1}{\alpha(x)} L(x, \dot{x}), \quad \forall (x, \dot{x}) \in \mathcal{A}.$$

The Lie derivative of the function $\tilde{L} : \mathcal{A} \to \mathbb{R}$ is: $(\mathfrak{L}_{\xi^c}\tilde{L}) = \mathfrak{L}_\xi(\frac{1}{\alpha})L + \frac{1}{\alpha}\mathfrak{L}_{\xi^c}(L)$. By Lemma 46, we find:

$$\mathfrak{L}_\xi \alpha = (\mathfrak{L}_\xi g^\xi)(\xi, \xi) + 2g^\xi(\mathfrak{L}_\xi \xi, \xi) = \mu g^\xi(\xi, \xi) + 0 = \mu\alpha. \tag{2.117}$$

Together with (2.116), this leads to:

$$(\mathfrak{L}_{\xi^c}\tilde{L}) = -\frac{1}{\alpha^2}\mu\alpha L + \frac{1}{\alpha}\mu L(\dot{x}) = 0,$$

meaning that ξ is a Killing vector field for $\tilde{L}$. This contradicts the hypothesis that ξ is essential. Hence, ξ must be lightlike at some $x \in M$. ■

Zeros of Killing vector fields on a Lorentz-Finsler space, [200].

Here are, for instance, two results due to Sanchez, holding on Lorentzian manifolds.

Theorem 48 *, [174]: Let (M, g) be a Lorentzian manifold with a non-spacelike (at any point) Killing vector field ξ. If $\xi_x = 0$ for some $x \in M$, then ξ vanishes identically.*

Theorem 49 *, [174]: If ξ is a Killing vector field on a Lorentzian manifold (M, g), admitting an isolated zero at some point $x \in M$, then, the dimension of M is even and ξ becomes timelike, spacelike and null on each neighborhood of x.*

Using Lemma 46, these results can easily be extended from Lorentzian spaces to Lorentz-Finsler ones[8] (of any dimension), as shown below.

Theorem 50 *Let (M, L) be a Lorentz-Finsler space, admitting a Killing vector field ξ with the property that $L(x, \xi(x)) \geq 0$, $\forall x \in M$. If $\xi = 0$ at one point $x \in M$, then ξ vanishes identically.*

Proof. Since the signature of g is assumed to be everywhere Lorentzian, it follows that the metric $g^{\xi} = g \circ \xi$ is Lorentzian, too, which makes (M, g^{ξ}) a Lorentzian manifold.

Further, as ξ is a Killing vector field for L, we get from Lemma 46 that ξ is a Killing vector field for the pseudo-Riemannian metric g^{ξ}, which, additionally, satisfies $g^{\xi}(\xi, \xi) = L \circ \xi \geq 0$; that is, ξ is non-spacelike for g^{ξ}. The statement now follows from Proposition 48. ∎

In a Finsler spacetime, Lorentzian signature of the Finsler metric tensor g is, by Definition 25, only ensured *inside* the timelike conic subbundle $\mathcal{T} \subset \mathcal{A}$; for L-lightlike vectors, g might very well not exist or be degenerate. Yet, the above result still works if we impose that $\partial\mathcal{T} \subset \mathcal{A}$. Under this assumption, the smoothness of L along $\partial\mathcal{T}$ ensures that g is defined, smooth and nondegenerate - hence, Lorentzian - on the whole set of *causal* vectors $\bar{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$. We thus obtain:

Corollary 51 *In a Finsler spacetime (M, L) such that $\bar{\mathcal{T}} \subset \mathcal{A}$, if a causal Killing vector field $\xi : M \to \bar{\mathcal{T}}$ vanishes at one point, then it must vanish identically.*

Finally, we can state:

Theorem 52 *If ξ is a Killing vector field for a Lorentz-Finsler space (M, L), admitting an isolated zero at some point $x \in M$, then, the dimension of M is even and $L \circ \xi$ takes all possible signs on each neighborhood of x.*

Proof. Assume ξ is a Killing vector field for (M, L), with an isolated zero at some $x \in M$; then, by Lemma 46, ξ is also a Killing vector for the Lorentzian metric g^{ξ} on M. But:

$$L \circ \xi = g^{\xi}(\xi, \xi), \tag{2.118}$$

in particular, the signs of $L \circ \xi$ and $g^{\xi}(\xi, \xi)$ coincide. The result now follows from Theorem 49. ∎

[8] In [200], Lorentz-Finsler spaces, as defined in Section 2.1.2 of this thesis, were called "Finsler spacetimes". Here, we preferred a more nuanced definition of the latter.

2.4 Inequalities from Finsler and Lorentz-Finsler norms

2.4.1 Introduction

The Cauchy-Schwarz inequality and the triangle inequality known for Riemannian spaces, together with their Lorentzian-reversed counterparts - which are basic results for both mathematics and physics - admit natural generalizations to Finsler, [26], respectively, Lorentz-Finsler spaces [1], [101], [143], [144].

In this section, which is a shortened version of my joint paper with N. Minculete and C. Pfeifer, [140], we will show Finsler geometry is behind some of the most notorious inequalities on $\mathbb{R}^n$: the arithmetic-geometric mean inequality together with its weighted version, Aczel's, Popoviciu's and Bellmann's inequalities. All these are nothing but reverse Cauchy-Schwarz, or reverse triangle inequalities for conveniently chosen Lorentz-Finsler functions. Similarly, Hőlder's and Minkowski's inequalities are obtained from the positive-definite (non-reversed) counterparts of these inequalities. Afterwards, the same method is put to work to construct completely new inequalities.

Moreover, in order to increase the applicability of the results, we will relax the usual assumptions on the pseudo-Finsler function L (or, accordingly, on F). Specifically, the Finslerian Cauchy-Schwarz inequality and its Lorentzian-reversed version are known in the literature, under the assumption that the Finslerian metric tensor g_v exists and everywhere positive definite for all nonzero vectors v, [26], respectively, of Lorentzian signature on the (strictly convex) set $F^{-1}([1,\infty))$, [1, 101, 143, 144]. Under these assumptions, the obtained inequalities are *strict*, i.e., equality only holds when the vectors v and w are collinear. In the following, we prove that:

- the respective inequalities still hold - just, non-strictly - if we allow g to be degenerate along some directions;

- also, for practical reasons, the strict convexity assumption on $F^{-1}([1,\infty))$ is replaced with the more relaxed one that F is defined on a convex conic domain $\mathcal{T}$.

The results below refer to Finsler (respectively, for Lorentz-Finsler) norms on the *vector space* $\mathbb{R}^{n+1}$ - but they can be extended in a straightforward manner to tangent bundles of $(n+1)$-dimensional smooth manifolds, where they will hold on each tangent space T_xM.

2.4.2 Finsler and Lorentz-Finsler functions on a vector space

On a pseudo-Finsler manifold (M,L), each partial function $F_x : \mathcal{A}_x \to [0,\infty)$, $v \mapsto F_x(v)$ is what we will call a *Finsler pseudo-norm* on any open, connected and conic subset $\mathcal{T}_x \subset \mathcal{A}_x$ where $L>0$. In the following, as already mentioned above, we will just consider this particular structure on a single, *fixed* real vector space, which we will identify as $\mathbb{R}^{n+1}$; this is why we will omit in the writing the dependence (if any) of F, $\mathcal{A}_x$ or $\mathcal{T}_x$ on the points of any manifold whatsoever and write simply, F, $\mathcal{A}$, respectively, $\mathcal{T}$.

Briefly, by a **Finsler pseudo-norm *on*** $\mathbb{R}^{n+1}$, we will understand a smooth, positively 1-homogeneous function $F : \mathcal{T} \to (0,\infty)$, $v \mapsto F(v)$, defined on an open, connected conic subset $\mathcal{T} \subset \mathbb{R}^{n+1}$, such that, at any $v \in \mathcal{T}$, the bilinear form $g_v : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$,

$$g_v(u,w) := \frac{1}{2}\frac{\partial^2 F^2}{\partial t \partial s}(v+tu+ws) \mid_{t=s=0} \tag{2.119}$$

is nondegenerate. Any Finsler pseudo-norm will be prolonged as 0 at $v = 0$.

Fixing an arbitrary basis $\{e_i\}_{i=\overline{0,n}}$ of $\mathbb{R}^{n+1}$, g_v will be written as:

$$g_v(u, w) = g_{ij}(v)u^i w^j, \qquad g_{ij}(v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}(v), \tag{2.120}$$

for all $u = u^i e_i$, $w = w^j e_j \in \mathbb{R}^{n+1}$ and $v = v^i e_i \in \mathcal{T}$. The following relations, following from the 1-homogeneity of F, will be repeatedly used in the following:

$$F(v) = \sqrt{g_v(v, v)}, \tag{2.121}$$

$$dF_v(w) = F_{\cdot i}(v)w^i = \frac{g_{ij}(v)v^j w^i}{F(v)} = \frac{g_v(v, w)}{F(v)}. \tag{2.122}$$

Particular cases: A Finsler pseudo-norm is called: a *(conic) Finsler norm* if g_v is positive definite at all $v \in \mathcal{T}$, respectively, a *Lorentz-Finsler norm*, if g_v has everywhere Lorentzian $(+, -, -, ..., -)$ signature on $\mathcal{T}$.

Remark: On a Finsler spacetime (M, L), all the considerations below will hold on the cone $\mathcal{T}_x$ of future-directed vectors at points $x \in M$. Yet, in a Finsler spacetime, we have one more property: $F_{|\partial\mathcal{T}} = 0$, which we will *not* assume it troughout this section, as it would just uselessly limit the range of allowed examples. Actually, in the following, we will relax the conditions on F even more, by allowing g_v to be degenerate (or even, not defined) along some directions.

2.4.3 Inequalities for (possibly, degenerate) Finsler and Lorentz-Finsler norms

Triangle inequality and its reverse.

The triangle inequality and its Lorentzian reverse, actually do not need any smoothness assumption on F. To be more precise, one can easily obtain the following result.

Lemma 53 *For any 1-homogeneous function $F : \mathcal{T} \to \mathbb{R}_+$ defined on some convex conic set $\mathcal{T} \subset \mathbb{R}^{n+1}$, the following statements are equivalent:*
(i) F is concave;
(ii) F obeys the reverse triangle inequality:

$$F(u + v) \geq F(u) + F(v), \qquad \forall u, v \in \mathcal{T}; \tag{2.123}$$

(iii) the set $F^{-1}([1, \infty))$ is convex.
The reverse triangle inequality is strict if and only if the concavity of F is strict.

Before proceeding to the proof, we note that the conicity and convexity of $\mathcal{T}$ ensure that for all $u, v \in \mathcal{T}$, the vectors $(1 - \alpha) u$, αv and $(1 - \alpha)u + \alpha v$, where $\alpha \in (0, 1)$, all remain in $\mathcal{T}$, that is, it makes sense to apply F to these vectors.
Proof. *(i) →(ii):* If F is concave, then, by definition:

$$F((1 - \alpha)u + \alpha v) \geq (1 - \alpha)F(u) + \alpha F(v), \tag{2.124}$$

for all $u, v \in \mathcal{T}$ and $\alpha \in [0,1]$. The triangle inequality is then the particular case thereof, obtained for $\alpha = 1/2$. Moreover, if F is *strictly* convex (i.e., (2.124) is strict), then (2.123) must be also strict.

(ii) → (i): Assume that F obeys (2.123) and pick two arbitrary vectors $u, v \in \mathcal{T}$. Then, for any $\alpha \in [0,1]$, we have, by (2.123)*:* $F((1-\alpha)u + \alpha v) \geq F((1-\alpha)u) + F(\alpha v)$, which, using the 1-homogeneity of F yields (2.124), meaning that F is concave.

If, moreover, (2.123) is strict, then, the above inequality is also strict, i.e., F is strictly convex.

(ii)→(iii): Assuming that the reverse triangle inequality (2.123) holds, pick two arbitrary vectors $v, w \in F^{-1}([1,\infty))$, that is, $F(u), F(v) \geq 1$, and an arbitrary $\alpha \in [0,1]$. Then,

$$F((1-\alpha)v + \alpha w) \geq F((1-\alpha)v) + F(\alpha w) = (1-\alpha)F(v) + \alpha F(w) \geq 1,$$

which means $F((1-\alpha)v + \alpha w) \in F^{-1}([1,\infty))$. Consequently, $F^{-1}([1,\infty))$ is convex.

(iii)→(i): The idea of the proof is similar to the one in the positive semi-definite case (see e.g., [102]). Assume $F^{-1}([1,\infty))$ is convex and pick two arbitrary vectors $v, w \in \mathcal{T}$. By the 1-homogeneity of F, it follows that the vectors $v' := \dfrac{v}{F(v)}$, $w' = \dfrac{w}{F(w)}$ obey $F(v') = F(w') = 1$, i.e., $v', w' \in F^{-1}([1,\infty))$. Set $\alpha := \dfrac{F(w)}{F(v) + F(w)} \in (0,1)$ and build the convex combination:

$$u := (1-\alpha)v' + \alpha w' = \frac{v + w}{F(v) + F(w)} \in \mathcal{T}.$$

Since $F^{-1}([1,\infty])$ is assumed to be convex, we have $u \in F^{-1}([1,\infty))$, i.e., $F(u) \geq 1$; taking into account the homogeneity of F, this gives: $F(v+w) \geq F(v) + F(w)$. ■

The above result extends a result in [1], by removing the restrictions on the continuity of F or on the boundary $\partial\mathcal{T}$.

Remark.

1. Similarly, one obtains the following equivalences (see also [26]):

   F is convex $\Leftrightarrow$ *$F^{-1}([0,1])$ is a convex set* $\Leftrightarrow$ F obeys the *triangle inequality:*

   $$F(v+w) \leq F(v) + F(w), \quad \forall v, w \in \mathcal{T}. \tag{2.125}$$

2. If, in addition, one assumes F is smooth, then one can speak about the Hessian $Hess(F)$, which is a powerful tool in characterizing the concavity/convexity of F, more precisely: convexity of F is equivalent to the fact that $Hess(F)$ is positive semidefinite, respectively, concavity of F is equivalent to the negative semidefiniteness of $Hess(F)$.

In the following, we will also relate these properties to the signature of the *Finslerian metric tensor g_v*.

Assume F is smooth and pick an arbitrary basis $\{e_i\}_{i=\overline{0,n}}$ of $\mathbb{R}^{n+1}$. Then, the components of the Hessian $Hess(F)$ are:

$$F_{\cdot i \cdot j}(v) = \frac{1}{F}[g_{ij}(v) - F_{\cdot i}(v)F_{\cdot j}(v)], \qquad \forall v \in \mathcal{T}. \tag{2.126}$$

This immediately leads to the identity:

$$F(v)F_{\cdot i \cdot j}(v)u^i u^j = g_{ij}(v)u^i u^j - (F_{\cdot i}(v)u^i)^2, \qquad \forall v \in \mathcal{T},\ \forall u \in \mathbb{R}^n, \tag{2.127}$$

which will serve in proving the following Lemma.

Lemma 54 *Consider a smooth positively 1-homogeneous function $F : \mathcal{T} \to \mathbb{R}$ defined on a conic domain $\mathcal{T} \subset \mathbb{R}^{n+1} \backslash \{0\}$. Then, at any $v \in \mathcal{T}$:*

1. *g_v has only one positive eigenvalue $\Leftrightarrow$ $Hess_v(F)$ is negative semidefinite.*
2. *g_v is positive semidefinite $\Leftrightarrow$ $Hess_v(F)$ is positive semidefinite.*

Proof. of 1:

$\rightarrow$: Fix an arbitrary $v \in \mathcal{T}$ such that g_v has only one positive eigenvalue and choose an orthogonal basis $\{e_i\}_{i=\overline{0,n}}$ for g_v, with $e_0 = v$. Since $g_v(e_0, e_0) = g_{ij}(v)v^i v^j = F^2(v) > 0$, it follows that all the other diagonal entries $g_{\alpha\alpha}(v) = g_v(e_\alpha, e_\alpha)$, $\alpha \neq 0$, are nonpositive.

Then, set $u = e_\alpha$. From (2.122), we find that $F_{\cdot i}(v)u^i = \dfrac{g_v(v,u)}{F(v)}$, i.e., the orthogonality condition $g_v(v,u) = 0$ can be re-expressed as $F_{\cdot i}(v)u^i = 0$; the latter, substituted into (2.127), gives

$$F_{\cdot i \cdot j}(v)u^i u^j = \frac{1}{F(v)} g_{ij}(v)u^i u^j = \frac{1}{F(v)} g_v(e_\alpha, e_\alpha), \tag{2.128}$$

which means: $F_{\cdot i \cdot j}(v)u^i u^j \leq 0$. Together with $F_{\cdot i \cdot j}(v)e_0^i e_0^j = F_{\cdot i \cdot j}(v)v^i v^j = 0$ (which holds by virtue of the 1-homogeneity of F), this implies that $Hess(F) = (F_{\cdot i \cdot j}(v))$ is negative semidefinite.

$\leftarrow$: Conversely, assume that $Hess_v(F)$ is negative semidefinite. Using the same g_v-orthogonal basis, we will have, again, (2.128), for all $u = e_\alpha$, $\alpha = \overline{1,n}$, which, taking into account that $(F_{\cdot i \cdot j}(v))$ is negative semidefinite, leads to: $0 \geq g_v(e_\alpha, e_\alpha)$. That is, g_v has at least n nonpositive eigenvalues. But, on the other hand, $g_v(e_0, e_0) = F^2(v) > 0$, i.e., the eigenvector $e_0 = v$ corresponds to a (unique) positive eigenvalue for g.

Statement 2. is proven similarly, taking into account that, this time, $F(v)F_{\cdot i \cdot j}(v)u^i u^j = g_{ij}(v)u^i u^j \geq 0$. ■

Putting together Lemmas 53 and 54 and the two above remarks, we find:

Proposition 55 ***(Triangle-type inequalities):*** *Consider a smooth, positively 1-homogeneous function $F : \mathcal{T} \to (0, \infty)$ defined on a convex conic domain $\mathcal{T} \subset \mathbb{R}^{n+1}$. Then:*

1. ***(The degenerate-Lorentzian case):*** *The following statements are equivalent:*
   - (a) *g_v has exactly one positive eigenvalue, for any $v \in \mathcal{T}$;*
   - (b) *the Hessian of F is negative semidefinite at all $v \in \mathcal{T}$;*
   - (c) *F is concave;*
   - (d) *the set $F^{-1}([1, \infty))$ is convex;*
   - (e) *F obeys the **reverse triangle inequality** (2.123).*
2. ***(The positive semidefinite case):*** *The following statements are equivalent:*
   - (a) *g_v is positive semidefinite for any $v \in \mathcal{T}$;*
   - (b) *the Hessian of F is positive semidefinite at all $v \in \mathcal{T}$;*
   - (c) *F is convex;*

(d) *the set* $F^{-1}([0,1])$ *is convex (where we have defined* $F(0) := 0$*);*

(e) F *obeys the* ***triangle inequality*** *(2.125).*

Strictness: The case when the concavity/convexity of F is strict - respectively, the above inequalities are strict - corresponds, see, e.g., [101], to the situation when g_v is nondegenerate.

Fundamental (Cauchy-Schwarz) inequality and its reverse

The Cauchy-Schwarz inequality can be similarly extended for degenerate metrics, as shown in the Theorem below.

Theorem 56 ***(Cauchy-Schwarz type inequalities):*** *Let* $\mathcal{T} \subset \mathbb{R}^{n+1}\backslash\{0\}$ *be a convex conic domain and* $F : \mathcal{T} \to (0,\infty)$ *a smooth, positively 1-homogeneous function. Then:*

1. ***(The degenerate-Lorentzian case):*** *If* g_v *has only one positive eigenvalue at all* $v \in \mathcal{T}$*, then, for any* $v, w \in \mathcal{T}$*, there holds the reverse Cauchy-Schwarz inequality:*

$$dF_v(w) \geq F(w). \tag{2.129}$$

2. ***(The positive semidefinite case):*** *If* g_v *is positive semidefinite at all* $v \in \mathcal{T}$*, then, there holds the Cauchy-Schwarz inequality:*

$$dF_v(w) \leq F(w). \tag{2.130}$$

If, in addition g_v *is everywhere nondegenerate, then the corresponding inequalities are strict.*

Proof. 1. The technique follows roughly the same steps as in the standard, positive definite Finsler case (see, e.g., [26], p. 8-9). Consider two arbitrary vectors $u, v \in \mathcal{T}$. Since $\mathcal{T}$ is convex and conic, it follows that $u + v \in \mathcal{T}$, i.e., $F(u + v)$ is well defined. Now, perform a Taylor expansion around v, with the remainder in Lagrange form:

$$F(u+v) = F(v) + F_{\cdot i}(v)u^i + \frac{1}{2}F_{\cdot i \cdot j}(v + \varepsilon u)u^i u^j. \tag{2.131}$$

From Lemma 54, we obtain that $F_{\cdot i \cdot j}$ is negative semidefinite, that is, $F_{\cdot i \cdot j}(v + \varepsilon u)u^i u^j \leq 0$ and therefore,

$$F(u+v) \leq F(v) + F_{\cdot i}(v)u^i. \tag{2.132}$$

Then, denoting $w := u + v$, the above becomes $F(w) \leq F(v) + F_{\cdot i}(v)(w^i - v^i)$. Using the 1-homogeneity of F, the terms $F(v)$ and $-F_{\cdot i}(v)v^i$ cancel out, which leaves us with:

$$F(w) \leq F_{\cdot i}(v)w^i, \tag{2.133}$$

which is the coordinate form of (2.129)

The strictness part was also proven in (*[101]*); we just sketch a proof here for completeness: If g_v is nondegenerate, then $F_{\cdot i \cdot j}$ has radical spanned by v, i.e., the equality $F_{\cdot i \cdot j}(v + \varepsilon u)u^i u^j = 0$ happens if and only if v and u are collinear; this leads to the strictness (2.132) and consequently, of (2.129).

Statement 2. is proven in a completely similar manner. ■

Here are some important remarks.

1. The name of Cauchy-Schwarz inequality for (2.130) is justified as follows. Using the identity $dF_v(w) = \dfrac{g_v(v,w)}{F(v)}$ (see (2.122)), this is equivalent to:

$$g_v(v,w) \leq F(v)F(w), \tag{2.134}$$

which is visibly a generalization of the Cauchy-Schwarz inequality $a(v,w) \leq \sqrt{a(v,v)}\sqrt{a(w,w)}$ holding for a usual scalar product $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Similarly, its Lorentzian reverse (2.129) can be written in the more familiar form:

$$g_v(v,w) \geq F(v)F(w). \tag{2.135}$$

2. The reverse Cauchy-Schwarz inequality loses strictness if F is degenerate-Lorentzian. To prove this, consider, for instance, $F : \mathcal{T} \to \mathbb{R}_+$, $F(v) = \sqrt{(v^0)^2 - (v^1)^2}$, where $\mathcal{T} \subset \mathbb{R}^4$ is the set of all vectors $v = (v^0, v^1, v^2, v^3)$ with $v^0 > 0$, $(v^0)^2 - (v^1)^2 > 0$. The corresponding metric tensor $g_v = diag(1,-1,0,0)$ is degenerate at all $v \in \mathcal{T}$. Picking $v = (1,0,1,0)$ and $w = (1,0,0,1)$, we have $g_v(v,w) = F(v)F(w) = 1$, while, obviously, v and w are not collinear. Obviously, the same happens in the positive semidefinite case.

Coordinate expression of the reverse Cauchy-Schwarz inequality: With respect to a given basis, (2.129) takes the form:

$$F_{\cdot i}(v)w^i \geq F(w). \tag{2.136}$$

2.4.4 Some remarkable examples

Here are some famous inequalities that can be obtained as either Cauchy-Schwarz type inequalities, or as triangle type ones, associated to Finsler or Lorentz-Finsler norms.

- **Aczél's inequality**, [2]:

$$(v^0w^0 - v^1w^1 - .... - v^nw^n)^2 \;\geq\; [(v^0)^2 - (v^1)^2 ... - (v^n)^2][(w^0)^2 - (w^1)^2 ... - (w^n)^2], \tag{2.137}$$

holding for all v, w belonging to the connected, convex conic set:

$$\mathcal{T} := \{v \in \mathbb{R}^n \mid \; v^0 \geq 0, \;\; (v^0)^2 - (v^1)^2 ... - (v^n)^2 \geq 0\}. \tag{2.138}$$

From a geometer's point of view, this is just the reverse Cauchy-Schwarz inequality (2.135) obtained for the orthonormal basis expression $(\eta_{ij}) = diag(1,-1,-1,...,-1)$ of the $(n+1)$-dimensional *Minkowski metric* η on $\mathbb{R}^{n+1}$ - or, equivalently, for the Lorentz-Finsler norm $F : \mathcal{T} \to \mathbb{R}_+$,

$$F(v) := \sqrt{\eta(v,v)} = \sqrt{(v^0)^2 - (v^1)^2 ... - (v^n)^2}.$$

The set $\mathcal{T}$ is interpreted as the future-pointing timelike cone of the Minkowski metric.

- **Popoviciu's inequality**, [165]:

$$a^0b^0 - a^1b^1 - ... - a^nb^n \geq \left[(a^0)^q - \left(a^1\right)^q - ... - (a^n)^q\right]^{\frac{1}{q}} \left[\left(b^0\right)^p - \left(b^1\right)^p - ... - (b^n)^p\right]^{\frac{1}{p}}, \tag{2.139}$$

where $p > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $(a^i), (b^i)$ belong to the convex conic domain:

$$\mathcal{T} := \{v \in \mathbb{R}^{n+1} \mid v^0, v^1, ..., v^n > 0, \left(v^0\right)^p - \left(v^1\right)^p - ... - (v^n)^p > 0\}. \tag{2.140}$$

This inequality can be obtained as a reverse Cauchy-Schwarz inequality, for the Lorentz-Finsler norm $F : \mathcal{T} \to \mathbb{R}^+$ defined by

$$F(v) := H(v)^{\frac{1}{p}}, \qquad H(v) = \left(v^0\right)^p - \left(v^1\right)^p - ... - (v^n)^p\ . \tag{2.141}$$

To check that F is indeed, a Lorentz-Finsler norm, one can use the Hessian $H_{\cdot i \cdot j}(v)$ (which is immediately seen to be Lorentzian on $\mathcal{T}$), together with (2.24). Then, the reverse Cauchy-Schwarz inequality $F(w) \leq F_{\cdot i}(v)w^i$ (which holds strictly on $\mathcal{T}$), followed by the substitutions:

$$q := \frac{p}{p-1}, \quad a^i := \left(v^i\right)^{p-1}, \quad b^j := w^j, \tag{2.142}$$

(in particular, $\dfrac{1}{p} + \dfrac{1}{q} = 1$), yields Popoviciu's inequality.

- **Bellman's inequality**:

$$(v_0^p - v_1^p - ... - v_n^p)^{1/p} + (w_0^p - w_1^p - ... - w_n^p)^{1/p} \leq [(v_0+w_0)^p - (v_1+w_1)^p - ... - (v_n+w_n)^p]^{1/p}, \tag{2.143}$$

holding (strictly) on $\mathcal{T}$, is then obviously the reverse triangle inequality applied to (2.141).

- **Hölder's inequality:**

$$a^0b^0 + a^1b^1 + ... + a^nb^n \ \leq \ \left[\left(a^0\right)^q + ... + (a^n)^q\right]^{\frac{1}{q}} \left[\left(b^0\right)^p + ... + (b^n)^p\right]^{\frac{1}{p}}, \tag{2.144}$$

which takes place for all $a^i, b^i > 0$, $i = \overline{0,n}$,, can be treated as the fundamental inequality of the Finsler norm:

$$F(v) = [\left(v^0\right)^p + ... + (v^n)^p]^{\frac{1}{p}}, \tag{2.145}$$

which is positive definite on $\mathcal{T} := \left\{v \in \mathbb{R}^{n+1} \mid v^i > 0, i = \overline{0,n}\right\}$. Substituting a, b, q as in (2.142) into the Cauchy-Schwarz inequality $F_{\cdot i}(v)w^i \leq F(w)$, one gets (2.144).

- **Minkowski's inequality:**

$$\left[\left(a^0 + b^0\right)^p + ... + (a^n + b^n)^p\right]^{\frac{1}{p}} \leq \left[\left(a^0\right)^p + ... + (a^n)^p\right]^{\frac{1}{p}} + \left[\left(b^0\right)^p + ... + (b^n)^p\right]^{\frac{1}{p}},$$

$\forall a^i, b^i > 0, i = \overline{0,n}, p > 1$ is just the triangle inequality for (2.145).

- **The arithmetic-geometric mean inequality**:
$$\frac{\alpha_0 + ... + \alpha_n}{n+1} \geq (\alpha_0\alpha_1...\alpha_n)^{\frac{1}{n+1}}, \quad \forall \alpha_i \in \mathbb{R}_+^*, \tag{2.146}$$
is obtained using the $(n+1)$-dimensional *Berwald-Moór* metric:
$$F : \mathcal{T} \to \mathbb{R}^+, \qquad F(v) = (v^0v^1...v^n)^{\frac{1}{n+1}}, \tag{2.147}$$
defined on the connected, convex conic set $\mathcal{T} := \left\{v \in \mathbb{R}^{n+1} \mid v^0, v^1, ..., v^n > 0\right\}$. As already mentioned above in Subsection 2.3.3, the metric tensor g_v associated to F is known to be of Lorentzian signature. A surprising fact is that its reverse Cauchy-Schwarz inequality (2.133) becomes, upon substituting $\alpha_i := \frac{w^i}{v^i}$, just (2.146).

- **The weighted arithmetic-geometric mean inequality:**
$$\sum_{i=0}^{n} a_i v^i \geq (v^0)^{a_0}(v^1)^{a_1}...(v^n)^{a_n}, \qquad \forall v^i > 0, \tag{2.148}$$
can be obtained similarly, using the function $F : \mathcal{T} \to \mathbb{R}^+$, defined by
$$F(v) = \left(v^0\right)^{a_0}\left(v^1\right)^{a_1}...\left(v^n\right)^{a_n}, \ \sum_{i=0}^{n} a_i = 1 \quad a_i \geq 0,$$
where, again, $\mathcal{T} = \left\{v \in \mathbb{R}^{n+1} \mid v^0, v^1, ..., v^n > 0\right\}$. A short direct calculation shows that F is a Lorentz-Finsler norm and its reverse Cauchy-Schwarz inequality (2.133) is just (2.148).

Here are also, just two examples of new inequalities that can be produced this way.

- **Kropina metric:** Start from a timelike Kropina-type deformation $F : \mathcal{T} \to \mathbb{R}_+$ of the Minkowski metric η :
$$F(v) := \frac{\eta_{ij}v^iv^j}{v^0} = \frac{1}{v^0}[(v^0)^2 - (v^1)^2 - .... - (v^1)^2],$$
where $\mathcal{T} := \left\{v \in \mathbb{R}^{n+1} \mid \eta_{ij}v^iv^j > 0, v^0 > 0\right\} \subset \mathbb{R}^{n+1}$, is the future-pointing timelike cone of η. The function F can be checked by direct computation, see [140] to be a Lorentz-Finsler norm. Then, F obeys the strict fundamental inequality (2.129), which becomes, after a short calculation:
$$2\eta(v,w) \ \geq \ \frac{w^0}{v^0}\eta(v,v) + \frac{v^0}{w^0}\eta(w,w), \qquad \forall v, w \in \mathcal{T}. \tag{2.149}$$

- **A Finslerian extension of Aczél's inequality (2.137)** is obtained by considering, on the entire space $\mathbb{R}^{n+1}$, a positive definite Finsler norm $\hat{F}$ and a 1-form $\rho = \rho_i dx^i$. Then, the mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}$,
$$F(v) = \sqrt{\rho^2(v) - \hat{F}^2(v)}$$
defines (see Theorem 4.1 in [101]), a (smooth, nondegenerate) Lorentz-Finsler norm on the convex conic domain $\mathcal{T} = \{v \in \mathbb{R}^{n+1} \mid \rho(v) > \hat{F}(v)\} \subset \mathbb{R}^{n+1}\backslash\{0\}$. The reverse Cauchy-Schwarz inequality then reads:
$$[\rho(v)\rho(w) - \hat{g}_v(v,w)]^2 \geq [\rho^2(v) - \hat{F}^2(v)][\rho^2(w) - \hat{F}^2(w)]. \tag{2.150}$$

Chapter 3

Finsler-based field theory

Pseudo-Riemannian geometry owes its tremendous flourishing to its main physical application, general relativity, in which it serves to describe one of the four fundamental physical interactions, gravity.

It is already commonplace that general relativity is, on a wide variety of physical scales, in excellent agreement with observations. Yet, there are problems, showing up at either very large scales and giving rise to the so-called dark energy/dark matter, or very small scales (its tension with quantum theory), which indicate that it is reasonable to look for a more general gravitational theory. A possible route, which we will take in this chapter, is to generalize the geometry underlying our gravitational models.

Such a more general geometry, which has the potential of producing realistic models, is precisely Finsler geometry; here is just a short and non-exhaustive list of features that recommend it (a more detailed discussion can be found in the review by Pfeifer, [159]):

- Finsler spacetimes are the most general spacetimes admitting a well defined notion of *arc length* (which is physically interpreted as *proper time*); thus, they are the most general spacetimes which satisfy the clock postulate stating that the time an observer measures between two events is given by the length of its worldline.

- Using the recent definitions of Finsler spacetimes (see Chapter 2), they also possess a smooth distribution of convex cones interpreted as future-pointing timelike cones - briefly, a well defined *cone structure*.

- They have well behaved *geodesic structure*, thus they allow for models obeying the *weak equivalence principle* stating that motion of free particles in a gravitational field takes place along geodesics of spacetime.

- What is generally no longer true in a Finslerian spacetime compared to a Lorentzian one, is local Lorentz invariance. This latter feature makes Finsler spacetimes particularly suitable for quantum gravity phenomenology models; actually, the relation between Lorentz symmetry breaking and Finsler geometry is a very active topic, [38], [81], [68], [75], [129].

Historically, the first to use geometry based on Finslerian line elements to describe physical interactions was Randers [168], in his search for a unified geometric description of gravity and electromagnetism. Later on, Finsler geometry (and, to some extent, its generalizations, Lagrange and generalized Lagrange geometry, introduced by R. Miron and his collaborators, [146]-[148]) were used in physics in various other contexts, for example: the geometric description of fields in media, [8], [47], [56], [106], [134], [158], [171], [222], the study of non-local Lorentz invariant extensions of fundamental physics, [185], [186], [178], [108]-[110], [36]-[38], and, finally, the search for an improved description of gravity [81], [124], [145], [172], [163], [93], [94], which might explain dark matter or dark energy geometrically, [90], [112], [130], [138], [139], [204].

For the latter topic, an extremely motivating example is the coupling of *kinetic gases* to Finsler spacetime geometry, proposed by us in [94], [95]. Yet, a first question is on how to proceed in our quest for a realistic Finslerian gravity model. So far, there exists a multitude of models, which mostly vary in:

- the way of obtaining the field equation: by variation from an action, [163], [93], [94], [209], by formal resemblance to the Einstein equations, e.g., [147], [195], or from further physical principles, e.g., [172].

- the choice of the fundamental variable, e.g., the Finsler function L, [172], [163], [93], the Finsler metric tensor g (e.g., [147], [209]) the nonlinear connection N, or a combination thereof, [103].

In the following, we will abide by two principles: 1) the field equations should be obtained by variational means and 2) the existence of a well-defined arc length on spacetime. The latter selects pseudo-Finsler geometry as the spacetime geometry and implies the *homogeneity* of certain degree of the related geometric objects, see Section 2.1.6.

Since these principles are, still, quite general, understanding the main features of the allowed models, is a first task to pursue. This is why this chapter introduces, in its first two sections, based on the paper [97], the general *mathematical framework for field theories* whose dynamical variables are geometric objects possessing a homogeneous dependence on direction. One such feature is the appearance of the novel, direction-dependent notion of *energy-momentum distribution tensor,* which, in the case of natural Lagrangians, will be proven, following a similar algorithm to the one in Section 1.3, to obey an *averaged conservation law.*

Then, within this framework, Section 3.3 presents the concrete Finslerian *model* which was first introduced in [93], [94]. Finally, Section 3.4 discusses the notion of cosmological symmetry in the class of Finsler spacetimes. All the results presented in this chapter are obtained by joint work with C. Pfeifer and M. Hohmann.

3.1 The general framework

3.1.1 Introduction

This section reproduces almost identically parts of the paper [97]; here are the main ideas to be discussed:

- *Establishing an appropriate class of configuration bundles, allowing us to treat (homogeneous) Finslerian geometric objects as sections and to consistently apply the tools of the calculus of variations.* Typically, in metric or metric-affine gravity theories, the configuration bundle Y sits over the spacetime manifold. Yet, in Finslerian field theories, all the typical geometric objects discussed in Chapter 2 have a nontrivial dependence on *tangent vectors* $\dot{x}$ to the spacetime manifold M. Moreover, this dependence obeys a property which cannot be ignored, which is positive *homogeneity* of some degree $k \in \mathbb{Z}$; let us recall that the 2-homogeneity of the Finsler function is essential in ensuring a well defined notion of arc length.

Thus, at first sight, one may think that the appropriate configuration bundles should sit over the tangent bundle TM. Yet, such a choice turns out to be highly problematic, due to homogeneity. The problem arises as, in this case, one cannot ensure the existence of *compactly supported variations* of geometric objects. This is seen as follows: since we want to stay in the class of k-homogeneous objects in $\dot{x}$, the considered variations must also be k-homogeneous in $\dot{x}$; but, once a k-homogeneous quantity is assumed to vanish at a boundary point $(x, \dot{x}) \in \partial D$ of the integration domain $D \subset TM$, then, by homogeneity, it will vanish along the entire ray $\{(x, \alpha\dot{x}) \mid \alpha > 0\}$ - i.e., typically, also inside D, which is not acceptable.

Another idea in the literature, see [160], is to consider as the base manifold X, the indicatrix bundle $L^{-1}(1)$ of our Finsler spacetime manifold (M, L). But, in this case, our integration domain $D \subset X$ will vary with L, which means that one cannot correctly define in this case, *fibered automorphisms* - since, as we have already seen in Subsection 1.1.3, for fibered automorphisms, the transformation of the base points (i.e., the transformation of D) cannot depend on the transformation of the field variables. Therefore, when considering theories having L-dependent quantities as our dynamical variables, $L^{-1}(1)$ or any of its subsets cannot serve as base manifold for our configuration bundle.

A possibility that does not suffer from any of these shortcomings - and which we will investigate here in detail - is to consider as the base manifold X, the positively *projectivized tangent bundle* (the projective sphere bundle) PTM^+, discussed in Section 2.2, which does not depend on the Finsler function or any derived object and allows one to correctly apply all the tools of the calculus of variations.

- *Analyzing the peculiar structure fibered manifolds* (Y, Π, PTM^+) *and of Lagrangians built upon them.* Fibered manifolds built upon PTM^+ have a quite sophisticated structure - if we are only to take into account that PTM^+ is a set of equivalence classes and a fibered manifold itself. This will of course impose conditions upon the constructed Lagrangians. For instance, any Lagrangian differential form λ^+ must become, whenever evaluated along sections (fields) $\gamma \in \Gamma(Y)$, a well defined differential form on PTM^+; in particular, this means it must be 0-homogeneous.

  Moreover, geometric objects that have a *nonzero* homogeneity degree in $\dot{x}$ - a notorious example is here the Finsler spacetime function L - cannot be directly regarded as mappings defined on PTM^+; thus, identifying them as sections of some PTM^+-based fibered manifold Y requires a careful construction. This is why we take some time to discuss the details - and prove that, using homogeneous local coordinates on PTM^+, local calculations are greatly simplified.

In the following, we will set for convenience, $\dim M = 4$; yet, the results below can be extended in a straightforward manner for arbitrary dimension. By "homogeneity" of geometric objects on TM, we will always mean *positive homogeneity in the tangent vectors* $\dot{x}$. Also, throughout the chapter, we will sometimes use a plus superscript $^+$ to designate geometric objects that are either defined on subsets of PTM^+, on bundles sitting over PTM^+; this is done especially in the cases when similar objects defined on Finslerian observer spaces $\mathcal{O}$ are also used, in order to avoid confusions.

3.1.2 Fibered manifolds over PTM^+

Consider a Finsler spacetime (M, L) and denote by (Y, Π, PTM^+) an arbitrary fibered manifold of dimension $7+m$. Then, Y will acquire a double fibered manifold structure:

$$Y \xrightarrow{\Pi} PTM^+ \xrightarrow{\pi_M} M. \tag{3.1}$$

As a consequence, Y will admit an atlas consisting of fibered charts (V, ψ), $\psi = (x^i, u^\alpha, z^\sigma)$, $i = 0,1,2,3$, $\alpha = 0,1,2$, $\sigma = 1,...,m$ on Y, that are adapted to both fibrations; thus, the two projections will be represented in these charts as:

$$\Pi : (x^i, u^\alpha, z^\sigma) \mapsto \left(x^i, u^\alpha\right), \quad \pi_M : (x^i, u^\alpha) \mapsto \left(x^i\right).$$

Further, corresponding to any induced local chart $(\Pi(V), \phi)$, $\phi = (x^i, u^\alpha)$ on PTM^+, we can introduce the homogeneous coordinates $\left(x^A\right) := (x^i, \dot{x}^i)$. This way, we obtain on $V = \Pi^{-1}(U^+)$ the coordinate functions

$$\tilde{\psi} := (x^i, \dot{x}^i, y^\sigma) = (x^A, y^\sigma)$$

on V, which we will call *fibered homogeneous coordinates*. The above introduced fiber coordinate y^σ of a point $p \in Y$ is typically not unique, as its relation to the usual coordinates $(x^i, u^\alpha, z^\sigma)$ typically depends on the choice of representative $(x, \dot{x})$ in the class of $[(x, \dot{x})]$. The precise form of this relation depends, obviously, on the concrete configuration manifold Y; just to give a hint for now, we will prove below in Section 3.1.4 that, in the particular case when Y is a space of k-*homogeneous* geometric objects, then, Y itself will be a set of equivalence classes and, on a coordinate chart with, say, $\dot{x}^3 \neq 0$, $\dot{x}^\alpha = \dot{x}^3 u^\alpha$, $\alpha = 1,2,3$, we will have:

$$\left(x^i, \dot{x}^i, y^\sigma\right) = (x^i,\ \dot{x}^3 u^0,\ \dot{x}^3 u^1,\ \dot{x}^3 u^2, \dot{x}^3;\ (\dot{x}^3)^k z^\sigma).$$

In fibered homogeneous coordinates, local sections of (Y, Π, PTM^+), say, $\gamma : W^+ \to Y$, $[(x, \dot{x})] \mapsto \gamma[(x, \dot{x})]$ (where $W^+ \subset PTM^+$ is open), are represented as:

$$\gamma : (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, y^\sigma(x^i, \dot{x}^i)). \tag{3.2}$$

The set of all such sections is denoted by $\Gamma(Y)$.

On the jet bundle J^rY, fibered charts $(V, \tilde{\psi})$ induce the fibered charts [1] $(V^r, \tilde{\psi}^r)$, with:

$$\tilde{\psi}^r = (x^i, \dot{x}^i, y^\sigma, y^\sigma{}_{,i}, y^\sigma{}_{\cdot i}, ..., y^\sigma{}_{\cdot i_1 \cdot i_2 ... \cdot i_r}),$$

where, for $k = 1, ..., r$ and $\gamma \in \Gamma(Y)$ locally represented as in (3.2),

$$y^\sigma{}_{,i_1 ... \cdot i_k}(x^j, \dot{x}^j) = \frac{\partial^k}{\partial x^{i_1} ... \partial \dot{x}^{i_k}}(y^\sigma(x^j, \dot{x}^j))$$

[1] We note that, since we are using *homogeneous* coordinates over each chart domain, the number of coordinate functions $(y^\sigma_{,i}, y^\sigma_{\cdot i})$ on the first jet bundle J^1Y is $8m$, not $7m$ as one would expect taking into account the dimension of the fibers of $J^1Y \to Y$. A similar remark applies to higher order jet bundles.

are all partial $x, \dot{x}$-derivatives up to the total order k. The canonical projections $\Pi^{r,s} : J^rY \to J^sY$, $J^r_{(x,\dot{x})}\gamma \mapsto J^s_{(x,\dot{x})}\gamma$ (with $r > s$), are then represented as:

$$\Pi^{r,s} : (x^i, \dot{x}^i, y^\sigma, y^\sigma{}_{,i_1}, ..., y^\sigma{}_{\cdot i_1 \cdot i_2 \dots \cdot i_r}) \mapsto (x^i, \dot{x}^i, y^\sigma, y^\sigma{}_{,i_1}, ..., y^\sigma{}_{\cdot i_1 \cdot i_2 \dots \cdot i_s}),$$

accordingly,

$$\Pi^r : J^rY \to PTM^+, \ (x^i, \dot{x}^i, y^\sigma, y^\sigma{}_{,i_1}, ..., y^\sigma{}_{\cdot i_1 \cdot i_2 \dots \cdot i_r}) \mapsto (x^i, \dot{x}^i).$$

As already seen in Section 1.1.4, in the calculus of variations, we need two classes of differential forms on J^rY, namely, horizontal and contact forms. In our case, these notions are translated as follows.

1. Π^r-*horizontal forms* $\rho \in \Omega_k(J^rY)$ are defined as forms that vanish whenever contracted with a Π^r-vertical vector field. In the local basis $(dx^i, d\dot{x}^i, dy^\sigma, ...dy^\sigma{}_{\cdot i_1 \dots \cdot i_r})$, they are expressed as:

$$\rho = \frac{1}{k!}\rho_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge d\dot{x}^{i_k}, \tag{3.3}$$

   where $\rho_{i_1 i_2 \dots i_k}$ are smooth functions of the coordinates on J^rY. Similarly, $\Pi^{r,s}$-horizontal forms, $0 \leq s \leq r$ are locally generated by wedge products of $dx^i, d\dot{x}^i, dy^\sigma, dy^\sigma{}_{,i} \dots, dy^\sigma{}_{\cdot i_1 \dots \cdot i_s}$.

2. *Contact forms* on J^rY are, by definition, forms $\rho \in \Omega_k(J^rY)$ that vanish along prolonged sections, i.e., $J^r\gamma^*\rho = 0$, $\forall \gamma \in \Gamma(Y)$. Standard examples of contact forms are the elements of the *local contact basis* $\{dx^i, d\dot{x}^i, \theta^\sigma, \theta^\sigma{}_{,i}, \theta^\sigma{}_{\cdot i}, ...\theta^\sigma{}_{\cdot i_1 \dots \cdot i_{r-1}}, dy^\sigma{}_{,i_1 \dots, i_r}, ...dy^\sigma{}_{\cdot i_1 \dots \cdot i_r}\}$ of $\Omega(J^rY)$, given by:

$$\theta^\sigma = dy^\sigma - y^\sigma{}_{,i} dx^i - y^\sigma{}_{\cdot i} d\dot{x}^i, \quad \theta^\sigma{}_{,i} = dy^\sigma{}_{,i} - y^\sigma{}_{,i,j} dx^j - y^\sigma{}_{,i \cdot j} d\dot{x}^j \qquad etc. \tag{3.4}$$

   Another important class of contact forms are *source forms* $\rho \in \Omega_8(J^rY)$, defined as $\Pi^{r,0}$-horizontal, 1-contact 8-forms on J^rY. In coordinates:

$$\rho = \rho_\sigma \theta^\sigma \wedge \mathrm{Vol}_0, \tag{3.5}$$

   where the expression

$$\mathrm{Vol}_0 := \mathbf{i}_{\mathbb{C}}(dx \wedge d\dot{x}) \tag{3.6}$$

   was introduced and discussed in Chapter 2, eq. (2.72).

Raising to $J^{r+1}Y$, any differential form $\rho \in \Omega_k(J^rY)$ can be uniquely decomposed as:

$$\left(\Pi^{r+1,r}\right)^* \rho = h\rho + p\rho,$$

where $h\rho$ is horizontal and $p\rho$ is contact. We recall that the horizontal component $h\rho$ is what survives of ρ when pulled back by prolonged sections $J^r\gamma$ (where $\gamma \in \Gamma(Y)$,) i.e.,

$$J^r\gamma^*\rho = J^{r+1}\gamma^*(h\rho). \tag{3.7}$$

The *horizontalization* morphism $h : \Omega(J^rY) \to \Omega(J^{r+1}Y)$, acts on the natural basis 1-forms by the rule (1.8), which in our case becomes:

$$hdx^i := dx^i, \ hd\dot{x}^i = d\dot{x}^i, \quad hdy^\sigma = y^\sigma{}_{,i} dx^i + y^\sigma{}_{\cdot i} d\dot{x}^i \quad etc. \tag{3.8}$$

Accordingly, for any smooth function f on J^rY, we obtain:

$$hdf = d_A f dx^A = d_i f dx^i + \dot{d}_i f d\dot{x}^i \in \Omega_1(J^{r+1}Y), \tag{3.9}$$

where $d_i f$ and $\dot{d}_i f$ represent total x^i- and, accordingly, total $\dot{x}^i$-derivatives (of order $r+1$). Using (3.7) for $\rho = df$, gives:

$$\partial_i(f \circ J^r\gamma) = d_i f \circ J^{r+1}\gamma, \quad \dot{\partial}_i(f \circ J^r\gamma) = \dot{d}_i f \circ J^{r+1}\gamma\,. \tag{3.10}$$

Total adapted derivatives. Alternatively, one may use a nonlinear connection on $\mathcal{A}^+ \subset PTM^+$, with local coefficients, say, $N^i{}_j$, to introduce the *total adapted derivative* operators

$$\boldsymbol{\delta}_i := d_i - N^j{}_i \dot{d}_j, \tag{3.11}$$

which help constructing manifestly covariant expressions (where we have identified $N^j{}_i$ with their pullbacks by Π^{r+1}). More precisely, using these operators, we can write (3.9) as

$$hdf = (\boldsymbol{\delta}_i f)dx^i + (\dot{d}_i f)\delta\dot{x}^i, \tag{3.12}$$

where $\delta\dot{x}^i = d\dot{x}^i + N^i{}_j dx^j$. If Y is a natural bundle over M and $f : J^rY \to \mathbb{R}$ is an invariant scalar (see Section 1.1.5), then $\boldsymbol{\delta}_i f$ and $\dot{d}_i f$ will transform, under coordinate changes induced from coordinate changes on M, as d-tensor components.

3.1.3 Fibered automorphisms

Taking into account the doubly fibered structure of the configuration bundle Y, we introduce:

Definition 57 (Automorphisms of Y) *: An automorphism of a fibered manifold (Y, Π, PTM^+) is a diffeomorphism $\Phi : Y \to Y$ such that there exists a fibered automorphism ϕ of (PTM^+, π_M, M) with $\Pi \circ \Phi = \phi \circ \Pi$.*

In particular, this means that there exists a diffeomorphism $\phi_0 : M \to M$ which makes the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\Phi"] \arrow[d, "\Pi"'] & Y \arrow[d, "\Pi"] \\ PTM^+ \arrow[r, "\phi"] \arrow[d, "\pi_M"'] & PTM^+ \arrow[d, "\pi_M"] \\ M \arrow[r, "\phi_0"] & M \end{tikzcd} \tag{3.13}$$

In fibered homogeneous coordinates, a fibered automorphism of Y is represented as:

$$\tilde{x}^i = \tilde{x}^i(x^j), \quad \dot{\tilde{x}}^i = \dot{\tilde{x}}^i(x^j, \dot{x}^j), \qquad \tilde{y}^\sigma = \tilde{y}^\sigma(x^i, \dot{x}^i, y^\mu).$$

An automorphism of Y is called *strict* if it covers the identity of PTM^+, i.e., $\phi = id_{PTM^+}$.

Generators of 1-parameter groups $\{\Phi_\varepsilon\}$ of automorphisms of Y are vector fields $\Xi \in \mathcal{X}(Y)$ that are projectable with respect to both projections Π and π_M; in coordinates, this is expressed as:

$$\Xi = \xi^i(x^j)\partial_i + \dot{\xi}^i(x^j, \dot{x}^j)\dot{\partial}_i + \Xi^\sigma(x^j, \dot{x}^j, y^\mu)\frac{\partial}{\partial y^\sigma}. \tag{3.14}$$

In particular, strict automorphisms are generated by Π-vertical vector fields $\Xi = \Xi^\sigma(x^j, \dot{x}^j, y^\mu)\dfrac{\partial}{\partial y^\sigma}$.

Given such a 1-parameter group $\{\Phi_\varepsilon\}$, any section $\gamma \in \Gamma(Y)$ is deformed into the section

$$\gamma_\varepsilon := \Phi_\varepsilon \circ \gamma \circ \phi_\varepsilon^{-1}.$$

In first approximation, if γ is locally represented as: $\gamma : \left(x^i, \dot{x}^i\right) \mapsto \left(x^i, \dot{x}^i, y^\sigma\left(x^i, \dot{x}^i\right)\right)$, then:

$$\gamma_\varepsilon : \left(x^i, \dot{x}^i\right) \mapsto \Big(x^i, \dot{x}^i, y^\sigma\left(x^i, \dot{x}^i\right) + \varepsilon(\tilde{\Xi}^\sigma \circ J^1\gamma)|_{(x^i, \dot{x}^i)} + \mathcal{O}(\varepsilon^2)\Big),$$

where

$$\tilde{\Xi}^\sigma := (\Xi^\sigma - \xi^i y^\sigma{}_{,i} - \dot{\xi}^i y^\sigma{}_{\cdot i}). \tag{3.15}$$

The functions $\tilde{\Xi} \circ J^1\gamma$, defined on each local chart in the domain of γ, are commonly denoted by δy^σ.

The automorphisms $\Phi_\varepsilon : Y \to Y$ are prolonged into automorphisms $J^r\Phi_\varepsilon$ of J^rY by the rule:

$$J^r\Phi_\varepsilon(J^r_{(x,\dot{x})}\gamma) := J^r_{\phi(x,\dot{x})}\gamma_\varepsilon.$$

The generator of the 1-parameter group $\{J^r\Phi_\varepsilon\}$ is called the r-th prolongation of the vector field Ξ and denoted by $J^r\Xi$.

3.1.4 Homogeneous geometric objects on TM as sections

A priori, k-homogeneous Finslerian geometric objects are local sections of some fiber bundle $\overset{\circ}{Y}$ sitting on $\overset{\circ}{TM}$. In the following, we will reinterpret these objects as sections of a bundle Y with base PTM^+, which, as we have argued above, is more appropriate in view of the calculus of variations. The construction of the configuration bundle (Y, Π, PTM^+) was inspired by the one made in [79, Sec 5.4] for the bundle of principal connections - and relies on factoring out an action of $(\mathbb{R}^*_+, \cdot)$, from both the total space and the base of the original bundle $\overset{\circ}{Y}$.

Consider a fiber bundle

$$\overset{\circ}{Y} \overset{\overset{\circ}{\Pi}}{\to} \overset{\circ}{TM},$$

with typical fiber Z. In the following, it will be convenient to abuse the notation by explicitly mentioning the base point of any element in $\overset{\circ}{Y}$, i.e., we will identify[2] elements $y \in \overset{\circ}{Y}$ as triples $(x, \dot{x}, y)$, where $(x, \dot{x}) = \overset{\circ}{\Pi}(y)$.

[2]Such an identification is always possible, via the fibered manifold isomorphism $(\overset{\circ}{\Pi}, id_{\overset{\circ}{Y}}) : \overset{\circ}{Y} \to \overset{\circ}{TM} \times_{\overset{\circ}{TM}} \overset{\circ}{Y}$ covering the identity of $\overset{\circ}{TM}$.

Assume that $(\mathbb{R}_+^*, \cdot)$ acts on $\overset{\circ}{Y}$ by some fibered automorphisms:

$$H: \mathbb{R}_+^* \times \overset{\circ}{Y} \to \overset{\circ}{Y}, \quad H(\alpha, \cdot) = H_\alpha \in Aut(\overset{\circ}{Y}) \tag{3.16}$$

of the form:

$$H_\alpha(x, \dot{x}, y) = \left(x, \alpha\dot{x}, \alpha^k y\right), \tag{3.17}$$

for some fixed $k \in \mathbb{R}$. In particular, each automorphism $H_\alpha \in Aut(\overset{\circ}{Y})$ covers the homothety $\chi_\alpha : \overset{\circ}{TM} \to \overset{\circ}{TM}$, $(x, \dot{x}) \mapsto (x, \alpha\dot{x})$, introduced in Section 2.1.6.

Note. In the following, we do not assume a specific form of the fiber Z of $\overset{\circ}{Y}$, we just assume that, for a given k, rescaling of any fiber element y by the power α^k, $\forall\alpha > 0$, makes sense. For instance, in the case of vector bundles over $\overset{\circ}{TM}$, this makes sense for any $k \in \mathbb{R}$, whereas for bundles whose fibers do not admit a rescaling of elements, one is forced to choose $k = 0$.

The quotient manifold Y. Now, consider the space of orbits of the action H, i.e., the set:

$$Y = \overset{\circ}{Y}_{/\sim}, \tag{3.18}$$

where the equivalence relation $\sim$ is given by:

$$(x, \dot{x}, y) \sim (x', \dot{x}', y') \Leftrightarrow \exists\alpha > 0 : \quad (x', \dot{x}', y') = H_\alpha(x, \dot{x}, y). \tag{3.19}$$

The action H can be easily shown to be free and proper (properness is proven by verifying that the mapping $f : \mathbb{R}_+^* \times \overset{\circ}{Y} \to \overset{\circ}{Y} \times \overset{\circ}{Y}$, $(\alpha, x, \dot{x}, y) \mapsto (x, \alpha\dot{x}, \alpha^k y, x, \dot{x}, y)$ is proper); as a consequence, the Quotient Manifold Theorem (see [128], Ch. 21) ensures that Y is a smooth manifold.

Moreover, $\overset{\circ}{Y}$ acquires the structure of a *principal bundle* over Y, with fiber $\mathbb{R}_+^*$ and projection

$$\mathrm{proj}_Y : \overset{\circ}{Y} \to Y, \quad (x, \dot{x}, y) \mapsto [(x, \dot{x}, y)]. \tag{3.20}$$

In the following, we show that Y is indeed the configuration space we are looking for.

Theorem 58 ***(Structure of the orbit space*** Y***):*** *Let* $\overset{\circ}{Y} \xrightarrow{\overset{\circ}{\Pi}} \overset{\circ}{TM}$ *be a fiber bundle with typical fiber* Z*, equipped with an action* H *of* $(\mathbb{R}_+^*, \cdot)$ *by rescaling, as in (3.16)-(3.17). Then:*

1. *The orbit space* $Y = \overset{\circ}{Y}_{/\sim}$ *of* H *is a fiber bundle over* PTM^+*, with typical fiber* Z.

2. k*-homogeneous sections* $f : \mathcal{Q} \to \overset{\circ}{Y}$*, where* $\mathcal{Q} \subset \overset{\circ}{TM}$ *is a conic subbundle, are in a one-to-one correspondence with local sections* $\gamma : \mathcal{Q}^+ \to Y$*, where* $\mathcal{Q}^+ = \pi^+(\mathcal{Q}) \subset PTM^+$.

Proof. 1. First, let us define the projection:

$$\Pi : Y \to PTM^+, \quad [(x, \dot{x}, y)] \mapsto [(x, \dot{x})]. \tag{3.21}$$

This mapping is independent of the choice of representatives in the class $[(x, \dot{x}, y)]$, as $\Pi[(x, \alpha\dot{x}, \alpha^k y)] = [(x, \alpha\dot{x})] = [(x, \dot{x})] = \Pi[(x, \dot{x}, y)]$; moreover, it is obviously a surjective submersion, which means that (Y, Π, PTM^+) is a fibered manifold.

It remains to construct a local trivialization of Y. With this aim, start with a given local trivialization of $\mathring{Y}$ and consider a chart domain $\mathring{V} := \mathring{\Pi}^{-1}(U) \subset \mathring{Y}$, such that there exists a diffeomorphism:

$$\mathring{V} \simeq U \times Z, \tag{3.22}$$

where $U \subset \mathring{T}M$ is a small enough coordinate neighborhood on which one of the coordinates $\dot{x}^i$ keeps a constant sign. But, on the one hand, as $(\mathring{T}M, \pi^+, PTM^+, \mathbb{R}_+^*)$ is a principal bundle, which gives a diffeomorphism

$$U \simeq U^+ \times \mathbb{R}_+^*,$$

where $U^+ = \pi^+(U) \subset PTM^+$. We thus get:

$$\mathring{V} \simeq (U^+ \times \mathbb{R}_+^*) \times Z.$$

On the other hand, using the principal bundle structure of $(\mathring{Y}, \mathrm{proj}_Y, Y, \mathbb{R}_+^*)$, we obtain, for small enough $\mathring{V}$ (i.e., for small enough U), the diffeomorphism:

$$\mathring{V} \simeq V \times \mathbb{R}_+^*,$$

where $V := \mathrm{proj}_Y(\mathring{V})$. This way, the given trivialization (3.22) of $\mathring{Y}$ can be written as follows:

$$\begin{tikzcd} V \times \mathbb{R}_+^* \arrow[r] \arrow[d, "\mathring{\Pi}"'] & (U^+ \times \mathbb{R}_+^*) \times Z . \arrow[dl, "\mathrm{proj}_1"] \\ U^+ \times \mathbb{R}_+^* & \end{tikzcd}$$

Assuming $\dot{x}^3$ has constant sign on U, a system of $\mathbb{R}_+^*$-adapted fibered coordinate functions on $\mathring{V}$ is of the form $(x^i, u^\alpha, \dot{x}^3, z^\sigma)$, where (x^i, u^α), with $u^\alpha = \dfrac{\dot{x}^\alpha}{\dot{x}^3}$, $\alpha = 0, 1, 2$ are coordinate functions on $U^+ \subset PTM^+$.

A local trivialization of Y is obtained by discarding the $\mathbb{R}_+^*$ factor in the above diagram:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\Pi"'] & U^+ \times Z . \arrow[dl, "\mathrm{proj}_1"] \\ U^+ & \end{tikzcd}$$

where Π is as in (3.21). Indeed, the mappings above are smooth and the top arrow is obviously a diffeomorphism; moreover, an elementary reasoning shows that $V = \Pi^{-1}(U^+)$, which completes the proof.

Using the above trivialization, the corresponding fibered coordinates on $V \subset Y$ are obtained by discarding the $\dot{x}^3$ coordinate from the coordinates $\left(x^i, u^\alpha, \dot{x}^3, z^\sigma\right)$ on $\overset{\circ}{V}$, i.e.,

$$\psi := \left(x^i, u^\alpha, z^\sigma\right), \quad \alpha = 0, 1, 2.$$

2. Let $f : \mathcal{Q} \to \overset{\circ}{Y}$, $(x, \dot{x}) \mapsto f(x, \dot{x}) \in \overset{\circ}{Y}_{(x,\dot{x})}$ be a k-homogeneous section, i.e., $f(x, \alpha\dot{x}) = \alpha^k f(x, \dot{x})$, $\forall \alpha > 0$ and define:

$$\gamma : \mathcal{Q}^+ \to Y, \quad \gamma[(x, \dot{x})] = [(x, \dot{x}, f(x, \dot{x}))].$$

The mapping γ is independent of the choice of representatives $(x, \dot{x}) \in [(x, \dot{x})]$ by virtue of the k-homogeneity of f. Moreover, $(\Pi \circ \gamma)[(x, \dot{x})] = [(x, \dot{x})]$ for all $(x, \dot{x}) \in \mathcal{Q}$, which makes γ a well defined local section of Y.

Injectivity of the correspondence $f \mapsto \gamma$ is immediate. To prove surjectivity, pick an arbitrary $\gamma \in \Gamma(Y)$ and define $f(x, \dot{x})$, for every representative $(x, \dot{x}) \in [(x, \dot{x})]$, as the third component y of the representative $(x, \dot{x}, y) \in \gamma[(x, \dot{x})]$; then, $f(x, \alpha\dot{x}) = \alpha^k y$ by the definition of equivalence classes in Y, which means that f is a k -homogeneous section of $\overset{\circ}{Y}$. ■

Homogeneous fibered coordinates on Y. Since we have now fixed, in (3.16), (3.17), the group action of $\mathbb{R}^*_+$ on $\overset{\circ}{Y}$, on each fibered chart domain (V, ψ), $\psi = \left(x^i, u^\alpha, z^\sigma\right)$ of Y as above, we can explicitly introduce *homogeneous fibered coordinates* as the local coordinates

$$\left(x^i, \dot{x}^i, y^\sigma\right) := (x^i,\ \dot{x}^3 u^\alpha,\ (\dot{x}^3)^k z^\sigma) \tag{3.23}$$

of an arbitrarily chosen representative of the class $[x, \dot{x}, y]$ where $u^\alpha = \frac{\dot{x}^\alpha}{\dot{x}^3}$, $\alpha = 0, 1, 2$. These are, obviously unique up to positive rescaling, i.e., $\left(x^i, \dot{x}^i, y^\sigma\right)$ and $(x^i, \alpha\dot{x}^i, \alpha^k y^\sigma)$ will represent the same class.

Examples.

1. **Finsler functions** $L : \mathcal{A} \to \mathbb{R}$. In this case, $\overset{\circ}{Y} = \overset{\circ}{TM} \times \mathbb{R}$ is a trivial line bundle, which means the configuration bundle $Y = \overset{\circ}{Y}_{/\sim}$ is the space of orbits of the Lie group action $H : \mathbb{R}^*_+ \times \overset{\circ}{Y} \to \overset{\circ}{Y}$ given by the fibered automorphisms:

$$H_\alpha : \overset{\circ}{Y} \to \overset{\circ}{Y}, \quad H_\alpha(x, \dot{x}, y) = (x, \alpha\dot{x}, \alpha^2 y), \quad \forall \alpha > 0. \tag{3.24}$$

This way, 2-homogeneous Finsler functions are identified with local sections γ

$$L \mapsto \gamma \in \Gamma(Y), \quad \gamma[(x, \dot{x})] = [x, \dot{x}, L(x, \dot{x})]. \tag{3.25}$$

In homogeneous fibered coordinates, the class $[(x, \dot{x}, L(x, \dot{x}))]$ is represented as $(x^i, \dot{x}^i, L(x, \dot{x}))$.

2. **0-homogeneous metric d-tensors** $g : \mathcal{A} \to T^0_2(\overset{\circ}{TM})$, $g_{(x,\dot{x})} = g_{ij}(x, \dot{x}) dx^i \otimes dx^j$ (see Section 2.1.6), are obtained as sections γ of the bundle $Y = \overset{\circ}{Y}_{/\sim}$, where $\overset{\circ}{Y} = T^0_2(\overset{\circ}{TM})$, where the Lie group action $H : \mathbb{R}^*_+ \times \overset{\circ}{Y} \to \overset{\circ}{Y}$ is given by

$$H_\alpha : \overset{\circ}{Y} \to \overset{\circ}{Y}, \quad H_\alpha(x, \dot{x}, y) = (x, \alpha\dot{x}, y), \quad \forall \alpha > 0.$$

In fibered homogeneous coordinates naturally induced by the coordinates $\left(x^i\right)$ on M, these sections are represented as $\gamma : \left(x^i, \dot{x}^i\right) \mapsto (x^i, \dot{x}^i, g_{ij}(x, \dot{x}))$.

Homogeneous d-tensors of any rank and any homogeneity degree can be treated similarly.

3.1.5 Finsler field Lagrangians, action, extremals

Taking into account that Finslerian geometric objects typically have a homogeneous dependence on direction - and this allows one to identify them as sections of fibered manifolds (Y, Π, PTM^+), let us introduce the following definition.

Definition 59 (Fields) *By a Finslerian field, we will understand a section γ of a fibered manifold (Y, Π, PTM^+) over the positively projectivized tangent bundle PTM^+.*

Having in mind that $\dim M = 4$ (that is, $\dim PTM^+ = 7$), a *field Lagrangian of order r* is a Π^r-horizontal 7-form $\lambda \in \Omega_7(J^rY)$. In homogeneous fibered coordinates, it can be expressed in two ways, as:

$$\lambda^+ = \mathcal{L}\mathrm{Vol}_0 = \Lambda d\Sigma^+, \tag{3.26}$$

where:

- $\mathrm{Vol}_0 = \mathbf{i}_{\mathbb{C}}(dx \wedge d\dot{x})$ and the expression $\mathcal{L}$ will be called the *Lagrangian density* corresponding to λ^+;
- $d\Sigma^+$ is an arbitrary *invariant* volume form on an appropriately chosen open subset $\mathcal{Q}^+ \subset PTM^+$; accordingly, $\Lambda = \Lambda(x^i, \dot{x}^i, y^\sigma, y^\sigma_{,i}, ..., y^\sigma_{\cdot i_1 \dots \cdot i_r})$ is called a *Lagrangian function.*

Example. If M is equipped with a pseudo-Finsler function $L : \mathcal{A} \to \mathbb{R}$, one can choose as $d\Sigma^+$ the canonical volume form (2.75) on the set $\mathcal{A}_0^+ \subset PTM^+$ of admissible directions along which $L \neq 0$; in this case, the relation between the Lagrangian function Λ and the Lagrange density $\mathcal{L}$ is:

$$\mathcal{L} = \Lambda \frac{|\det g|}{L^2}. \tag{3.27}$$

The fact that the base manifold of our configuration bundle is PTM^+ entails the *0-homogeneity* of all Finsler field Lagrangians λ - where, by 0-homogeneity of a geometric object on J^rY, we will mean that the homogeneous coordinate expression of respective object formally corresponds to a positively 0-homogeneous object on $\overset{\circ}{TM}$.

More precisely, using any globally defined, invariant volume form $d\Sigma^+$ on PTM^+ - which in particular, means that it is at least well defined locally on PTM^+, i.e., it is 0-homogeneous, we find the following result.

Proposition 60 *In homogeneous local coordinates corresponding to any fibered chart (V^r, ψ^r) on Y, any Finsler field Lagrangian function $\Lambda : V^r \to \mathbb{R}$ as in (3.26) must obey:*

$$\dot{x}^i \dot{d}_i \Lambda = 0\,. \tag{3.28}$$

Proof. Pick an arbitrary section of Π, say, $\gamma : U \to Y$, where $U \subset PTM^+$ is a local chart domain. The function $\Lambda \circ J^r\gamma$ is then defined on a subset of PTM^+, hence, it must be 0-homogeneous in $\dot{x}$; that is,

$$\dot{x}^i \dot{\partial}_i(\Lambda \circ J^r\gamma) = 0.$$

But, from (3.10), we find: $\dot{\partial}_i(\Lambda \circ J^r\gamma) = (\dot{d}_i\Lambda) \circ J^{r+1}\gamma$, which, substituted into the above relation and using the arbitrariness of γ, leads to (3.28). ∎

The formulation of fields as local sections of a configuration bundle (Y, Π, PTM^+), allows us now to straightforwardly apply the coordinate-free formulation of the calculus of variations for Finsler field Lagrangians presented in Chapter 1.

- The **action** attached to the Lagrangian (3.26) and to a piece $D^+ \subset PTM^+$ is the function $S_{D^+} : \Gamma(Y) \to \mathbb{R}$, given by:

$$S_{D^+}(\gamma) = \int\limits_{D^+} J^r\gamma^*\lambda^+.$$

- **Variations as Lie derivatives.** The *variation* of the action under the flow $\{\Phi_\varepsilon\}$ of a doubly projectable vector field $\Xi \in \mathcal{X}(Y)$ is given by the Lie derivative

$$\delta_\Xi S_D(\gamma) = \int\limits_{D^+} J^r\gamma^*\mathfrak{L}_{J^r\Xi}\lambda^+. \tag{3.29}$$

- **Critical sections.** A field $\gamma \in \Gamma(Y)$, $[(x,\dot{x})] \mapsto \gamma[(x,\dot{x})]$ on a Finsler spacetime is a *critical section* for S, if for any piece $D^+ \subset PTM^+$ and for any Π-vertical vector field Ξ such that $supp(\Xi \circ \gamma) \subset D^+$, there holds: $\delta_\Xi S_D(\gamma) = 0$.

- **Euler-Lagrange form and Noether currents.** For any Lagrangian $\lambda^+ \in \Omega_7(Y)$, there exists a unique source form $\mathcal{E}(\lambda^+) \in \Omega_8(J^{s+1}Y)$ of order $s+1 \leq 2r$, called the *Euler-Lagrange form* of λ^+, such that:

$$J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda^+) = J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}(\mathcal{E}(\lambda^+)) - d(J^s\gamma^*\mathcal{J}^\Xi), \tag{3.30}$$

  for some $\mathcal{J}^\Xi \in \Omega_6(J^sY)$. The *Noether current* 6-form $\mathcal{J}^\Xi$ is only unique up to a total derivative; in integral form, this reads:

$$\int\limits_{D^+} J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda^+) = \int\limits_{D^+} J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}(\lambda^+) - \int\limits_{\partial D^+} J^s\gamma^*\mathcal{J}^\Xi. \tag{3.31}$$

  In the local contact basis, $\mathcal{E}(\lambda^+)$ is described as:

$$\mathcal{E}(\lambda^+) = \mathcal{E}_\sigma\theta^\sigma \wedge \mathrm{Vol}_0,$$

  where $\mathcal{E}_\sigma$ are the Euler-Lagrange expressions (1.34) of λ^+.

3.2 The energy-momentum distribution tensor

3.2.1 Introduction

As already pointed out in Chapter 1, general covariance of Lagrangians leads to a notion of energy-momentum tensor, respectively, to an energy-momentum balance law. As stated in [84], the energy-momentum tensor "measures the response of fields to compactly supported diffeomorphisms of spacetime".

Passing to Finslerian field theories, in order to measure such a response, we must first take into account that our configuration manifolds do not have as their base the spacetime manifold M, but its positively projectivized tangent bundle PTM^+; that is, canonical lifts of spacetime diffeomorphisms will be, by the nature of our configuration spaces, *double lifts*. This will lead, on the one hand to the novel notion of PTM^+-based ($\dot{x}$-dependent) *energy-momentum distribution tensor.* On the other hand, we prove that, due to the fact that general covariance of Lagrangians is based on M, which is a space of lower dimension than PTM^+, the resulting energy-momentum balance law will be a "weaker" one, more precisely, it will not be a pointwise one, but an *averaged* one - expressed as the vanishing of an integral over $\dot{x}$. The construction presented below, which is part of the paper [97], adapts to the Finslerian context the ideas in Section 1.3.

To identify the energy-momentum tensor in our construction of field theories on Finsler spacetimes, we need some preparations:

1. *Lifts of diffeomorphisms* ϕ_0 of M into doubly fibered automorphisms of the configuration manifold Y, that cover the natural lifts[3] of ϕ_0 to PTM^+, see the diagram (3.13).
2. *A splitting of the total Lagrangian* λ^+ *of the theory into a background (vacuum) Lagrangian* λ_g^+ *and a matter one* λ_m^+ and, accordingly, of the variables of the theory into background and dynamical ones. The background Lagrangian will only depend on the background variables (e.g., metric components, or a Finsler function etc.), whereas the matter Lagrangian λ_m^+ will depend on all the variables of the theory. In other words, denoting the background coordinates by y_B^σ and *non-background* or *dynamical* variables by y_D^σ, we have:

$$\lambda^+(y_B^\sigma, ..., y_{B,i\dots j}^\sigma, y_D^\sigma, ...y_{D,i\dots j}^\sigma) = \lambda_g^+(y_B^\sigma, ..., y_{B,i\dots j}^\sigma) + \lambda_m^+(y_B^\sigma, ..., y_{B,i\dots j}^\sigma, y_D^\sigma, ...y_{D,i\dots j}^\sigma).$$

Then, under the assumption that the matter Lagrangian λ_m^+ is generally covariant, it will be invariant under any one-parameter group of canonical lifts of diffeomorphisms of M, thus giving rise to conserved Noether currents $\mathcal{J}^\Xi$ (where Ξ is the canonical lift to Y of a diffeomorphism generating vector field $\xi_0 \in \mathcal{X}(M)$). Roughly speaking, the energy-momentum tensor will be given by the correspondence $\xi_0 \mapsto \mathcal{J}^\Xi$.

Assumption. In the case of Finsler spacetimes, the fundamental background variable is the Finsler Lagrangian L itself - which we will thus assume in the following. This way, our configuration space will be a fibered product

$$Y := Y_g \times_{PTM^+} Y_m$$

[3] Such lifts exist, e.g. when Y is a bundle of k-homogeneous d-tensors, which is the $\mathbb{R}_+^*$-orbit space of a bundle $\overset{\circ}{Y}$ of d-tensors on $\overset{\circ}{TM}$. Diffeomorphisms ϕ_0 of M are naturally lifted into fibered automorphisms $d\phi_0$ of TM and further, by tensor lifting to $\overset{\circ}{Y}$; finally, the lifts $PTM^+ \to Y$ are obtained by projectivisation.

over PTM^+, where $Y_g = (\overset{\circ}{TM} \times \mathbb{R})_{/\sim}$ was constructed in Section 3.3.2 and Y_m is both a fiber bundle over PTM^+ and a natural bundle over M. Under this assumption, we will show that any vector field $\xi_0 \in \mathcal{X}(M)$ admits a canonical lift $\Xi \in \mathcal{X}(Y)$.

As Y sits over PTM^+, it acquires a double fibered manifold structure:

$$Y \xrightarrow{\Pi} PTM^+ \xrightarrow{\pi_M} M;$$

we denote the homogeneous coordinates corresponding to a doubly fibered chart on Y by $(x^i, \dot{x}^i, \hat{L}, y^\sigma_D)$, where $y_B = \hat{L}$ is the coordinate on the fiber of Y_g and y^σ_D are local coordinates on the fiber of Y_m.

These being said, consider a Lagrangian $\lambda^+_m \in \Omega(J^rY)$ of order r,

$$\lambda^+_m = \mathcal{L}_m(x^i, \dot{x}^i, \hat{L}, \hat{L}_{,i}, \hat{L}_{\cdot i}, ..., \hat{L}_{\cdot i_1 \dots i_r}, y^\sigma_D, ..., y^\sigma_{D \cdot i_1 \dots i_r})\mathrm{Vol}_0,$$

which will be interpreted as the *matter Lagrangian*. We will assume that λ^+_m is *natural (generally covariant),* meaning that, for any compactly supported vector field $\xi_0 \in \mathcal{X}(M)$ on spacetime, λ^+_m is invariant under the flow of the canonical lift of ξ_0 i.e.: $\mathfrak{L}_{J^r\Xi}\lambda^+_m = 0$. Applying the horizontalization operator h, this gives:

$$h\mathfrak{L}_{J^r\Xi}\lambda^+_m = 0. \tag{3.32}$$

In the following, we will explore in detail the consequences of this invariance of λ^+_m.

3.2.2 Construction of the energy momentum distribution tensor

Canonical lifts of $\xi_0 \in \mathcal{X}(M)$ to Y.

Assume $\{\phi_{0,\varepsilon}\}$ is a 1-parameter group of compactly supported diffeomorphisms of M, generated by $\xi_0 \in \mathcal{X}(M)$, $\xi_0 = \xi^i\partial_i$. Then:

1. Each $\phi_{0,\varepsilon}$ is first naturally lifted to TM, as $\phi_\varepsilon := d\phi_{0,\varepsilon}$. The generator of $\{\phi_\varepsilon\}$ is the complete lift $\xi = \xi_0^{\mathbf{c}} \in \mathcal{X}(\overset{\circ}{TM})$ of ξ_0 :

$$\xi = \xi^i\partial_i + \dot{\xi}^i\dot{\partial}_i, \qquad \dot{\xi}^i = \xi^i{}_{,j}\dot{x}^j. \tag{3.33}$$

   The complete lift ξ is 0-homogeneous, which means that we can identify it with a vector field on PTM^+, see Section 2.2.2.

2. Further, taking into account that $Y_g = (\overset{\circ}{TM} \times \mathbb{R})_{/\sim}$ is obtained as a quotient space of the *trivial* bundle $\overset{\circ}{TM} \times \mathbb{R}$, the canonical lift $\Phi_{g,\varepsilon} : Y_g \to Y_g$ of ϕ_ε is also a trivial one i.e., it acts on the fiber variable $\hat{L}$ as the identity:

$$\Phi_{g,\varepsilon}[(x, \dot{x}, \hat{L})] = [(\phi_\varepsilon(x, \dot{x}), \hat{L})].$$

   The above mapping is well defined, i.e., independent of the choice of the representative of the class $[(x, \dot{x}, \hat{L})]$, due to the linearity of ϕ_ε in $\dot{x}$. Moreover, since all $\Phi_{g,\varepsilon}$ act trivially on $\hat{L}$, the generator ξ is canonically lifted into a vector field $\Xi_g \in \mathcal{X}(Y_g)$, with vanishing $\dfrac{\partial}{\partial \hat{L}}$ component:

$$\Xi_g = \xi^i\partial_i + \dot{\xi}^i\dot{\partial}_i + 0\frac{\partial}{\partial \hat{L}}.$$

3. According to our first assumption at the beginning of this section, ξ can also be canonically lifted into some vector field $\Xi_m \in \mathcal{X}(Y_m)$, $\Xi = \xi^i \partial_i + \dot{\xi}^i \dot{\partial}_i + \Xi^\sigma \dfrac{\partial}{\partial y^\sigma_D}$. All in all, we obtain that the canonical lift of $\xi_0 \in \mathcal{X}(M)$ to the fibered product $Y = Y_g \times_{PTM^+} Y_m$ is expressed in a fibered chart by adding to ξ the contributions describing the transformation of each of the fiber variables:
$$\Xi = \xi^i \partial_i + \dot{\xi}^i \dot{\partial}_i + 0 \frac{\partial}{\partial \hat{L}} + \Xi^\sigma \frac{\partial}{\partial y^\sigma_D}, \tag{3.34}$$
where Ξ^σ are functions of the coordinates $x^i, \dot{x}^i, y^\sigma_D, ..., y^\sigma_{D\cdot i_1 \dots i_r}$ and of a finite number of the derivatives of ξ^i.

First variation formula.

Recalling that our configuration bundle Y is the fibered product $Y := Y_g \times_{PTM^+} Y_m$, the Euler-Lagrange form $\mathcal{E}(\lambda^+_m)$ will be split into a Y_g and a Y_m-component as:
$$\mathcal{E}(\lambda^+_m) = \mathcal{E}_g(\lambda^+_m) + \mathcal{E}_m(\lambda^+_m).$$

Denoting the order of $\mathcal{E}(\lambda^+_m)$ by $s+1$ (where, obviously, $s+1 \leq 2r$), in the local contact basis of $\Omega(J^{s+1}Y)$, the above source forms are written as:
$$\mathcal{E}_g(\lambda^+_m) = \frac{\delta \mathcal{L}_m}{\delta \hat{L}} \theta \wedge \mathrm{Vol}_0, \qquad \mathcal{E}_m(\lambda^+_m) = \frac{\delta \mathcal{L}_m}{\delta y^\sigma_D} \theta^\sigma_D \wedge \mathrm{Vol}_0, \tag{3.35}$$
where $\theta = d\hat{L} - \hat{L}_{,i} dx^i - \hat{L}_{\cdot i} d\dot{x}^i$ and $\theta^\sigma_D = dy^\sigma_D - y^\sigma_{D,i} dx^i - y^\sigma_{D\cdot i} d\dot{x}^i$.

Now, using $h\mathfrak{L}_{J^r\Xi}\lambda^+_m = 0$, we find:
$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda^+_m) + h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_m(\lambda^+_m) - hd\mathcal{J}^\Xi = \ 0.$$

On-shell for the dynamical variables y^σ_D (that is, along sections $\gamma := (L, \gamma_m)$ such that the "matter component" $\gamma_m : PTM^+ \to Y_m, \left(x^i, \dot{x}^i\right) \mapsto \left(x^i, \dot{x}^i, y^\sigma_D\left(x^i, \dot{x}^i\right)\right)$, is critical for λ^+_m), the $\mathcal{E}_m$-term above vanishes, which leaves us with:
$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda^+_m) - hd\mathcal{J}^\Xi \ \simeq_{\gamma_m} 0, \tag{3.36}$$
where $\simeq_{\gamma_m}$ means equality on-shell for the matter component γ_m.

The energy-momentum distribution tensor.

The surviving Euler-Lagrange component $h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda^+_m)$ in (3.36) can be split into a linear expression in ξ and a divergence expression; the latter will couple with $hd\mathcal{J}^\Xi$ into a boundary term and will provide the building block of the energy-momentum distribution tensor Θ. This is seen in the following result.

Theorem 61 ***(Existence of the energy-momentum distribution tensor):*** *For any natural Finsler field Lagrangian $\lambda_m^+ \in \Omega_7(J^rY)$, there exist unique $\mathcal{F}(M)$-linear mappings $\Theta : \mathcal{X}(M) \to \Omega(J^{s+1}Y)$, $\mathcal{B} : \mathcal{X}(M) \to \Omega(J^{s+2}Y)$, with Π^{s+1}, respectively, Π^{s+2}-horizontal values, where $s+1 \leq 2r$ is the order of the Euler-Lagrange form $\mathcal{E}(\lambda_m^+)$, such that, for any $\xi_0 \in \mathcal{X}(M)$:*

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = \mathcal{B}(\xi_0) + hd\Theta(\xi_0). \tag{3.37}$$

Proof. We will first construct Θ and $\mathcal{B}$ in a fibered chart and then show that the obtained expressions are independent of the choice of this chart. In any fibered chart, $\mathcal{E}_g$ is expressed as:

$$\mathcal{E}_g(\lambda_m^+) = \frac{\delta \mathcal{L}_m}{\delta \hat{L}}\,\theta \wedge \mathrm{Vol}_0 =: -\frac{1}{2}\mathfrak{T}\,\hat{L}^{-1}\theta \wedge d\Sigma^+, \tag{3.38}$$

where $d\Sigma^+$ is the pullback of an (arbitrary) invariant volume form on PTM^+. The coefficient $\mathfrak{T}$ can easily be seen to be a 0-homogeneous scalar invariant; 0-homogeneity is ensured by the factor $\hat{L}^{-1}$, as both $\hat{L}^{-1}\theta$ and $d\Sigma^+$ are 0-homogeneous, whereas invariance of $\mathfrak{T}$ under lifted spacetime diffeomorphisms follows from the invariance of λ_m^+.

Then, for any vector field $\Xi \in \mathcal{X}(Y)$, we find:

$$\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = (-\frac{1}{2}\mathfrak{T}\hat{L}^{-1}\mathbf{i}_{J^{s+1}\Xi}\theta)d\Sigma^+ + \frac{1}{2}\mathfrak{T}\hat{L}^{-1}\theta \wedge \mathbf{i}_{J^{s+1}\Xi}d\Sigma^+.$$

The last term is a multiple of θ, hence it is a contact form and will not contribute to the action integral; the remaining component is the horizontal component $h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+)$ and can be expressed (up to a pullback by $\Pi^{s+2,s+1}$ of the right hand side) as:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = -\frac{1}{2}(\mathfrak{T}\,\hat{L}^{-1}\mathbf{i}_{J^{s+1}\Xi}\theta)d\Sigma^+. \tag{3.39}$$

Further, using $\theta = d\hat{L} - \hat{L}_{,i}dx^i - \hat{L}_{\cdot i}d\dot{x}^i = d\hat{L} - \boldsymbol{\delta}_i\hat{L}dx^i - \hat{L}_{\cdot i}\boldsymbol{\delta}\dot{x}^i$ and $\boldsymbol{\delta}_i\hat{L} = 0$, we find:

$$\theta = d\hat{L} - \hat{L}_{\cdot i}\delta\dot{x}^i,$$

where we recall that $\delta\dot{x}^i = d\dot{x}^i + G^i{}_j dx^j$. Inserting into θ the lift $\xi = \xi^i\partial_i + (\xi^i{}_{,j}\dot{x}^j)\dot{\partial}_i$ of ξ_0, a brief calculation leads to:

$$\mathbf{i}_{J^r\Xi}\theta = -2\dot{x}_i\nabla\xi^i.$$

We can thus rewrite (3.39) as:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = \mathfrak{T}\,\hat{L}^{-1}\dot{x}_i\nabla\xi^i d\Sigma^+.$$

Taking into account that $\nabla\dot{x}_i = 0$ and $\nabla\hat{L} = 0$, this can be uniquely split into a linear term in ξ^i and a divergence expression:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = [\nabla(\mathfrak{T}\hat{L}^{-1}\dot{x}_i\xi^i) - \xi^i\dot{x}_i\hat{L}^{-1}\nabla\mathfrak{T}]d\Sigma^+. \tag{3.40}$$

Then, using $\dot{x}^i{}_{|j} = 0$ and $\nabla = \dot{x}^j D_{\boldsymbol{\delta}_j}$, we can rearrange the divergence term as

$$\nabla(\mathfrak{T}\hat{L}^{-1}\dot{x}_i\xi^i) = (\mathfrak{T}\hat{L}^{-1}\dot{x}_i\dot{x}^j\xi^i)_{|j}, \tag{3.41}$$

which suggests the notation:

$$\Theta^j{}_i := \mathfrak{T}\hat{L}^{-1}\dot{x}^j\dot{x}_i \tag{3.42}$$

Since $\mathfrak{T}$ is a scalar invariant, the functions $\Theta^j{}_i$, defined on the given fibered chart, transform under induced fibered coordinate changes as the components of a tensor on M (equivalently, as d-tensor components on TM). Also, noticing that the last term in (3.40) can be written as:

$$\xi^i\dot{x}_i\hat{L}^{-1}\nabla\mathfrak{T} = \xi^i\dot{x}_i\hat{L}^{-1}\dot{x}^j\mathfrak{T}_{|j} = \Theta^j{}_{i|j}\xi^i,$$

it follows that the mappings $\Theta : \mathcal{X}(M) \to \Omega_6(J^{s+1}Y)$ and $\mathcal{B} : \mathcal{X}(M) \to \Omega_7(J^{s+2}Y)$ given by

$$\Theta(\xi_0) \quad = \quad (\Theta^j{}_i\xi^i)\mathbf{i}_{\delta_j}d\Sigma^+; \tag{3.43}$$

$$\mathcal{B}(\xi_0) \quad = \quad -\Theta^j{}_{i|j}\xi^i d\Sigma^+. \tag{3.44}$$

are globally well defined, i.e., independent of the chosen coordinate charts. Moreover, they have Π^{s+1} (respectively, Π^{s+2})-horizontal values, they are both linear in ξ and obey (3.37), which completes the proof of the existence. Uniqueness of $\mathcal{B}$ and Θ follows from the uniqueness of the splitting (3.40) and the arbitrariness of ξ^i. ∎

Notes. The invariant scalar $\mathfrak{T}$ will act as the source term for Finsler gravity equations having L as the dynamical variable. Its precise expression depends on the chosen volume form; for instance, if $d\Sigma^+$ is the canonical volume form (2.75), then:

$$\mathfrak{T} = -2\frac{\hat{L}^3}{|\det g|}\frac{\delta\mathcal{L}_m}{\delta\hat{L}}. \tag{3.45}$$

Further, identifying, by abuse of notation, the Reeb vector field $\ell^+ = l^i\delta_i \in \mathcal{X}(\mathcal{A}_0^+)$ with the vector field on $J^{s+1}Y$ obtained by replacing δ_i with the formal total adapted derivative $\boldsymbol{\delta}_i$, i.e., with: $l^i\boldsymbol{\delta}_i \in \mathcal{X}(J^{s+1}Y)$, and the values $\omega^+_{[(x,\dot{x})]}$ of the Hilbert form $\omega^+ : \mathcal{A}_0^+ \to \Omega_1(M)$ with their pullbacks to $J^{s+1}Y$, we can also provide a coordinate-free formula for Θ.

Proposition 62 ***(Coordinate-free expression of*** Θ***).*** *The energy-momentum distribution* Θ : $\mathcal{X}(M) \to \Omega(J^{s+1}Y)$ *can be expressed as:*

$$\Theta = \mathfrak{T}\omega^+ \otimes \mathbf{i}_{\ell^+}d\Sigma^+. \tag{3.46}$$

Proof. In homogeneous fibered coordinates, Θ is expressed as:

$$\Theta = \Theta^i{}_j dx^j \otimes \mathbf{i}_{\boldsymbol{\delta}_i}d\Sigma^+, \tag{3.47}$$

where, according to (3.42), $\Theta^j{}_i := \mathfrak{T}\hat{L}^{-1}\dot{x}^j\dot{x}_i$.

A quick computation shows that, regardless of the sign of $\hat{L}$, one can write $\hat{L}^{-1}\dot{x}^i\dot{x}_j = \hat{F}_{\cdot j}l^i$, where $\hat{F} = \sqrt{|\hat{L}|}$, which leads us to: $\Theta = \mathfrak{T}(\hat{F}_{\cdot j}dx^j) \otimes \mathbf{i}_{l^i\boldsymbol{\delta}_i}d\Sigma^+$. The statement then follows by realizing that $\omega^+ = \hat{F}_{\cdot j}dx^j$. ∎

The above theorem justifies the following definitions.

Definition 63 (Energy-momentum distribution tensor) *The energy-momentum distribution tensor associated to a natural Lagrangian λ_m^+ on a bundle $Y = Y_g \times_{PTM^+} Y_m$, which is natural over a Finsler spacetime M, is the $\mathcal{F}(M)$-linear mapping $\Theta : \mathcal{X}(M) \mapsto \Omega_6(J^{s+1}Y)$ defined by (3.46).*

Definition 64 (Energy-momentum scalar) *We call the function $\mathfrak{T} : J^{s+1}Y \to \mathbb{R}$, defined by the relation*

$$\mathcal{E}_g(\lambda_m^+) = -\frac{1}{2}\mathfrak{T}\ \hat{L}^{-1}\theta \wedge d\Sigma^+,$$

where $d\Sigma^+$ is the pullback to $J^{s+1}Y$ of an invariant volume form on PTM^+, the energy-momentum scalar corresponding to $d\Sigma^+$.

The $\mathcal{F}(M)$-linear mapping $\mathcal{B} : \mathcal{X}(M) \mapsto \Omega_7(J^{s+2}Y)$ defined by (3.37)-(3.44) will be called, similarly to Section 1.3, the *balance function* and it will serve to naturally express the energy-momentum conservation (or energy-momentum balance) law.

3.2.3 The averaged energy-momentum conservation law

Denote by $\mathcal{T}^+ \subset PTM^+$ the set of timelike directions of a given Finsler spacetime (M, L) and consider local sections $\gamma \in \Gamma(Y)$ such that:

$$\mathrm{supp}(J^r\gamma^*\lambda_m^+) \subset \mathcal{T}^+.$$

This way, it makes sense to integrate the form $J^{s+1}\gamma^*\mathbf{i}_{J^s\Xi}\mathcal{E}_g(\lambda_m^+)$ on the entire set $\mathcal{T}_x^+ = \mathcal{O}_x^+$ of timelike directions at x.

Consider a piece $D_0 \subset M$ and denote by

$$\mathcal{T}^+(D_0) := \underset{x\in D_0}{\cup}\mathcal{T}_x^+ = \underset{x\in D_0}{\cup}\mathcal{O}_x^+,$$

the set of all timelike directions (equivalently, of all observer directions) corresponding to points of D_0. Then, (3.37) becomes, with $\gamma := (L, \gamma_m)$:

$$\int\limits_{\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) = \int\limits_{\mathcal{T}^+(D_0)} J^{s+2}\gamma^*\mathcal{B}(\xi_0) + \int\limits_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\Theta(\xi_0). \tag{3.48}$$

On-shell for γ_m, we have, according to (3.36): $J^{s+1}\gamma^*\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}_g(\lambda_m^+) - J^{s+1}\gamma^* d\mathcal{J}^\Xi \ \simeq_{\gamma_m} 0$, which, substituting into the above relation, gives:

$$\int\limits_{\mathcal{T}^+(D_0)} J^{s+2}\gamma^*\mathcal{B}(\xi_0) + \int\limits_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*(\Theta(\xi_0) - \mathcal{J}^\Xi) \ \simeq_{\gamma_m} 0. \tag{3.49}$$

We are now able to prove the following result.

Theorem 65 *Consider a bundle Y_m over PTM^+, which is natural over M, and an arbitrary section $\gamma = (L, \gamma_m) \in \Gamma(Y_g \times_{PTM^+} Y_m)$ with $\mathrm{supp}(J^r\gamma^*\lambda_m^+) \subset \mathcal{T}^+$. Then, the following statements hold:*

1. ***Averaged energy-momentum conservation law:*** *At any $x \in M$ and in any corresponding fibered chart:*

$$\int_{\mathcal{T}_x^+} ({\Theta^j}_{i|j} \circ J^{s+1}\gamma) d\Sigma_x^+ = 0, \tag{3.50}$$

   where $d\Sigma^+ =: d^4x \wedge d\Sigma_x^+$.

2. ***Relation to Noether currents:*** *For any $\xi_0 \in \mathcal{X}(M)$:*

$$\int_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\Theta(\xi_0) = \int_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\mathcal{J}^\Xi, \tag{3.51}$$

   where Ξ denotes the canonical lift of ξ_0 to Y.

Proof. 1. Fix a point $x_0 \in M$. Consider an arbitrary piece $D_0 \subset M$ containing x_0 as an interior point and an arbitrary $\xi_0 \in \mathcal{X}(M)$ with support contained in D_0.

Now, let us have a look at the boundary term in (3.49). Since the support of the integrand, at every $x \in M$, is strictly contained in $\mathcal{T}_x^+$, the only possible nonzero values are obtained at points $[(x, \dot{x})]$ with $x \in \partial D_0$. But, at these points, ξ_0 identically vanishes (hence also $\Xi = 0$, since Ξ is built from ξ and its derivatives), which means that this boundary term is actually zero. It follows:

$$\int_{\mathcal{T}^+(D_0)} J^{s+2}\gamma^*\mathcal{B}(\xi_0) \simeq_{\gamma_m} 0. \tag{3.52}$$

In coordinates, this is:

$$\int_{\mathcal{T}^+(D_0)} ({\Theta^j}_{i|j} \circ J^{s+1}\gamma)\xi^i d\Sigma^+ \simeq_{\gamma_m} 0.$$

Squeezing D_0 around x_0 such that D_0 is contained into a single chart domain, the above integral can be written as an iterated integral $\int_{D_0} \xi^i (\int_{\mathcal{T}_x^+} ({\Theta^j}_{i|j} \circ J^{s+1}\gamma) d\Sigma_x^+) d^4x$, which, taking into account the arbitrariness of ξ^i, leads to the result.

2. follows then immediately from (3.49) and 1. ■

Relation (3.51) says that, the energy-momentum tensor $\Theta(\xi_0)$ is, at least up to a term which does not contribute to the integral (3.51)), the conserved Noether current $\mathcal{J}^\Xi$ - i.e. (see also [84]), it gives the correct notions of energy and momentum of the system under discussion.

Remarks.

1. Taking into account that $\mathcal{O}_x^+ = \mathcal{T}_x^+$ and Proposition 1, the averaged conservation law can be rewritten as an integral over $\mathcal{O}_x$:

$$\int_{\mathcal{O}_x} ({\Theta^j}_{i|j} \circ J^{s+1}\gamma) d\Sigma_x = 0, \tag{3.53}$$

   where $d\Sigma_x = (\pi^+)^* d\Sigma_x^+$ is the pullback of the volume form $d\Sigma_x^+ \in \Omega_3(\mathcal{T}_x^+)$ by the restriction to $\mathcal{O}_x$ of the projection $\pi^+ : \overset{\circ}{T}M \to PTM^+$.

2. As already stated in the Introduction of this section, due to the fact that naturality of Lagrangians comes from M, which is a space of lower dimension than the one of the space PTM^+ on which the action integral is considered, in the above relation, integration over $\mathcal{T}_x^+$ cannot be removed, i.e., we can typically only establish an *averaged* conservation law. This is a distinctive feature of Finslerian field theory.

Energy-momentum tensor density on M. The mapping $\Theta : \mathcal{X}(M) \to \Omega(J^{s+1}Y)$ gives rise to an energy-momentum tensor *density* on M, by averaging over observer (or timelike) directions $\mathcal{O}_x^+ = \pi^+(\mathcal{O}_x)$. Consider an arbitrary fibered chart on Y; $\Theta(\xi) = \Theta^i{}_j \xi^j \otimes \mathbf{i}_{\delta_i} d\Sigma^+$. Then, for any section $\gamma \in \Gamma(Y)$ such that $\mathrm{supp}(J^r\gamma^*\lambda_m^+) \subset \mathcal{T}^+$, set

$$\mathcal{T}^i_{\ j}(x) := \int\limits_{\mathcal{O}_x^+} (\Theta^i{}_j \circ J^{s+1}\gamma)_{|(x,\dot{x})} d\Sigma_x^+, \qquad \forall x \in M\,. \tag{3.54}$$

Under the above assumption this integral is finite, so the result is well defined. Moreover, given the expression of $d\Sigma_x^+$, the functions $\mathcal{T}^i_{\ j}(x)$ represent the components of a tensor density on M.

3.3 A concrete model: Finsler gravity sourced by a kinetic gas

3.3.1 Introduction

As a concrete *model* for a field theory on Finsler spacetimes, we discuss in the dynamics of a Finsler spacetime sourced by a kinetic gas introduced in [93], [94], [95]. We first discuss the purely geometric (vacuum) field theory, where the Finsler spacetime function L itself is the dynamical field, and then add a matter Lagrangian as source of these dynamics.

The section follows the jet bundle formulation in [97], with details from our earlier papers [93], [94], [95] incorporated.

Vacuum action and vacuum field equation, [93]. The first field equation which took the Finsler function L as fundamental variable was formulated by Rutz in 1993, [172], as follows.

It was argued that, in vacuum, the trace of the geodesic deviation operator should vanish. This argument was applied in the pseudo-Riemannian case by Pirani, [164], to obtain the Einstein vacuum field equations. Similarly, Rutz's equation postulates that the trace of the Finslerian geodesic deviation operator (2.38) - which is nothing but the Finsler-Ricci scalar (2.39), must vanish:

$$R_0 = 0. \tag{3.55}$$

Using the canonical variational completion algorithm introduced in Section 1.2.3, we will prove below that Rutz's equation is not variational and find the variationally completed equation. This equation turns out to be similar to the one found in [163] on the unit tangent bundle and (to a certain extent) in [59] for $\overset{\circ}{T}M$-smooth positive definite Finsler functions. The important new ingredients are here that, on the one hand, the Lagrangian was obtained *from a physical principle* and by variational completion algorithm and, on the other hand, the corresponding variational problem is *consistently and rigorously formulated*, also in indefinite signature. To the best of our knowledge, this is the first example of a Finslerian gravitational field equation abiding by these two ideas.

Coupling to matter, [94]. As already mentioned in the beginning of this chapter, in the theory of kinetic gases, the function encoding the properties of the gas is a function defined on the tangent (or, equivalently, on the cotangent) bundle of spacetime, called the *1-particle distribution function (1PDF)* φ, see, e.g., [175]. In general relativity, the gravitational field of a kinetic gas is described in terms of the so-called *Einstein-Vlasov equations* [6], which are the usual Einstein field equations with energy-momentum tensor components obtained by *averaging* (i.e., integrating) φ over observer velocities. Yet, when the gravitational field of a kinetic gas is derived via the Einstein-Vlasov equations, the information about the velocity distribution of the gas particles is averaged out and therefore lost. We show that Finsler geometry allows one to derive the gravitational field of a kinetic gas *directly* from its 1PDF, taking the velocity distribution fully into account. We conjecture that this refined approach may account for the observed dark energy phenomenology.

3.3.2 Finsler vacuum action from variational completion

Vainberg-Tonti Lagrangian for Rutz's equation.

We have shown above that, for theories using the 2-homogeneous Finsler function $L : \mathcal{A} \to \mathbb{R}$ as the dynamical variable, the appropriate configuration bundle is the bundle

$$Y_g := (\overset{\circ}{T}M \times \mathbb{R})_{/\sim}$$

obtained, [97], as the quotient space of the trivial bundle $\overset{\circ}{T}M \times \mathbb{R}$ with respect to the action (3.24) of $(\mathbb{R}^*_+, \cdot)$. This way, Finsler spacetime functions $L : \mathcal{A} \to \mathbb{R}$ are in a one-to-one correspondence with local sections $\gamma \in \Gamma(Y_g)$; more precisely, in homogeneous fibered coordinates $(x^i, \dot{x}^i, \hat{L})$ on Y_g, L is described as the principal component of the section $\gamma : (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, L(x^i, \dot{x}^i))$, that is,

$$L = \hat{L} \circ \gamma. \tag{3.56}$$

In the following, by g, G^i, R, R_0 etc., we will mean the *formal* Finslerian geometric objects obtained by replacing L with $\hat{L}$ in the usual formulas; taking into account that R and R_0 are of fourth order in $\hat{L}$, all these objects will be identified as objects on the fourth order jet bundle J^4Y_g.

Also, we will fix the invariant volume form, as:

$$d\Sigma^+ := \frac{|\det g|}{\hat{L}^2} \mathbf{i}_{\mathbb{C}}(dx \wedge d\dot{x}) \in \Omega_7(J^4Y_g). \tag{3.57}$$

In order to apply the canonical variational completion algorithm to Rutz's equation, we must first write down a generally covariant *source form:*

$$\mathcal{E} = \mathcal{F}\hat{L}^{-1}\, \theta \wedge d\Sigma^+ \in \Omega_8(J^4Y_g), \tag{3.58}$$

where $\mathcal{F} = \mathcal{F}(x^i, \dot{x}^i, \hat{L}, \hat{L}_{,i}, \hat{L}_{\cdot i}, ...\hat{L}_{\cdot i_1 \dots \cdot i_4})$, whose vanishing is equivalent to Rutz's equation[4]. Since

[4]The factor $\hat{L}^{-1}$ was introduced, just as in the previous section, to adjust the homogeneity degree of θ; thus, along sections L, $\mathcal{F}$ will become a 0-homogeneous function in $\dot{x}$.

any differential form on J^4Y_g must be 0-homogeneous, it turns out that $\mathcal{E}$ must be (up to multiplication by a constant):

$$\mathcal{E} = R_0\hat{L}^{-1}\ \theta \wedge d\Sigma^+ = (\hat{L}^{-3}R_0\,|\det g|)\ \theta \wedge \mathrm{Vol}_0 \in \Omega(J^4Y_g). \tag{3.59}$$

Corresponding to any coordinate chart domain of J^4Y_g, the Vainberg-Tonti Lagrangian is $\Lambda_g = \mathcal{L}_g \mathrm{Vol}_0$, where:

$$\mathcal{L}_g = \hat{L}\int_0^1 (\hat{L}^{-3}R_0\,|\det g|) \circ \chi_u\ du, \tag{3.60}$$

and χ_u denotes the fiber homothety (1.62), expressed, in fibered homogeneous coordinates, as:

$$\chi_u : (x^i, \dot{x}^i, \hat{L}, \hat{L}_{,i}, \hat{L}_{\cdot i}, ...\hat{L}_{\cdot i_1 \dots \cdot i_r}) \mapsto (x^i, \dot{x}^i, u\hat{L}, u\hat{L}_{,i}, u\hat{L}_{\cdot i}, ...u\hat{L}_{\cdot i_1 \dots \cdot i_r}).$$

Under the homotheties χ_u, the formal metric tensor components obey $g_{ij} \circ \chi_u = ug_{ij}$, $g^{ij} \circ \chi_u = u^{-1}g^{ij}$; hence, the geodesic spray coefficients G^i, the canonical nonlinear connection coefficients $G^i{}_j$ and the curvature components $R^i{}_{jk}$ (see Section 2.1.4) remain invariant. This implies that $R_0 = \hat{L}^{-1}R^i{}_{ji}$ transforms as: $R_0 \circ \chi_u = u^{-1}R_0$, which, together with $|\det g| \circ \chi_u = u^4|\det g|$, gives:

$$(\hat{L}^{-3}R_0\,|\det g|) \circ \chi_u = \hat{L}^{-3}R_0\,|\det g|\,,$$

leading to the desired Lagrangian density

$$\mathcal{L}_g = \hat{L}^{-2}R_0\,|\det g| \int_0^1 du = \frac{R_0\,|\det g|}{\hat{L}^2}.$$

We have thus obtained the Vainberg-Tonti Lagrangian of Rutz's equation as:

$$\lambda_g^+ = R_0\frac{|\det g|}{\hat{L}^2}\mathrm{Vol}_0 = R_0 d\Sigma^+. \tag{3.61}$$

Note. The obtained Lagrange density $\mathcal{L}_g$ coincides along sections γ with the ones suggested by Pfeifer and Wohlfarth in [163] on the indicatrix bundle and, respectively by Chen and Shen, [59] for positive definite Finsler spaces - here, derived by the means of canonical variational completion.

The Finsler gravity action on PTM^+ and its variation

Consider an arbitrary piece $D^+ \subset PTM^+$. Without loss of generality, we are looking for functions L that are smooth and do not vanish on the preimage $(\pi^+)^{-1}(D^+) \subset \overset{\circ}{T}M$. The action associated to the Lagrangian (3.61) and to D^+ is the mapping $S_{D^+} : \Gamma(Y_g) \to \mathbb{R}$, $\gamma \mapsto S_{D^+}(\gamma)$ given by:

$$S_{D^+}(\gamma) = \int\limits_{D^+} J^4\gamma^*\lambda_g^+ = \int_{D^+} \frac{(R_0\,|\det g|)}{\hat{L}^2} \circ J^4\gamma\ \mathrm{Vol}_0. \tag{3.62}$$

In order to determine the corresponding Euler-Lagrange equation, variation with respect to *vertical* vector fields $\Xi =: 2\hat{v}\dfrac{\partial}{\partial \hat{L}} \in \mathcal{X}(Y_g)$ such that $supp(\Xi \circ \gamma)$ is strictly contained in D^+, is sufficient. This way, the flow $\{\Phi_t\}$ consists of strict automorphisms of Y_g, giving rise to the deformed sections $\gamma_t = \Phi_t \circ \gamma$. In coordinates:

$$\gamma_t : \left(x^i, \dot{x}^i\right) \mapsto \left(x^i, \dot{x}^i, \bar{L}\left(x, \dot{x}\right)\right). \tag{3.63}$$

The Euler-Lagrange form $\mathcal{E}(\lambda_g^+) = E\hat{L}^{-1}\theta \wedge d\Sigma^+$ will be identified by direct computation from the first variation formula

$$\delta S_{D^+}(\gamma) := \frac{d}{dt}|_{t=0}(S_{D^+}(\gamma_t)). \tag{3.64}$$

Theorem 66 *, [93] The (unique) Euler-Lagrange expression of λ_g^+ is:*

$$E := \frac{1}{2}g^{ij}(\hat{L}R_0)_{\cdot i \cdot j} - 3R_0 - g^{ij}(P_{i|j} - P_iP_j + (\nabla P_i)_{\cdot j}) \in \mathcal{F}(J^6Y_g). \tag{3.65}$$

Proof. To keep notations short, let us momentarily designate by a bar quantities evaluated along the deformed section γ_t; e.g., $\bar{L} := \hat{L} \circ \gamma_t$, $\bar{L}_{\cdot i} := \hat{L}_{\cdot i} \circ \gamma_t$ etc. and without bars, the same quantities evaluated along the undeformed section γ; that is, $L_{\cdot i}$, g_{ij} etc. will for now denote the usual, TM-based Finslerian geometric objects attached to the pseudo-Finsler function $L = \hat{L} \circ \gamma$. With these notations, we have:

$$\delta S_{D^+}(\gamma) = \frac{d}{dt}|_{t=0}\left(\int_{D^+} \frac{(\bar{R}_0\,|\det \bar{g}|)}{\bar{L}^2}\ \mathrm{Vol}_0\right) = \frac{d}{dt}|_{t=0}\left(\int_{D^+} (\bar{L}^{-3}\bar{R}\,|\det \bar{g}|)\ \mathrm{Vol}_0\right).$$

Using the product rule and the identity $|\det \bar{g}|)\ \mathrm{Vol}_0 = \bar{L}^2 d\Sigma^+$, this is:

$$\delta S_{D+}(\gamma) = \left(I_1 + I_2 + I_3\right), \tag{3.66}$$

as follows:

$$I_1 = \int_{D^+} \frac{d\bar{L}^{-3}}{dt}|_{t=0} RL^2 d\Sigma^+, \quad I_2 = \int_{D^+} L^{-1}\frac{d\bar{R}}{dt}|_{t=0} d\Sigma^+, \quad I_3 = \int_{D^+} \frac{L^{-1}R}{|\det g|}\frac{d(\det \bar{g})}{dt}|_{t=0}\ d\Sigma^+. \tag{3.67}$$

To evaluate each of these integrals, denote:

$$2v := \frac{d\bar{L}}{dt}|_{t=0} = 2\hat{v} \circ \gamma,$$

Thus, $v = v(x, \dot{x})$ is a 2-homogeneous function on some domain of TM, which vanishes, together with its derivatives of any order, along directions corresponding to the boundary ∂D^+. The first integral is then easily seen to be

$$I_1 = -\int_{D^+} 3\frac{R}{L}\frac{2v}{L}\ d\Sigma^+. \tag{3.68}$$

To evaluate I_2, we write:

$$\bar{L}_{\cdot i} \overset{t^1}{\simeq} L_{\cdot i} + 2tv_i, \quad \bar{g}_{ij} \overset{t^1}{\simeq} g_{ij} + tv_{ij}, \quad \bar{g}^{ij} \overset{t^1}{\simeq} g^{ij} - tv^{ij},$$

where: $v_i := v_{\cdot i}$, $v_{ij} := v_{\cdot i \cdot j}$, $v^{ij} := g^{mi}g^{nj}v_{mn}$ and the symbol $\overset{t^1}{\simeq}$ means equality modulo higher than the first power in t. As a consequence:

$$2\bar{G}^i = \frac{1}{2}\bar{g}^{ij}(\dot{x}^k \bar{L}_{\cdot j,k} - \bar{L}_{,j}) \overset{t^1}{\simeq} 2G^i + tg^{ij}(\dot{x}^k v_{j,k} - v_{,j} - 2G^k v_{jk}).$$

Since v_i are d-tensor components, it makes sense to rewrite the above in terms of the Chern-Rund covariant derivatives: $v_{i|j} = v_{i,j} - G^k{}_j v_{ik} - \Gamma^k{}_{ij} v_k$, which gives (see also [59] for the variation of G^i):

$$2\bar{G}^i \overset{t^1}{\simeq} 2G^i + 2tA^i, \quad \text{with} \quad A^i = \frac{1}{2}g^{ij}(\nabla v_j - v_{|j}), \tag{3.69}$$

Then, we find by differentiation: $\bar{G}^i{}_j \overset{t^1}{\simeq} G^i{}_j + tA^i{}_{\cdot j}$ and accordingly, after a brief computation using $G^i{}_{j\cdot k} = \Gamma^i{}_{jk} + P^i{}_{jk}$:

$$\bar{R}^i{}_{jk} = R^i{}_{jk} + t(A^i{}_{\cdot j|k} - A^i{}_{\cdot k|j} + A^l{}_{\cdot j}P^i{}_{kl} - A^l{}_{\cdot k}P^i{}_{jl})$$

Taking the trace $i = j$, contracting with $\dot{x}^k$, and taking into account that $P^i{}_{jk}\dot{x}^k = 0$, $A^l{}_{\cdot k}\dot{x}^k = 2A^l$, we get

$$\bar{R} \overset{t^1}{\simeq} R + t(\nabla A^i{}_i - 2A^i{}_{|i} - 2A^l P_l)$$

respectively,

$$I_2 = \int_{D^+} L^{-1}(\nabla A^i{}_i - 2A^i{}_{|i} - 2A^l P_l)\; d\Sigma^+ .$$

The term $\nabla A^i{}_i$ is, see (2.78), a boundary term, which we can neglect; also, $L_{|i} = 0$ and the divergence formula (2.76) give $L^{-1}A^i{}_{|i} = \mathrm{div}(L^{-1}A^i\delta_i) + L^{-1}A^i P_i$, that is,

$$I_2 = -\int_{D^+} \frac{4}{L} A^l P_l \; dV_0^+ .$$

Using the definition (3.69) of A^i, we can expand the integrand as

$$-4L^{-1}A^i P_i = 2\left[(L^{-1}vP^i)_{|i} - L^{-1}vP^i{}_{|i} - \nabla(L^{-1}v_i P^i) + L^{-1}v_i \nabla P^i\right],$$

which, after a series of integration by parts using (2.76),(2.77), yields:

$$I_2 = \int_{D^+} (P^i P_i - P^i{}_{|i} - g^{ij}(\nabla P_i)_{\cdot j}) \frac{2v}{L}\; dV_0^+ . \tag{3.70}$$

For the integral I_3, the derivative formula for the determinant $\frac{d}{dt}\det\bar{g} = \bar{g}^{ij}\frac{d\bar{g}_{ij}}{dt}\det\bar{g}$, followed by integration by parts leads to:

$$I_3 = \int_{D^+} \frac{R}{L} g^{ij} v_{ij}\; d\Sigma^+ = \int_{D^+} \frac{1}{2} g^{ij} R_{\cdot i \cdot j} \frac{2v}{L}\; d\Sigma^+.$$

Collecting the results and reverting to $L = \hat{L}\circ\gamma, v = \hat{v}\circ\gamma$ etc., we get (3.65). ∎

Along critical sections, we must thus have $E \circ J^6 Y_g = 0$, which gives:

Corollary 67 ***(Vacuum field equation):*** *Critical functions L of the Finsler gravity action (3.62) are given by the equation:*

$$\frac{1}{2}g^{ij}(LR_0)_{\cdot i\cdot j} - 3R_0 - g^{ij}(P_{i|j} - P_iP_j + (\nabla P_i)_{\cdot j}) = 0, \tag{3.71}$$

where R_0 is the Finsler-Ricci scalar of L and P_i are the components of the trace of the Landsberg tensor of L.

Once a solution L of this equation is found, the equation holds on the set of admissible non-lightlike vectors $\mathcal{A}_0^+$ of L.

Particular case: Lorentzian metrics. For the class of Lorentzian (quadratic) Finsler spacetime functions $L : TM \to \mathbb{R}$,

$$L(x, \dot{x}) = a_{ij}(x)\dot{x}^i\dot{x}^j, \tag{3.72}$$

one obtains that: *L obeys the Finsler vacuum equation (3.71) $\Leftrightarrow$ a obeys the vacuum Einstein equations $r_{ij} = 0$.*

This is easily seen taking into account that, for semi-Riemannian metrics, we have (see Section 2.1.5): $P_i = 0$ and $R = -r_{ij}\dot{x}^i\dot{x}^j$. Hence, in this case, equation (3.71) becomes (after multiplication by L):

$$0 = 3R - \frac{L}{2}g^{ij}R_{\cdot i\cdot j} = -3r_{ij}\dot{x}^i\dot{x}^j + (a_{ij}\dot{x}^i\dot{x}^j)r. \tag{3.73}$$

Assuming that L is a solution of (3.73), then, differentiating twice with respect to $\dot{x}$, we find: $3r_{ij} - a_{ij}r = 0$, which, contracting the equation with a^{ij}, implies $r = 0$; substituting back into the above relation, we find: $r_{ij} = 0$. Conversely, if a obeys $r_{ij} = 0$, then, $R = -r_{ij}\dot{x}^i\dot{x}^j = 0$, which immediately leads to (3.73).

Actually, there is more to it. Using equation (2.82) in Chapter 2, a direct computation shows that our Lagrangian function, which is the Finsler-Ricci scalar $R_0 = -r_{ij}L^{-1}\dot{x}^i\dot{x}^j$ of L is, up to a divergence term and multiplication by a constant, nothing but the scalar curvature r of a :

$$R_0 = -\frac{1}{4}r + \operatorname{div} Y,$$

where $Y = -\frac{1}{8}(La^{ij}(R_0)_{\cdot j})\dot{\partial}_i \in \mathcal{X}(PTM^+)$.

3.3.3 Action for a kinetic gas

The 1PDF and the action on the observer space.

Fix, for the moment, the Finsler spacetime structure (M, L).

The notion of kinetic gas. A *kinetic gas* is a collection of a large number of particles which propagate through spacetime on piecewise normalized geodesics $c : [a, b] \to M$, $\tau \mapsto c(\tau)$; here, the arc length parameter $\tau = s$ is denoted as τ and physically interpreted as proper time.

In the language of Finsler spacetimes (M, L), this means, on the one hand, that the tangent vectors $\dot{c}(\tau) := \frac{dc}{d\tau}(\tau)$ are future-directed unit timelike vectors, i.e., elements of the *observer space*:

$$\dot{c}(\tau) \in \mathcal{O}_{c(\tau)} \subset \mathcal{T}_{c(\tau)}.$$

On the other hand, geodesic motion means that the lifted trajectories $C(\tau) = (c(\tau), \dot{c}(\tau))$ are integral curves of the *Reeb vector field* $\ell = l^i \delta_i$ (which can be regarded as a vector field on $\mathcal{O}$, see Section 2.2.3). Assuming, for simplicity, that all the particles have mass m, their trajectories are critical points of the action integral[5]:

$$S[c] = -m \int_{\tau_1}^{\tau_2} C^*\omega = -m \int_{\tau_1}^{\tau_2} F(c(\tau), \dot{c}(\tau)) d\tau \; = -mt, \tag{3.74}$$

where $\omega = F_{\cdot i} dx^i$ is the Hilbert form. The number $t = \tau_2 - \tau_1$ denotes the proper time passing along a particle trajectory between the points $c(\tau_1)$ and $c(\tau_2)$.

The restriction of the Hilbert form $\omega \in \Omega_1(\overset{\circ}{TM})$ to $\mathcal{O}$, which is nothing but the pullback by the diffeomorphism $\pi^+_{|\mathcal{O}} : \mathcal{O} \to \mathcal{T}^+$ of the contact structure ω^+ on $\mathcal{T}^+ \subset PTM^+$, defines a contact structure on $\mathcal{O}$; hence all the construction in Subsection 2.2.3 can be carried out with no modification on $\mathcal{O}$.

The 1-particle distribution function. Instead of describing the motion of all particles individually, the kinetic gas theory employs the so-called *1-particle distribution function (1PDF).* This is typically defined as a scalar function of the particle positions and velocities:

$$\varphi : \mathcal{O} \to \mathbb{R}; \quad (x, \dot{x}) \mapsto \varphi(x, \dot{x}),$$

allowing one to express number $N[\sigma]$ of particle trajectories crossing an oriented (6-dimensional) hypersurface $\sigma \subset \mathcal{O}$ as the integral:

$$N[\sigma] = \int_\sigma \varphi \Omega \, ; \tag{3.75}$$

here, Ω is the canonical volume form on σ :

$$\Omega = \mathbf{i}_\ell d\Sigma = \frac{1}{3!} d\omega \wedge d\omega \wedge d\omega, \tag{3.76}$$

obeying the relations:

$$d\Omega = 0, \quad d\Sigma = \omega \wedge \Omega \, . \tag{3.77}$$

Remarks.

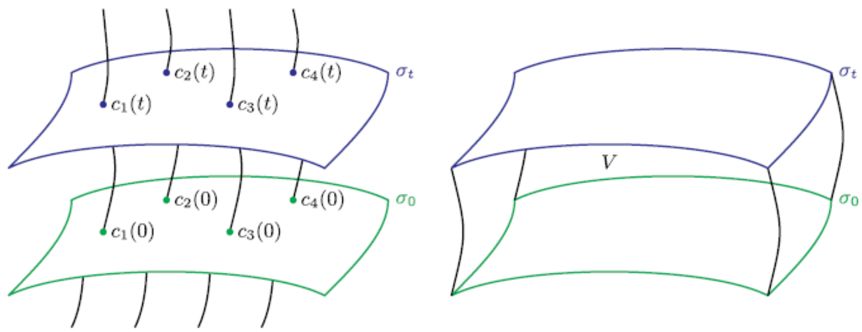
1. Since in all practical physical situations, there will always be gas particles with a finite maximal velocity, we assume in what follows that for all $x \in M$, the partial function $\varphi_x = \varphi(x, \cdot) : \mathcal{O}_x \to \mathbb{R}$ has compact support.

2. The 1PDF is *a priori* defined for normalized (unit) vectors, but it can be naturally prolonged into a 0-homogeneous function $\varphi : \mathcal{T} \to \mathbb{R}$ on the entire future timelike cone $\mathcal{T}$. Alternatively (as we will actually do in the next subsection), one can also identify φ as a function defined on $\mathcal{O}^+ = \mathcal{T}^+$ and, accordingly, regard $N[\sigma]$ as an integral on the hypersurface $\sigma^+ = \pi^+(\sigma)$ of PTM^+- with the advantage that σ^+ does not depend on the Finsler spacetime function L.

[5] Here, a sign correction is introduced. Namely, the action presented here is *minus* the one introduced in the paper [94]; this sign thus agrees with the one in [126] and will give in the next subsection, the correct (plus) sign for the components of the energy-momentum tensor density.

Collisionless gases: Liouville equation.

Choose a hypersurface $\sigma_0 \subset \mathcal{O}$ as above and fix an arbitrary $t > 0$. We obtain a family of hypersurfaces σ_s by following the flow of the Reeb vector field $\ell = l^i \delta_i$ from each point in σ_0, for one and the same parameter $s \in (0, t)$. This family of hypersurfaces spans a volume $D = \bigcup_{s \in (0,t)} \sigma_s$, see the picture below for a sketch.



Kinetic gas: volume spanned by particle worldlines

The difference between the number of particles on σ_0 and σ_t is given by

$$N[\sigma_t] - N[\sigma_0] = \int_D \ell(\varphi)\, d\Sigma\,.$$

In the particular case of collisionless gases, there holds $N[\sigma_t] - N[\sigma_0] = 0$, which, taking into account the arbitrariness of t and σ_0 (hence, of D) gives the *Liouville equation:*

$$\ell(\varphi) = 0. \tag{3.78}$$

Equivalently, using the dynamical covariant derivative $\nabla = \dot{x}^i D_{\delta_i}$ attached to the canonical non-linear connection of (M, L), this can be written as:

$$\nabla\varphi = 0 \tag{3.79}$$

The above equation is interpreted as follows: *for collisionless gases, φ is constant along lifted geodesics $C : \tau \mapsto (c(\tau), \dot{c}(\tau))$* of spacetime.

The above reasoning, which is similar to the one made for Lorentzian spacetimes (M, a) in [175], already hints that φ may be, up to a constant rescaling, the energy-momentum scalar $\mathfrak{T}$. This intuition, yet, has to be checked; to this aim, we will construct a Lagrangian for the kinetic gas, starting from a physical principle.

The action S_{gas}. Assume, for the beginning, that the kinetic gas consists of P particles of the same mass m. The action of the gas in a compact domain $D \subset \mathcal{O}$ constructed as above, is obtained by summing the actions (3.74) for each individual particle:

$$S_{gas} = -mPt = -mP \int_0^t F(c\,(s)\,, \dot{c}\,(s))ds.$$

Expressing the total number P of particles as $P = N[\sigma_t]$, we obtain:

$$S_{gas} = -m \int_0^t \left(\int_{\sigma_s} \varphi \Omega \right) ds = -m \int_D \varphi \Omega \wedge \omega,$$

which, using the identity $d\Sigma = \Omega \wedge \omega$, gives:

$$S_{gas} = -m \int_D \varphi d\Sigma. \tag{3.80}$$

Using the assumption of compact support of each partial function $\varphi_x : \mathcal{O}_x \to \mathbb{R}$, we can rewrite S_{gas} as an iterated integral, as follows. Suppose that the piece $D \subset \mathcal{O}$ is of the form $D = \bigcup_{x \in D_0} V_x$ (where $D_0 \subset M$ and $V_x \subset \mathcal{O}_x$, $x \in D$ are compact) and V_x is large enough as to contain the support of φ_x. Then:

$$S_{gas} = -m \int_D \varphi d\Sigma = -m \int_{D_0} \left(\int_{V_x} \varphi_x(\dot{x}) d\Sigma_x \right) d^4x = -m \int_{D_0} \left(\int_{\mathcal{O}_x} \varphi_x(\dot{x}) d\Sigma_x \right) d^4x\,, \tag{3.81}$$

where $d\Sigma_x = \mathbf{i}_{\partial_0} \mathbf{i}_{\partial_1} \mathbf{i}_{\partial_2} \mathbf{i}_{\partial_3} d\Sigma = \dfrac{|\det g|}{L^2} \mathbf{i}_{\mathbb{C}}(d^4\dot{x})$.

If the particles have different masses m_i, then the action S_{gas} will be written as a finite sum of actions corresponding to each mass m_i.

Translating φ and S_{gas} into the jet bundle language

Now, we will allow the Finsler function L to vary; otherwise stated, we will regard it as a component $L = \hat{L} \circ \gamma$ of a local section $\gamma \in \Gamma(Y_g)$, $[(x, \dot{x})] \mapsto [(x, \dot{x}, L(x, \dot{x}))]$, where $Y_g := (\overset{\circ}{T}M \times \mathbb{R})_{/\sim}$ is as in the previous subsection.

- The first thing to note is that the particle number $N(\sigma)$ is independent of the Finsler spacetime function L, as it only depends on the trajectories of the particles and on the chosen hypersurface. But, having a look at the integral (3.75), this implies that $\varphi : \mathcal{T} \to \mathbb{R}$ (or, equivalently, its PTM^+-version $\varphi \circ (\pi^+|_{\mathcal{O}})^{-1}$) *does* depend on L and its derivatives up to order two - as the volume form $d\Sigma$ depends on these. Therefore, to be completely honest, we should actually write:

  $$\varphi = \varphi(x, \dot{x}, L(x, \dot{x}), L_{,i}(x, \dot{x}), ..., L_{\cdot i \cdot j}(x, \dot{x})), \tag{3.82}$$

  which suggests that φ arises from a function on some jet bundle $J^r Y_g$, with $r \geq 2$. Since we want to couple the obtained Lagrangian to the vacuum one λ_g, which is of order 4, it will be convenient to set $r := 4$. Thus, by formally replacing L in the above relation with the coordinate function $\hat{L}$, we get the homogeneous fibered coordinate expression

  $$\varphi^+ : \left(x^i, \dot{x}^i\right) \mapsto (x^i, \dot{x}^i, \hat{L}, \hat{L}_{,i}, ..., \hat{L}_{\cdot i \cdot j})$$

  of a well defined function $\varphi^+ : J^4 Y_g \to \mathbb{R}$ - more precisely, $\varphi^+(J^4_{[(x,\dot{x})]}\gamma) := \varphi(x, \dot{x})$. This way, the function φ in (3.82) and the section $\gamma \in \Gamma(Y_g)$ are related by:

  $$\varphi = \varphi^+ \circ J^4\gamma \circ \pi^+|_{\mathcal{O}}. \tag{3.83}$$

- The number of particles $N(\sigma)$ crossing the hypersurface $\sigma \subset \mathcal{O}$ can be now rewritten as:

$$N(\sigma) = \int\limits_{\sigma^+} (\varphi^+ \circ J^4\gamma)\Omega^+, \tag{3.84}$$

  where $\sigma^+ = \pi^+(\sigma) \subset PTM^+$ and $\Omega^+ = \mathbf{i}_{\ell^+} d\Sigma^+$.

- Finally, we obtain for S_{gas} in (3.80):

$$S_{gas} = -m \int_{D^+} (\varphi^+ \circ J^4\gamma) d\Sigma^+ = -m \int\limits_{D^+} J^4\gamma^* \lambda_m^+, \tag{3.85}$$

  which corresponds to the *matter Lagrangian*:

$$\lambda_m^+ \in \Omega_7(J^4 Y_g), \qquad \lambda_m^+ := -\varphi^+ d\Sigma^+. \tag{3.86}$$

Remark. Kinetic gases are a very peculiar case, allowing for a simple structure of the configuration bundle, which is just $Y = Y_g$ (let us recall that, in general, when discussing energy-momentum tensors, this is generally *a fibered product* $Y = Y_g \times_{PTM^+} Y_m$).

3.3.4 Field equation and energy-momentum distribution tensor

The field equation.

Summarizing the results in the above subsections, the total Lagrangian of our model is:

$$\lambda^+ = \frac{1}{2\kappa^2}\lambda_g^+ + \lambda_m^+, \tag{3.87}$$

where κ is a constant and $\lambda_g^+ = R_0 d\Sigma^+$ is the vacuum Lagrangian (3.61). Variation of the corresponding action with respect to L with respect to vertical vector fields $\Xi \in \mathcal{X}(Y_g)$ leads to the following result.

Theorem 68 *The Euler-Lagrange equation attached to the Lagrangian (3.87) is*[6]*:*

$$\frac{1}{2} g^{ij}(LR_0)_{\cdot i \cdot j} - 3R_0 - g^{ij}(P_{i|j} - P_i P_j + (\nabla P_i)_{\cdot j}) = \kappa^2 m \varphi\,. \tag{3.88}$$

Proof. The proof below is a transcription in the jet bundle language of the reasoning made in [94].

The variation of the λ_g^+-part of the action is known from (3.65), therefore, we only need to vary its matter part S_{gas}, (3.85).

Consider a Π-vertical vector field $\Xi = 2\hat{v}\dfrac{\partial}{\partial \hat{L}} \in \mathcal{X}(Y_g)$, such that $\Xi \circ \gamma$ has support strictly contained in D^+. Then, the variation of S_{gas} is:

$$\delta S_{gas} = -m \int\limits_{D^+} J^4\gamma^* \mathfrak{L}_{J^4\Xi} \lambda_m^+,$$

[6] The different sign compared to the one in the paper [94] is obtained due to the choice, here, of the minus sign in λ_m^+. Also, a small correction (the factor m, which was lost in [94]) is introduced here.

which, using $\lambda_m^+ = \varphi^+ d\Sigma^+ = \varphi^+ \Omega^+ \wedge \omega^+$, can be rewritten as:

$$\delta S_{gas} = -m \int_{D^+} J^4\gamma^* \mathfrak{L}_{J^4\Xi}(\varphi^+ \Omega^+ \wedge \omega^+).$$

The factor $\varphi^+\Omega^+$ is insensitive to variations in L (since its integral on *any* hypersurface $\sigma^+ \subset PTM^+$, expressing the number $N(\sigma)$ of particle trajectories crossing σ, does not depend on L), i.e., it is invariant under the flow of Ξ, that is,

$$\delta S_{gas} = -m \int_{D^+} J^4\gamma^* [\varphi^+ \Omega^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+].$$

Substituting $\Omega^+ = \mathbf{i}_\ell d\Sigma^+$ into the wedge product $\Omega^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+$, we find:

$$\Omega^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+ = (\mathbf{i}_\ell d\Sigma^+) \wedge \mathfrak{L}_{J^4\Xi}\omega^+ = \mathbf{i}_\ell(d\Sigma^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+) + \left(\mathbf{i}_\ell \mathfrak{L}_{J^4\Xi}\omega^+\right) d\Sigma^+.$$

But, $\mathfrak{L}_{J^4\Xi}\omega^+ = J^4\Xi(\hat{F}_{\cdot i})dx^i$ is a multiple of dx^i, meaning that $d\Sigma^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+ = 0$. Thus, the first term in the right hand side above vanishes and we remain with:

$$\Omega^+ \wedge \mathfrak{L}_{J^4\Xi}\omega^+ = \left(\mathbf{i}_\ell \mathfrak{L}_{J^4\Xi}\omega^+\right) d\Sigma^+.$$

The 0-form $\mathbf{i}_\ell \mathfrak{L}_{J^4\Xi}\omega^+ = l^i J^4\Xi(\hat{F}_{\cdot i})$ can be obtained directly, by applying $J^4\Xi$ to the ratio $\hat{F}_{\cdot i} = \dfrac{\hat{L}_{\cdot i}}{2\hat{L}^{1/2}}$ and using the local expression $J^4\Xi = 2\hat{v}\dfrac{\partial}{\partial \hat{L}} + 2(d_i\hat{v})\dfrac{\partial}{\partial \hat{L}_{,i}} + 2(\dot{d}_i\hat{v})\dfrac{\partial}{\partial \hat{L}_{\cdot i}} + (...)$, together with the 2-homogeneity of $\hat{v}$, as:

$$\mathbf{i}_{\ell^+}\mathfrak{L}_{J^4\Xi}\omega^+ = \frac{\hat{v}}{\hat{L}}.$$

Substituting into the action, we find:

$$\delta S_{gas} = -m \int_{D^+} J^6\gamma^*(\varphi^+ \frac{\hat{v}}{\hat{L}} d\Sigma^+), \tag{3.89}$$

where the integrand (which is actually, of order 2) was pulled back to J^6Y_g to match the order of the variation of λ_g^+. Using formula (3.65) for the variation of the vacuum action, we now get the statement. ■

The energy momentum distribution tensor of a kinetic gas.

The action S_{gas} (and, accordingly, the matter Lagrangian λ_m^+) is, by construction, generally covariant. That is, it will lead to an energy-momentum distribution tensor Θ, obeying the averaged conservation law (3.50).

As the boundary terms in δS_{gas} vanish, we can write: $\delta S_{gas} = \int_{D^+} J^6\gamma^* \mathbf{i}_{J^6\Xi}\mathcal{E}(\lambda_m^+)$; then, (3.89) together with $\Xi = 2\hat{v}\dfrac{\partial}{\partial \hat{L}}$, identify the Euler-Lagrange form as:

$$\mathcal{E}(\lambda_m^+) = -\frac{m}{2}\varphi^+ \hat{L}^{-1}\theta \wedge d\Sigma^+ \in \Omega_8(J^6Y_g).$$

Using (3.38), (3.42), we obtain the energy-momentum scalar:

$$\mathfrak{T} := m\varphi^+$$

and accordingly, the components of the energy-momentum tensor distribution Θ as:

$$\Theta^i{}_j = m\varphi^+ \ l^i l_j. \tag{3.90}$$

Averaging over observer directions will provide the corresponding energy-momentum density on M (3.54), with components:

$$\mathcal{T}^i_j(x) := m \int\limits_{\mathcal{O}_x^+} (\varphi^+ l^i l_j) \circ J^6\gamma \ d\Sigma_x^+ = \ m \int\limits_{\mathcal{O}_x} \varphi l^i l_j d\Sigma_x; \tag{3.91}$$

we recall that $\varphi(x,\cdot)$ is compactly supported, i.e., the above integrals are finite.

The averaged conservation law (3.50) holds at all $x \in M$, i.e.,

$$\int\limits_{\mathcal{T}_x^+} (\Theta^j{}_{i|j} \circ J^6\gamma) d\Sigma_x^+ = 0. \tag{3.92}$$

In our case, we easily find: $\Theta^i{}_{j|i} = m(\varphi^+ \ l^i l_j)_{|i} = m\varphi^+_{|i} l^i l_j = (\nabla\varphi^+) l_j$, which, rewriting the above as an integral over $\mathcal{O}_x$ and simplifying the factor m, gives the *averaged conservation law for a kinetic gas* as a system of 4 equations:

$$\int\limits_{\mathcal{O}_x} (\nabla\varphi) l_j d\Sigma_x^+ = 0. \tag{3.93}$$

Particular cases:

1. **Collisionless gases.** In this case, we have already seen that φ is subject to the *Liouville equation* $\nabla\varphi = 0$. In other words, the Liouville equation is nothing else than a *pointwise covariant conservation law* of Θ:
$$\Theta^i{}_{j|i} \circ J^4\gamma = 0.$$
2. **Lorentzian spaces.** On a Lorentzian manifold (M, a), the functions:
$$T^i{}_j = \frac{1}{\sqrt{|\det a|}} \mathcal{T}^i_j \tag{3.94}$$
represent the components of a tensor of type (1,1) on M. In this case, Levi-Civita covariant derivatives are, [62], just the integrals of the Chern covariant derivatives, that is,
$$T^i_{j;i} = |\det a|^{-1/2} \int\limits_{\mathcal{O}_x^+} J^{s+1}\gamma^* \Theta^i{}_{j|i}(x,\dot{x}) d\Sigma_x.$$
Thus, the averaged energy-momentum conservation law (3.50) reads
$$T^i{}_{j;i} = 0.$$

Remark. In the particular case of kinetic gases on a Lorentzian spacetime, our expression (3.91) of the energy-momentum density $\mathcal{T}$ reduces to the known one, see [175]. Yet, on general Finsler spacetimes, the Finslerian metric tensor g depends on tangent vectors $\dot{x} \in T_xM$, hence formally writing (3.94) would *not* define any tensor on M; all we can get is an energy-momentum tensor *density* on M, by averaging over observer directions, as in (3.54) and, accordingly, the averaged conservation law (3.53) of Θ.

3.4 Cosmologically symmetric Finsler spacetimes

3.4.1 Introduction

The cosmological (Copernic) principle underlying modern cosmology states that, at the largest scales, the Universe is *homogeneous* (intuitively, "the same at all points") and *isotropic* ("the same in all directions").

To translate the Copernic principle into a geometric language, one first needs a spacetime manifold M that can be foliated into "spatial slices" playing the role of the Universe in the above statement. This is achieved via a so-called *global time function* $t : M \to \mathbb{R}$, whose level hypersurfaces $t = const.$ are the said spatial slices. Homogeneity and isotropy are then defined in terms of a *group of isometries* acting transitively on each of these hypersurfaces and containing a local *isotropy group* which acts transitively on spatial directions at each point.

In the class of Lorentzian (semi-Riemannian) spacetimes, the only possible metrics obeying the above requests are the celebrated *Friedmann-Lemaître-Robertson* (FLRW) metrics, having spatial slices of constant sectional curvature $k \in \{-1, 0, 1\}$ and locally given in spherical coordinates (t, r, θ, φ) by:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right), \tag{3.95}$$

where $a = a(t)$ is a smooth real function called the *scale factor.*

But, the notion of isometry is also well defined in Finsler spacetime geometry, which means that the same ideas can be applied to Finsler geometry. Below, we apply the above geometric formulation of the Copernic principle to identify the algebra of generators of cosmological symmetry in the Finslerian case. Surprisingly or not, we find that these generators are the same as in the Lorentzian case - i.e., the corresponding isometries are spatial rotations and the so-called *quasi-translations.*

After having identified the symmetry generators, integration of the resulting Killing equations straightforwardly leads to the most general form of cosmologically symmetric Finsler functions; yet, this class is still a very large one, i.e., the cosmological symmetry demand leaves room, in Finsler spacetime geometry, for a much wider choice than in Lorentzian geometry.

As we have already seen above, a class of Finsler spacetime geometries, which can be regarded as closest to pseudo-Riemannian geometry, are the so-called *Berwald spacetimes.* In this particular case, we were able to completely classify Finsler functions with cosmological symmetry.

The ideas presented below are part of the paper [96].

3.4.2 Axioms of Finslerian cosmological symmetry

Consider a Finsler spacetime (M, L), with admissible set $\mathcal{A} \subset \overset{\circ}{TM}$ and future-pointing timelike conic bundle $\mathcal{T}$ (see Section 2.1 for the definitions of $\mathcal{A}$ and $\mathcal{T}$).

Time function and spatial slices. We assume in the following that (M, L) possesses a smooth *global time function* $t : M \to \mathbb{R}$, which assigns to each point $p \in M$ a "time stamp" and whose differential dt satisfies $dt(X) > 0,\ \forall X \in \mathcal{T} \cup \partial\mathcal{T}$. The level sets of the time function:

$$\Sigma_T := \{p \in M |\ t(p) = T = constant\} \tag{3.96}$$

are interpreted as equal-time *spatial hypersurfaces*; as, by definition, $\dim M = 4$, all spatial hypersurfaces Σ_T are 3-dimensional. We assume that all the sets Σ_T are connected.

The definition below, [96] naturally extends to the Finslerian realm the definition of cosmologically symmetric Lorentzian spacetimes in [219], [220].

Definition 69 ***(Cosmologically symmetric Finsler spacetimes):*** *A Finsler spacetime (M, L) equipped with a global time function $t : M \to \mathbb{R}$ is said to admit* cosmological symmetry *if it is:*

(i) ***spatially homogeneous****, i.e. there exists a Lie Group G of isometries of (M, L) acting transitively on each spatial slice Σ_T;*

(ii) ***spatially isotropic,*** *i.e., the stabilizer (the isotropy group) at each point $p \in M$:*

$$G_p = \{\psi \in G | \psi(p) = p\}$$

acts transitively on the projective tangent spaces $PT_p\Sigma_T$ of Σ_T.

Interpretation of the axioms. Spatial homogeneity means that, for each two points q_1 and q_2 in Σ_T, there exists an isometry of (M, L) bringing q_1 to q_2, whereas the spatial isotropy request says that, having fixed a point $p \in \Sigma_T$, any two lines in the tangent plane $T_p\Sigma_T$ - i.e., any two elements[7] $[v_1], [v_2] \in PT_p\Sigma_T$ - can be mapped isometrically into each other.

3.4.3 Identifying the symmetry generators

Assume, in the following, that (M, L) is a cosmologically symmetric Finsler spacetime, with cosmological symmetry group $(G, \cdot)$, and fix a spatial slice Σ_T as in (3.96). We start by some remarks that will be used repeatedly in the following:

1) [128]: On any Lie group $(G, \cdot)$, the connected component G' of the identity element (the *identity component)*, is a Lie subgroup.

2) [150]: If a Lie group $(G, \cdot)$ acts transitively on a connected manifold M, then its identity component G' still acts transitively on M.

3) In determining the Lie algebra of generators of a Lie group $(G, \cdot)$, it is only the identity component G' that has a contribution.

4) If a group acts effectively[8] on a set, then any of its subgroups acts effectively on that set.

[7] Elements of the projective tangent space $PT_p\Sigma_T$ are lines $[v] = \{\alpha v \mid \alpha \in \mathbb{R}\}$ directed by tangent vectors $v \in T_p\Sigma_T$.

[8] A group action is said to be effective if the only element of the group providing a trivial action on all points, is the identity element.

Taking into account the first three statements above, we can assume with no loss of generality that G is *connected*, i.e., it coincides with its identity component. In particular, this means that its action on M can only consist of *orientation-preserving transformations* $\psi : M \to M.$ The latter statement follows immediately having in mind that, on the one hand, the correspondence $\psi \mapsto J(\psi) : Diff(M) \to (0, \infty) \cup (-\infty, 0)$, attaching to a given diffeomorphism $\psi : M \to M$ its Jacobian determinant $J(\psi)$, is continuous (thus, it maps connected sets of transformations ψ into connected subsets of $\mathbb{R}$) - and, on the other hand, our subset of interest contains $\psi = id_M$.

The homogeneity demand makes each spatial slice (Σ_T, G) a *homogeneous space*, which is thus (see, e.g., [128]) diffeomorphic to the quotient $\Sigma_T = G/G_p$ and the following relation holds:

$$\dim G = \dim G_p + \dim \Sigma_T = \dim G_p + 3. \tag{3.97}$$

We will now prove two lemmas to identify the dimension of the groups G and G_p.

Lemma 70 *On a cosmologically Finsler spacetime* (M, L), *the dimension of the spatial isotropy group* G_p *at each point* $p \in M$ *is at most 3.*

Proof. Fix an arbitrary point $p \in M$. Since the full symmetry group G acts by Finslerian *isometries* on the 3-dimensional spatial slice Σ_T at p, we have, see Section 2.3.3:

$$\dim G \leq \frac{3\,(3+1)}{2} = 6.$$

The statement then follows from (3.97). ■

Lemma 71 *The dimension of the isotropy group* G_p *is at least 3, at any* $p \in M$.

Proof. Fix an arbitrary $p \in M$. We will proceed in several steps.

- *Step 1. Identify a connected quotient group* G_p' *of* G_p *that acts transitively and effectively on* $PT_p\Sigma_T$:

  Let us first write down explicitly the action of G_p on the projectivized tangent space $PT_p\Sigma_T$. Elements $\psi \in G_p$ act on tangent vectors $v \in T_p\Sigma_T$ via the linear tangent map $d\psi_p$; on elements $[v] = \{\alpha v \mid \alpha \in \mathbb{R}\} \in PT_p\Sigma_T$ of the projectivized tangent space $PT_p\Sigma_T$, G_p acts by the rule:

  $$(\psi, [v]) \mapsto \left[d\psi_p\,(v)\right];$$

  (which is independent of the representative of the class $[v]$ by virtue of the linearity of $d\psi_p$).

  This action is, by the isotropy hypothesis, transitive.

  Further, any Lie group action on a manifold gives rise to an *effective* Lie group action on the respective manifold, by factorizing away the elements which provide trivial actions. In our case, denoting by Id_p the subgroup of G_p whose elements act trivially on $PT_p\Sigma_T$, we obtain that the quotient group

  $$G_p' = G_p/Id_p \tag{3.98}$$

  still acts effectively on $PT_p\Sigma_T$.

To identify the subgroup Id_p, we note that the statement $\psi \in Id_p$ is equivalent to the fact that, for any $v \in T_p\Sigma_T$, there holds $[d\psi(v)] = [v]$, equivalently:

$$d\psi_p(v) = \alpha v,$$

for some $\alpha \in \mathbb{R}$. Using the above remark that ψ must be orientation-preserving, it follows that, in the above equality, $\alpha > 0$, hence we can use the positive 2-homogeneity of L to obtain: $L(p, \alpha v) = \alpha^2 L(p, v)$.

But, on the other hand, each ψ is, by hypothesis, an *isometry* of (M, L), which means: $L\left(p, d\psi_p(v)\right) = L(p, v)$. We thus get:

$$L(p, v) = L\left(p, d\psi_p(v)\right) = L(p, \alpha v) = \alpha^2 L(p, v),$$

which ultimately gives $\alpha = 1$ and therefore,

$$\psi \in Id_p \Leftrightarrow \quad d\psi_p(v) = v, \quad \forall v \in T_p\Sigma_T.$$

In other words, Id_p consists of those $\psi \in G$ whose differential at p is the identity of $T_p\Sigma_T$. Accordingly, the quotient group G'_p is obtained by identifying as a single element those diffeomorphisms $\psi, \psi' \in G$ which have the same values at p and the same differentials $d\psi_p$ - but whose higher order derivatives at p might differ.

Since we have only factorized away *trivial* actions, the action of G'_p on $PT_p\Sigma_T$ is also, still transitive.

We can again consider with no loss of generality that G'_p is connected; elsewhere, we can just rebrand as G'_p its connected component of the identity (which, using the above Remark 2, still acts transitively and effectively on $PT_p\Sigma_T$). This way, G'_p acts transitively and effectively on $PT_p\Sigma_T$, as required.

- *Step 2. Show that any maximal compact subgroup H_p of G'_p still acts transitively on $PT_p\Sigma_T$:*

  Choosing an arbitrary basis of $T_p\Sigma_T$, we can identify $T_p\Sigma_T$ with $\mathbb{R}^3$ and, accordingly, the projectivized tangent space $PT_p\Sigma_T$ with the projective plane $P\mathbb{R}^3$. We are thus able to use the following result, [154], p. 398: *If a connected Lie group acts transitively on a compact manifold with finite fundamental group, then any maximal compact subgroup also acts transitively on the respective manifold.*

  The projective plane is connected, compact and with fundamental group $\mathbb{Z}_2$. Therefore: if H_p is a maximal compact subgroup of G'_p, then H_p also acts transitively on $PT_p\Sigma_T$. Moreover, using Remark 4 above, this action is also effective.

- *Step 3. Show that* $\dim H_p = 3$.

  This follows immediately from another result in [154], p. 398-401, stating that: *any connected, compact Lie group acting transitively and effectively on the projective plane $P\mathbb{R}^3$ is isomorphic to $SO(3)$.*

  The group H_p is, by construction, compact and acts transitively and effectively on $PT_p\Sigma_T \simeq P\mathbb{R}^3$. Again, assuming it is not connected, its identity component, say, H'_p, still acts transitively and effectively on $P\mathbb{R}^3$; moreover, as any connected component is closed, this makes H'_p a closed subset of a compact set - that is, H'_p is also compact. Applying the mentioned result, we get

$$H'_p \simeq SO(3).$$

- *Step 4: Evaluation of* $\dim G_p$.

  Summing up, we have:

  $$\dim G_p \geq \dim G'_p \geq \dim H_p = \dim H'_p = 3,$$

  which completes the proof.

■

From Lemma 70 and Lemma 71 we now immediately find:

Theorem 72 *On a Finsler spacetime obeying the cosmological principle, the dimension of the isotropy group G_p is 3 and the dimension of the full symmetry group G is equal to 6.*

As we have seen above, we can assume with no loss of generality that G_p is connected - elsewhere, we will rebrand as G_p its identity component. This way, we obtain:

Proposition 73 *On cosmologically symmetric Finsler spacetimes, the isotropy group G_p is compact.*

Proof. Denote by $\mathcal{G}$ any maximal compact subgroup of G_p. We proceed similarly to the reasoning in Lemma 71: as G_p is connected and $PT_p\Sigma_T \simeq P\mathbb{R}^3$ has finite fundamental group, it follows that $\mathcal{G}$ must still act transitively and effectively on $PT_p\Sigma_T$. But, this entails that the identity component $\mathcal{G}'$ of $\mathcal{G}$ (which is, again, compact), acts transitively and effectively on $PT_p\Sigma_T \simeq P\mathbb{R}^3$, therefore:

$$\mathcal{G}' \simeq SO(3),$$

in particular, $\dim \mathcal{G}' = \dim \mathcal{G} = 3$.

The statement then follows from Cartan's classification theorem, [154], p. 389, stating that: *any connected Lie group is the direct product between one of its maximal compact subgroups and a Euclidean space*.

Taking into account that G_p was assumed to be connected and $\dim G_p = 3$, we conclude that $G_p = \mathcal{G}'$, i.e. G_p itself is compact. ■

The proof of the above Proposition points out that (the identity component of) G_p is actually, $SO(3)$.

The symmetry generators

To explicitly determine the generators of the groups G and G_p, we use the above Proposition stating that the isotropy group G_p is compact. This way, each spatial slice Σ_T is a homogeneous manifold having a compact isotropy group; thus, see [107] (p. 154), it must admit a G-invariant Riemannian metric h.

But, the G-invariance of h means that the generators of our group G are also Killing vector fields of h. We thus have at hand a 3-dimensional Riemannian manifold (Σ_T, h) admitting a 6-dimensional group of isometries, which means that(Σ_T, h) is actually, *maximally symmetric.*

In particular, h has constant scalar curvature k - and it allows one, see [220] to identify the generators of G. In local spherical coordinates (r, θ, ϕ) given by h on Σ_T, these read:

- generators of the isotropy group (which are the elements of the Lie algebra $so(3)$):

$$X_1 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad X_2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad X_3 = \partial_\phi;$$

- generators of the so-called *quasi translations*:

$$\begin{aligned} X_4 &= \sqrt{1-kr^2}\sin\theta\cos\phi\partial_r + \frac{\sqrt{1-kr^2}}{r}\sin\theta\cos\phi\partial_\theta - \frac{\sqrt{1-kr^2}}{r}\frac{\sin\phi}{\sin\theta}\partial_\phi, \\ X_4 &= \sqrt{1-kr^2}\sin\theta\sin\phi\partial_r + \frac{\sqrt{1-kr^2}}{r}\sin\theta\sin\phi\partial_\theta + \frac{\sqrt{1-kr^2}}{r}\frac{\cos\phi}{\sin\theta}\partial_\phi, \\ X_6 &= \sqrt{1-kr^2}\cos\theta\partial_r - \frac{\sqrt{1-kr^2}}{r}\sin\theta\partial_\theta. \end{aligned} \tag{3.99}$$

Substituting the above vector fields into the Finsler Killing equations $X_I^{\mathbf{c}}(L) = 0, \quad I = 1, ..., 6,$ where $X^{\mathbf{c}} = X^i\partial_i + \dot{x}^j X^i{}_{,j}\dot{\partial}_i$ yields the most general spatially homogeneous and isotropic Finsler Lagrangian, as:

$$L(t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}) = L(t, \dot{t}, w), \quad w^2 = \frac{\dot{r}^2}{1-kr^2} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \tag{3.100}$$

Note. The general solution (3.100) of the equations $X_I^{\mathbf{c}}(L) = 0$ was already already determined by C. Pfeifer, in his thesis [160], based on an intuition, namely, by *postulating* X_I as the symmetry generators; in the paper [96], we proved, starting from an axiomatic definition of homogeneity and isotropy, that X_I are, indeed, the generators of cosmological symmetry for Finsler spacetimes.

3.4.4 Classification of cosmologically symmetric Berwald spacetimes

The Berwald condition

Besides the classical characterizations of Berwald-type spaces, there exists a newer characterization of Berwald spaces by Pfeifer, Fuster and Heefer, [162], which will prove extremely useful in the following. This starts from the remark that any pseudo-Finsler function $L : \mathcal{A} \to \mathbb{R}$ can be written, in any local chart, as

$$L(x, \dot{x}) = \tilde{L}(x, \dot{x})\Omega(x, \dot{x}), \tag{3.101}$$

where $\tilde{L}(x, \dot{x}) = \tilde{g}_{ij}(x)\dot{x}^i\dot{x}^j$ is an arbitrary pseudo-Riemannian[9] Finsler function and $\Omega = \Omega(x, \dot{x}) : \mathcal{A} \to \mathbb{R}$ is a 0-homogeneous function in $\dot{x}$; outside the null cone $\tilde{L}(x, \dot{x}) = 0$, the function Ω is also smooth. Using this remark, the following result is known:

Theorem 74 ***(The Berwald condition,*** *[162]): A pseudo-Finsler space* (M, L), *with admissible set* $\mathcal{A} \subset \overset{\circ}{TM}$, *is of Berwald type if and only if there exists a quadratic pseudo-Finsler function* $\tilde{L}(x, \dot{x}) = \tilde{g}_{ij}(x)\dot{x}^i\dot{x}^j$ *and a symmetric* $(1, 2)$*-type tensor* $D = D^i{}_{jk}(x)\partial_i \otimes dx^k \otimes dx^j$ *on* M *such that, at any point* $(x, \dot{x}) \in \mathcal{A}$ *such that* $\tilde{L}(x, \dot{x}) \neq 0$ *and in any local chart around it:*

$$\partial_i\Omega - (\tilde{\gamma}^j{}_{ik}\dot{x}^k)\dot{\partial}_j\Omega = D^j{}_{ik}\dot{x}^k\left(\dot{\partial}_j\Omega + \frac{2\tilde{\dot{x}}_j\Omega}{\tilde{L}}\right), \tag{3.102}$$

[9] We will avoid here denoting pseudo-Riemannian metrics by a, in order not to interfere with the classical notation of the scale factor $a = a(t)$.

where Ω *is as in (3.101),* $\tilde{\gamma}^i{}_{jk}$ *are the Christoffel symbols of* $\tilde{g}$ *and* $\widetilde{\dot{x}}_j = \dot{x}^i \tilde{g}_{ij}(x)$.

Once the above result holds for one pseudo-Riemannian Finsler function $\tilde{L}$, it holds for any other such function (just, with a different D, see our paper [96] for a justification), which means that we are actually free to choose $\tilde{L}$. These being said, let us choose as our $\tilde{L}$, the most general homogeneous and isotropic quadratic Finsler Lagrangian

$$\tilde{L}(x, \dot{x}) = (\dot{t}^2 + \sigma a(t)^2 w^2), \tag{3.103}$$

where $\sigma \in \{-1, 1\}$ is a sign factor, corresponding to either positive definite ($\sigma = 1$), or Lorentzian ($\sigma = -1$) signature, the function $a = a(t)$ in (3.103) is a smooth function, called the *scale function* and w is as in (3.100).

Choosing $\sigma = 1$ in (3.103) has the advantage that the set of null vectors of the chosen metric does not interfere with our result (anyway, as we will see below, the final result will not depend on the choice of σ).

To evaluate the Berwald condition, let us first write L as in (3.101):

$$L(t, \dot{t}, w) = (\dot{t}^2 + \sigma a(t)^2 w^2)\Omega(t, \dot{t}, w).$$

On the conic bundle $\mathcal{T}$ of L, where $\dot{t} \neq 0$, this expression can be more conveniently rewritten in terms of the 0-homogeneous variable

$$s := w/\dot{t}$$

as:

$$L(t, \dot{t}, w) = \dot{t}^2(1 + \sigma a(t)^2 s^2)\Omega(t, 1, s).$$

The second ingredient in the Berwald condition (3.102) is the $(1, 2)$-tensor field D. The most general spatially homogeneous and isotropic such tensor field that is symmetric in its vector arguments, has the following nonzero components, see for example [91]:

$$D^t{}_{tt} = b(t),\ D^t{}_{rr} = \frac{c(t)}{1 - kr^2},\ D^t{}_{\theta\theta} = r^2 c(t),\ D^t{}_{\phi\phi} = r^2 \sin^2\theta c(t)\,, \tag{3.104}$$

$$D^r{}_{rt} = D^r{}_{tr} = D^\theta{}_{\theta t} = D^\theta{}_{t\theta} = D^\phi{}_{\phi t} = D^\phi{}_{t\phi} = d(t)\,. \tag{3.105}$$

where $b(t), c(t), d(t)$ are arbitrary functions of t.

Using the above expressions in (3.102) yields two independent equations, which need to be solved to determine $\Omega(t, s) := \Omega(t, 1, s)$:

$$\begin{cases} M(t,s)\Omega(t,s) + N(t,s)\dfrac{\partial}{\partial s}\Omega(t,s) = 0, \\ \dfrac{\partial}{\partial t}\Omega(t,s) + P(t,s)\Omega(t,s) + Q(t,s)\dfrac{\partial}{\partial s}\Omega(t,s) = 0, \end{cases} \tag{3.106}$$

where

$$M(t,s) := 2\frac{c(t) + \sigma a(t)^2 d(t)}{1 + \sigma s^2 a(t)^2}, \qquad N(t,s) := \frac{a'(t) - a(t)\left\{s^2\left[c(t) - \sigma a(t) a'(t)\right] - d(t)\right\}}{s a(t)} \tag{3.107}$$

$$P(t,s) := -2\frac{b(t) + \sigma s^2 a(t)^2 d(t)}{1 + \sigma s^2 a(t)^2} \qquad Q(t,s) := s\frac{a(t)\left[b(t) - d(t)\right] - a'(t)}{a(t)}. \tag{3.108}$$

Solving the Berwald condition

- If $N \neq 0$, then we can divide the first equation in (3.106) by N to get:

$$\frac{M(t,s)}{N(t,s)}\Omega(t,s) + \frac{\partial}{\partial s}\Omega(t,s) = 0\,. \tag{3.109}$$

Substituting M and N from (3.107), this equation can be explicitly integrated to give:

$$\Omega(t,s) = f(t)\frac{a'(t) - a(t)\left\{s^2\left[c(t) - \sigma a(t)a'(t)\right] - d(t)\right\}}{1+\sigma s^2 a(t)^2}.$$

Then, plugging Ω into the Finsler Lagrangian $L = \dot{t}^2(1+\sigma s^2 a(t)^2)\Omega(t,s)$ yields a quadratic expression in $\dot{t}$ and w :

$$L = I(t)\dot{t}^2 - J(t)w^2,$$

(with $I(t) = f(t)(a'(t) + a(t)d(t))$ and $J(t) = a(t)f(t)\left[c(t) - \sigma a(t)a'(t)\right]$).

That is, in the case $N \neq 0$, the obtained solution L defines a *pseudo-Riemannian* Finsler spacetime metric - which thus must coincide, after a coordinate change, with the FLRW one.

- If $N = 0$ and $M \neq 0$, then the first equation in (3.106) implies immediately that $\Omega(t,s) = 0$ and thus the Finsler Lagrangian $L = \tilde{L}\Omega$ is identically zero. Hence, in this case, L does not define a spacetime structure.

- Finally, the only case that leads to nontrivial Finslerian solutions is $M = N = 0$, as we will see below.

Setting $M = N = 0$, we find from the definitions (3.107) of M and N :

$$c(t) + \sigma a(t)^2 d(t) = 0, \qquad a'(t) - a(t)\left\{s^2\left[c(t) - \sigma a(t)a'(t)\right] - d(t)\right\} = 0.$$

Since s and t are independent variables, the coefficient of s^2 in the latter equation must vanish, which immediately implies $c(t) = \sigma a(t)a'(t)$. Plugging this into what is left of the above equations yields $d(t) = -\frac{a'(t)}{a(t)}$.

The first equation (3.106) is, in our case, trivially satisfied, whereas the second one gives:

$$\frac{\partial}{\partial t}\Omega(t,s) + sb(t)\frac{\partial}{\partial s}\Omega(t,s) - \frac{2(b(t) - \sigma s^2 a(t)a'(t))}{1+\sigma s^2 a(t)^2}\Omega(t,s) = 0\,. \tag{3.110}$$

This is a first order quasilinear PDE, with the general solution:

$$\Omega(t,s) = \frac{B(t)^2}{s^2 a(t)^2 + \sigma} f(sB(t)^{-1}), \qquad B(t) := \exp(\int_{t_0}^{t} b(\tau)d\tau,$$

where f is an arbitrary smooth function and $t_0 \in \mathbb{R}$. Accordingly, we find the Finsler spacetime function $L = \sigma\dot{t}^2\left(s^2 a(t)^2 + \sigma\right)\Omega$, as:

$$L = \sigma\dot{t}^2 B(t)^2 f(sB(t)^{-1}). \tag{3.111}$$

In the above, we can see that the sign factor σ (as well as the lower integration point t_0 in the expression of B) can, without loss of generality, be absorbed into the free function f.

One further step of simplification can be done by changing the coordinate t into:

$$\tilde{t}(t) := \int_0^t B(\lambda) d\lambda,$$

which implies $\dfrac{d\tilde{t}}{dt} = B(t)$; in the newly induced tangent bundle coordinates, $\dot{\tilde{t}} = \dot{t}B(t)$ and $\tilde{s} = w/\dot{\tilde{t}}$, (3.111) becomes

$$L(\tilde{t}, \dot{\tilde{t}}, w) = \dot{\tilde{t}}^2 f(\tilde{s})\,. \tag{3.112}$$

The above results are summarized in the following theorem:

Theorem 75 ***(Classification of cosmologically symmetric Berwald spacetimes):*** *If a Finsler spacetime Lagrangian L is of Berwald type and admits cosmological symmetry, then it falls into one of the following classes:*

1. *pseudo-Riemannian (quadratic in $\dot{x}$), in which case it is, up to a t coordinate redefinition, given by the Friedmann-Lemaitre-Robertson-Walker metric*

$$L = \dot{t}^2 - a\,(t)\,w^2, \qquad w^2 = \frac{\dot{r}^2}{1 - kr^2} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2); \tag{3.113}$$

2. *nontrivially Finslerian, in which case it is of the form (3.112).*

In the nontrivial Finslerian case, the function f in (3.112) can be chosen freely (just, taking care that the resulting function L obeys the Finsler spacetime axioms in Section 2.1) and must be determined from the gravitational field equation.

Chapter 4

Outlook and perspectives

This chapter briefly lists some future research directions, which are based on the results we have presented above and which we will pursue in the future.

4.1 A geometric toolkit for the calculus of variations

4.1.1 Hamilton-de Donder equations for higher order field theories

There are multiple ideas in the literature, on how to define a Hamiltonian field theory, see, e.g., [79], [121]. One such possibility, introduced by D. Krupka [118] (and extending an older idea by Dedecker, [64]), relies on Lepage equivalents of Lagrangians. Given a Lagrangian λ of order r over a fibered manifold (Y, π, X) and an arbitrary Lepage equivalent θ_λ of λ (say, of order s), a local section δ of the fibered manifold (J^sY, π^s, X) is called a *Hamilton extremal* of θ_λ, if, for any π^s-vertical vector field ξ on J^sY :

$$\delta^* \mathbf{i}_\xi d\theta_\lambda = 0. \tag{4.1}$$

Thus, the Hamilton equation (4.1) depends not only on the Lagrangian λ, but also on the choice of the Lepage equivalent θ_λ. In particular, it is not generally guaranteed that Lagrangians producing the same Euler-Lagrange equation (1.42) would generally also produce the same Hamilton equation - let alone the question of whether such a Hamilton equation is equivalent to the Euler-Lagrange one.

This lack of uniqueness is, yet, eliminated if the ($\mathbb{R}$-linear) mapping $\lambda \mapsto \theta_\lambda$ has the *closure property* discussed in Section 1.4, as, in this case, (1.128) ensures that, for all equivalent Lagrangians λ, λ', the resulting Hamilton equation (4.1) will be the same.

In Section 1.4, we have proposed two notions of local Lepage equivalent θ_λ for general Lagrangians λ of arbitrary order $r \geq 1$, possessing the closure property. Both these notions are constructed as

$$\theta_\lambda = \Theta_{\bar{\lambda}} + d\alpha, \tag{4.2}$$

i.e., by adding an exact form $d\alpha$ to the principal (Poincaré-Cartan) form $\Theta_{\bar{\lambda}}$ of an appropriately chosen equivalent Lagrangian $\bar{\lambda} = \lambda - hd\alpha$; this guarantees that all equivalent Lagrangians to λ will lead to the same $d\theta$:

$$d\theta_\lambda = d\Theta_{\bar{\lambda}}. \tag{4.3}$$

Moreover, basing our construction on the *principal* (Poincaré-Cartan) Lepage equivalent as in (4.2)-(4.3) has promising features. It is known, [122], that, under certain *regularity conditions* on the Lagrangian function $\mathcal{L}$, the Hamilton equation (4.1) for the principal Lepage equivalent Θ_λ - called the *Hamilton-de-Donder equation* - becomes, indeed, equivalent to the Euler-Lagrange equation of λ, which is what one would expect from a "correct" Hamilton equation.

The above mentioned regularity conditions (involving the second order derivatives of the Lagrangian density function $\mathcal{L}$) are, yet, quite restrictive. Thus, in the future, we plan to explore, using Lepage forms as in (4.2), the possibility of extending the Hamilton-de-Donder procedure to *general* Lagrangians, of any order.

As a remark, the above construction of θ_λ allows, at least in principle, for other choices of $\bar{\lambda}$ in (4.3); an interesting such choice for $\bar{\lambda}$ might be, for instance, the so-called *augmented Lagrangian* equivalent to λ, defined by Fatibene, Ferraris and Francaviglia, [71] for the cases when a standard background (a "vacuum state", e.g., Minkowski metric in the case of general relativity) can be fixed.

4.1.2 Extending the Vainberg-Tonti Lagrangian construction

The canonical variational technique, as introduced in Section 1.2.3 is a powerful tool, allowing one to establish whether a given differential system is locally variational or not - and, in the negative case, to transform it into a variational one. The algorithm relies on the Vainberg-Tonti Lagrangian, which can be attached, on vertically star-shaped chart domains, to any source form.

But, on the one hand, the condition of vertical star-shapedness of the domain is a limitation to the applicability of the algorithm[1]. Another issue is that the Vainberg-Tonti Lagrangian of a source form does not necessarily have the same *symmetries* with the respective source form.

Both the above problems suggest that it would be interesting to find an "improved" Vainberg-Tonti Lagrangian, with the following property: if the given source form admits a specified 1-parameter group of symmetries, then, the Lagrangian should also be invariant under this group. This way, once given a set of variational equations that are known to have a certain local 1-parameter group of symmetries (i.e., they satisfy the so-called *Noether-Bessel-Hagen equations*, [114]), one would obtain a Lagrangian which is invariant to (at least) this specified 1-parameter group; also, for a non-variational such system, one would obtain a variational completion possessing the given symmetries.

Such a Lagrangian could be constructed, for instance, by replacing the group of fiber homotheties with a different local Lie group. Of course, the question is whether such a Lagrangian can be constructed in general; moreover, one may obtain, for a non-variational source form, in principle, a variational completion which differs from the canonical one. These problems are to be studied in the near future.

4.1.3 Energy-momentum tensors and gravitational energy-momentum pseudo-tensor

The general construction presented in Section 1.3 leaves room for at least two questions, which we plan to investigate in the future:

1. Can a similar notion (and a similar energy-momentum balance law) be defined in the case when the differential index of the background manifold $Y^{(b)}$ is greater than 1?

[1] A quick fix, which works in a lot (though, not in all) of the cases, was presented in Section 1.2.3 and consists in considering the integral that defines the Vainberg-Tonti Lagrangian as a limit.

An affirmative answer would be relevant for purely affine gravity theories, where the index of the lifting is 2.

2. Obtaining a general construction of a conserved gravitational *energy-momentum pseudotensor* in general gravity theories. Such a construction, which consists in "completing" the balance term $\mathcal{B}(\xi)$ up to an exact form by a canonically added term, was first made by Landau and Lifshitz, [126], in the case of general relativity; extensions of this procedure exist for the case of *metric* theories, see, e.g., the paper by Capozziello, Capriolo and Transirico, [55]. Yet, to the best of our knowledge, in the case of arbitrary (not necessarily metric) backgrounds, an extension of the Landau-Lifshitz pseudotensor is not known yet.

4.2 Finsler spacetimes

The main problem we plan to focus on, in the near future, is the study of several *special classes of Finsler spacetimes* which are relevant for solving the Finsler gravity field equation (3.88).

4.2.1 Spacetimes with (α, β)-metric

Spaces with (α, β)-metrics, obtained by deforming a pseudo-Riemannian metric α on a given manifold M using a 1-form $\beta \in \Omega_1(M)$, are a most immediate class of examples of pseudo-Finsler spaces - and it includes Randers, Bogoslovsky-Kropina and Kundt spaces discussed in Section 2.1.3. The case when α is positive definite has been (and is still being) quite intensively studied, see, e.g., [21], [58], [63], [173]. Yet, in Lorentzian signature, apart from the purely computational results that can be immediately extended from the positive definite case, very little is known.

A first question that arises is the one of the precise conditions to be satisfied, such that the resulting "deformed" pseudo-Finsler (α, β)-metric defines a Finsler spacetime according to Definition 25, Sec. 2.1.2; some first results in this directions were obtained in the cases of Randers and Bogoslovsky-Kropina cases, yet, a much more general study is needed. But also other questions, such as the one of (α, β)-metrics possessing given symmetries (e.g., cosmological, or spatial spherical symmetry) are relevant.

4.2.2 Finsler spacetime functions with $\dot{x}$-compactly supported Finsler-Ricci scalar

We discussed in detail in Sec. 3.3.3 that a kinetic gas couples to Finsler gravity and that, in this case, the 1PDF of the gas sources the Finsler gravity equation (3.88). An immediate consequence is the restriction of the right hand side of this equation to each the (projectivized) tangent space PT_xM^+, $x \in M$, has compact support; we refer to this property briefly, as *$\dot{x}$-compact support.* Since, on the other hand, the Finsler-Ricci scalar R_0 is a basic ingredient of the left hand side of the field equation, it is natural to look for solutions L with the property that R_0 has $\dot{x}$-compact support. In particular, we plan to:

1. Determine the conditions for L such that R_0 has $\dot{x}$-compact support.

2. Find, if not all, at least large enough classes of Finsler spacetimes of Berwald type, respectively, of weakly Landsberg type, with $\dot{x}$-compactly supported R_0.

3. In particular, try to classify all Finsler spacetime functions with identically vanishing R_0. In positive definite Finsler geometry, a famous result by Akbar-Zadeh, [26], says that, under some supplementary assumptions (forward geodesic completeness, boundedness of the Cartan tensor), $R_0 = 0$ implies that the space is, actually, flat. The question is whether such a result can be extended to Finsler spacetimes.

   A related question is the existence of non-weakly Landsberg Finsler spacetime functions with $R_0 = 0$. This would be also relevant as it would simplify the field equation and also could provide a direct physical interpretation of the trace of the Landsberg tensor.

4.2.3 Compactly supported deviations from Lorentzian metrics a.

Finsler spacetime functions of the form

$$L(x, \dot{x}) := a_x(\dot{x}, \dot{x}) + h(x, \dot{x}), \tag{4.4}$$

such that the the 2-homogeneous partial function $h(x, \cdot) : T_xM \to \mathbb{R}$ is smooth on $T_xM\backslash\{0\}$ and has $\dot{x}$-compact support, are especially interesting for our Finslerian gravity model, as they lead to a Landsberg tensor whose trace $trace(P)$ has $\dot{x}$-compact support. If, additionally, the Lorentzian metric a is Ricci-flat, then R_0 is also $\dot{x}$-compactly supported. Moreover, these metrics will be *smooth* on the whole slit tangent bundle $\overset{\circ}{T}M$, which simplifies a lot of considerations.

Understanding in depth the geometry of Finsler spacetimes (4.4) is one of the topics we want to pursue in the next years. In particular, we plan to find:

- all (or at least large enough classes of) nontrivial examples of spacetimes (4.4) of *Berwald type*;

- curvature properties of metrics (4.4), especially in the case when L and a have specific symmetries, e.g., cosmological or spherical symmetry.

4.2.4 Weakly Landsberg spacetimes. Weak unicorns

The weakly Landsberg assumption $trace(P) = 0$ greatly simplifies the Finsler gravity field equation (3.88). So, it is natural look for weakly Landsberg spacetimes obeying some additional conditions. For instance:

1. *Cosmologically symmetric weakly Landsberg spacetimes*; this is actually already work in progress, jointly with C. Pfeifer, A. Fuster and S. Heefer. It turns out that in cosmological symmetry, the set of four weakly Landsberg equations $P_i = 0, i = 0, ..., 3$, actually reduce to a single (though, rather complicated) third order partial differential equation. A future task is to find, if not the most general solution of this equation, at least large enough classes of solutions.

2. *Weak unicorns*. In Finsler geometry, a *unicorn* is defined as a Landsberg metric which is non-Berwaldian. These metrics are particularly interesting, as they are more general than Berwald ones, while still leading to a simple form of the field equation (3.88). The name is justified as such examples are quite rare - actually, it is still an open problem whether for (properly, i.e., $\overset{\circ}{T}M$-smooth) Finsler metrics, these do exist at all; all the known examples of unicorns so far have non-admissible directions at each point of M, see [187] for a recent review of unicorns in (positive definite) Finsler geometry.

Passing to spacetime signature, most examples are *not* smooth on the entire slit tangent bundle $\overset{\circ}{TM}$, one can therefore expect weak unicorns to exist - and even more, *weak* (or *generalized*) unicorns, i.e., weakly Landsberg metrics that are not Berwald. Yet, this case is very little studied; to the best of our knowledge, only one particular class of weak spacetime unicorns, due to Asanov, [12] is known - and it definitely deserves a much deeper study.

In the near future, we plan to determine weak *spacetime* unicorns - or prove their non-existence (if the case), at least for some specific classes of metrics, e.g., of Randers, Bogoslovsky-Kropina, or Kundt type etc..

4.2.5 Smooth Berwald spacetime functions

Since in the case of Finsler spacetime metrics that are smooth on the entire slit tangent bundle $\overset{\circ}{TM}$, the question of the metrizability of the affine connection (see Section 2.3.2) of Berwald spacetimes is still open, we plan to also investigate this problem.

4.3 Finslerian field theory

4.3.1 Solutions of the Finslerian field equation

So far, to the best of our knowledge, for the Finslerian vacuum equation, only a very few, Berwald-type solutions of the *vacuum* field equation are known, belonging to the VGR (Bogoslovsky-Kropina), [78], [77], or Randers type (the latter are completely classified in [89]), or a class of static Berwald ones, [52]. In the nearest future, we plan to determine other (not necessarily vacuum) solutions, in three cases: spherical symmetry, cosmological symmetry and linearized perturbations of Ricci-flat Lorentzian metrics.

Vacuum spatially spherically symmetric solutions

In general relativity, spatially spherically symmetric solutions of the vacuum Einstein equations and, in particular, the *Schwarzschild metric*, model the gravitational field around a massive spherically symmetric source such as a star or a black hole. According to a famous result by Birkhoff, the Schwarzschild metric is the unique solution of the vacuum Einstein equations which is: spatially spherically symmetric, *static* (i.e., it possesses a timelike Killing vector field which is orthogonal to a family of hypersurfaces) and *asymptotically flat*.

Similarly, spatially spherically symmetric solutions of the Finslerian vacuum field equation (3.71) are candidates to model the matter around a massive gravity source, which we hope could explain at least a part of the observed dark matter phenomenology. But, a Schwarzschild-type solution of this equation - let alone its uniqueness or non-uniqueness is not yet known[2].

For a Finsler spacetime (M, L) equipped with a time function $t : M \to \mathbb{R}$ as defined in Section 3.4, *spatial spherical symmetry* is well defined, more precisely, it can be understood as $SO(3)$-invariance of the conic Finsler function induced by L on each spatial slice $t = const$.

[2]For Rutz's equation $R = 0$ a Birkhoff-type theorem is known to hold, [172]. Yet, our equation (3.71), which is the canonical variational completion of Rutz's equation, is a much more complicated one.

The most general form of spatially spherically symmetric Finsler spacetime functions is already known in the literature, [160]. That is, in finding a solution, we will restrict our attention to Finsler functions of this form.

I.A. Berwald-type solutions.

A greatly simplifying assumption is to look for vacuum solutions of Berwald type. In this direction, here are some concrete problems that we plan to solve in the near future:

1. Classifying, if possible, all Berwald-type Finsler spacetime functions admitting spherical symmetry. This is already work in progress, using a similar technique to the one presented in Section 3.4.

2. Finding all possible spatially spherically symmetric, static[3] and asymptotically flat solutions of the vacuum Finslerian field equation (3.71) of Berwald type. Since the Schwarzschild metric is obviously such a solution, the question is whether there exist other, nontrivially Finslerian, ones. This would actually solve the question on whether Birkhoff's Theorem stating the uniqueness of the Schwarzschild solution extends to the Berwald-Finsler context.

3. In the case when nontrivially Finslerian solutions exist, an interesting problem is the study of the behavior of *geodesics* of the obtained solutions; these could explain, for instance, rotational curves of galaxies - thus, providing a geometric alternative to dark matter, as pointed out in the paper by Chang and Li, [57].

4. Finding other (e.g., not necessarily static) spherically symmetric solutions of the vacuum equation (3.71).

I. B. Non-Berwaldian solutions.

In particular, we plan to investigate the existence (or non-existence) of vacuum solutions with zero Ricci scalar $R_0 = 0$, yet, nonzero trace of the Landsberg tensor. Obtaining such a solution would an important step towards a deeper understanding of the physical interpretation of the Landsberg tensor.

Cosmologically symmetric solutions of the Finsler field equation

Interpreting the matter content of the Universe as a kinetic gas, rather than as a perfect fluid, allows one to investigate the influence of the distribution of kinetic energies of the gas particles on the resulting gravitational field, as discussed in Section 3.3.3. We conjectured in [94], [95] that this may account for the dark energy phenomenology. In order to verify this conjecture, we plan to:

1. Expand the Finslerian field equation (3.88), for the particular class of cosmologically symmetric Finsler spacetime functions. As the general form of cosmologically symmetric Finsler functions is known and the most general form of cosmologically symmetric 1-particle distribution functions φ can be obtained similarly, this is just a routine task - yet, with quite deep implications. Namely, the obtained equation will represent a Finslerian analogue of the *Friedmann equations* in general relativity - which predict the behavior of the scale function $a = a(t)$ of the universe.

[3]A Finslerian analogue of staticity was defined in the paper by Caponio and Stancarone, [54].

2. Find an *exact solution* of the Finslerian field equation, with right hand side given by a concrete cosmologically symmetric 1-particle distribution function. For this exact solution, one can study the behavior of the obtained "scale function". As we expect this behavior to differ from the one obtained in general relativity, we expect this difference to account for at least a part of the dark energy in the universe.

3. Study the behavior of timelike, respectively, lightlike *geodesics* for the obtained solution. These will model trajectories of massive bodies, respectively, of light rays in our model.

Linearized Finslerian perturbations of Lorentzian metrics.

We plan to investigate first order perturbations of the Minkowski metric η and of Ricci-flat Lorentzian metrics on $M = \mathbb{R}^4$ (such as the Schwarzschild metric), which would thus model a "weakly Finslerian" gravitational field and Finslerian gravitational waves. Here are some problems we plan to study in the near future:

1. Linearization of the Finsler field equation and obtaining a Finsler post-Newtonian formalism.

   The latter is obtained, after having linearized the field equation around the Minkowski background, by also taking a Taylor expansion of the left hand side of the field equation, around zero *spatial* velocity and then identifying the corresponding coefficients, order by order. This topic is already work in progress, with C. Pfeifer and M. Hohmann.

2. Linearized perturbations of Minkowski metric on $\mathbb{R}^4$ - that is, finding solutions of the Finslerian field equation (3.88), expressible as $L(x,\dot{x}) = \eta(\dot{x},\dot{x}) + \epsilon h(x,\dot{x})$, with $\epsilon^2 \simeq 0$. In this case, the corresponding 1-particle distribution function φ has to be also of the same magnitude order as ϵ; this would thus describe a "sparse" kinetic gas, generating a weak gravitational field. For instance, one can use as φ a curved spacetime extension of the *Maxwell-Jüttner distribution*, see, e.g., [191].

3. Linearized perturbations of the Schwarzschild metric. Such a (non-vacuum) solution of the Finsler gravity equation (3.88) would model a region of spacetime possessing gravity generated by a massive source lying outside that region, e.g., a star or a black hole situated in the center of a galaxy, together with some other smaller sources situated *inside* that region - such as planets in the solar system, or stars in a galaxy.

4.3.2 Finslerian equation and the Einstein-Vlasov equations

In general relativity, the energy-momentum tensor of a kinetic gas is obtained, see [175], by integrating the 1-particle distribution function φ over $\dot{x} \in \mathcal{O}_x$, as in eq. (3.91) - which is always possible, as $\varphi(x,\cdot)$ is assumed to have compact support, or at least, to be integrable on the observer spaces $\mathcal{O}_x$. The *Einstein-Vlasov equations* are then the Einstein equations with the right hand side given by (3.91); thus, the Einstein-Vlasov equations are second order PDE's having as unknown functions, the components of a Lorentzian metric tensor a on the spacetime manifold M.

On the other hand, the Finsler gravity equation (3.88) is a single equation, having as unknown, a scalar function $L : TM \to \mathbb{R}$. As already noticed above, in the non-vacuum case, the right hand side $\kappa^2 m\varphi$ of this equation has $\dot{x}$-compact support, which means that the same must hold for its left hand side. That is, we expect non-vacuum solutions to be non-Riemannian. Yet, once we have a solution obtained for a realistic φ, by applying to both hand sides of the field equation the same procedure as in (3.91) in the left hand side of the equation, we obtain an equation of the form:

$$\tilde{\mathcal{G}}^i{}_j = 8\pi\kappa\tilde{\mathcal{T}}^i{}_j,$$

relating two tensor field densities of type (1,1) over M.

This way, if, on the given manifold M, there exists a "fiducial" Lorentzian metric $a : M \to T_2^0 M$ modeling the given physical situation (e.g., the Friedmann-Lemaître-Robertson-Walker metric, in cosmology), we plan to compare the obtained tensor density $\tilde{\mathcal{G}}$ to the (densitized) Einstein tensor of a. The obtained difference will represent corrections to the Einstein equations obtained from velocity averaging which could account, in cosmology, for at least a part of the observed dark energy phenomenology.

4.3.3 Finsler geometry as the geometry of modified dispersion relations. Cotangent bundle formulation of Finsler field theory

In physics, a *dispersion relation* is a relation of the form $H(x,p) = const.$ (where $H : T^*M \to \mathbb{R}$ is a smooth enough function), to be satisfied by the 4-momenta p of physical particles; the function H is interpreted as the Hamiltonian of the given physical system, [159].

In special and general relativity, dispersion relations for a free particle are given by a Hamiltonian H which is quadratic in p. In the case when H is non-quadratic (possibly, not even 2-homogeneous), the resulting dispersion relation is called *modified.* Modified dispersion relations are a most prominent way to describe possible observables from *quantum gravity*, [68], [75] - and they naturally lead to Finsler geometry, [82], [166]. Therefore, we plan to study the following.

1. *The cotangent bundle formulation of Finsler field theory framework.* Traditionally, Finsler geometry is formulated on the tangent bundle of the spacetime manifold. But, for most physical applications (kinetic gases or modified dispersion relations are here just some first examples), the language of *momenta*, rather than then one of velocities, seems more appropriate. Hence, for the physics community, it would be both more accessible and more useful to have a formulation of our framework in terms of momenta - i.e., a reformulation on the cotangent bundle of the spacetime manifold.

   A cotangent bundle version of Finsler geometry, having as a central piece a *2-homogeneous* Hamiltonian, instead of a 2-homogeneous Lagrangian, obtained via the Legendre transformation from the usual one, is since long known from the work of R. Miron and collaborators, see. e.g., [148]. Yet, in the case of non-homogeneous Hamiltonians, the relation between the given Hamiltonian and the corresponding Finsler spacetime function (obtained via the so-called *Helmholtz action, [159]*) is a more sophisticated one, hence obtaining a clear relation between the basic Finslerian geometric notions attached to the resulting Finsler function L and the given Hamiltonian function is still to be done.

   A next step would be a cotangent bundle reformulation of the *Finslerian field equation* (3.88).

2. Modified dispersion relations give rise to a pseudo-Riemannian metric $g = g(p)$ on each cotangent space T_x^*M, i.e., to a *curved momentum space* at each point $x \in M$. Hence, understanding in depth the geometry of these curved momentum spaces, appears as an interesting topic.

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