Geometric Methods of Finsler-Based Field Theory

-Habilitation Thesis-

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Main goals and motivation

Beauty is the first test: there is no permanent place in the world for ugly mathematics (G.H. Hardy)

Main goals:

◇ develop a general geometric framework for Lagrangian field theories based on Finsler geometry;

◇ explore other applications, in more general field theories, of the newly developed geometric tools.

Motivation of our study: extending general relativity so as to address:

◇ the dark energy&dark matter problem:

◇ tensions with quantum mechanics.

"Who ordered Finsler?"

◇ **In physics:**

- **most general geometry with a well defined notion of arc length** ($\sim$ *proper time*);

- quantum gravity phenomenology (*modified dispersion relations*)

- description of wave propagation in media

- **kinetic description of gases** ($\longrightarrow$ gravitational field generated by multiple sources, moving with different velocities).

◇ **In pure mathematics:**

- Lorentz-Finsler geometry is: little explored, strikingly different form positive definite one and... beautiful.

General structure:

Chapter 1: A geometric toolkit for the calculus of variations

Chapter 2: Geometry of Finsler spacetimes

Chapter 3: Finsler-based field theory

Chapter 4: Outlook and perspectives

1 A geometric toolkit for the calculus of variations

1.1. Preliminaries

Main refs.: Krupka 2015; Giachetta, Mangiarotti&Sardanashvili 2009.

Fibered manifold: a triple (Y, π, X) with:
X, Y - smooth manifolds ($\dim X = n$, $\dim Y = n + m$)
$\pi : Y \to X$ – surjective submersion

Fibers: $Y_x = \pi^{-1}(x))$
Fibered charts on Y: (V, ψ), $\psi = (x^A, y^\sigma)$ - such that $\pi : (x^A, y^\sigma) \mapsto (x^A)$

Interpretation in physics:
Y - *configuration space*, X - *parameter space* (usually - spacetime)
Local sections $\gamma \in \Gamma(Y)$, $\gamma : (x^i) \mapsto (x^A, y^\sigma(x^A))$ - **fields**

Arena for field theory: the jet bundles $(J^r Y, \pi^r, X)$.

Lagrangian of order r := a (π^r)-horizontal form $\lambda \in \Omega_n(J^r Y)$:

$$\lambda = \mathcal{L} d^n x,$$

with: $\mathcal{L} = \mathcal{L}(x^A, y^\sigma, y^\sigma_i, ..., y^\sigma_{i_1 ... i_n}), \qquad d^n x := dx^1 \wedge ... \wedge dx^n.$

Action: $S_D : \Gamma(Y) \to \mathbb{R}$:

$$S_D(\gamma) = \int_D J^r \gamma^* \lambda,$$

where $D \subset X$ - **piece** (=compact n-dim. submanifold with boundary).

Variations of S_D - from 1-parameter groups $\{\Phi_\varepsilon\}$ of **fibered automorphisms**

$$\begin{array}{ccc} Y & \xrightarrow{\Phi_\varepsilon} & Y \\ {}^{\pi}\downarrow & & \downarrow^{\pi} \\ X & \xrightarrow{\varphi_\varepsilon} & X \end{array} \quad \Rightarrow \quad \Phi_\varepsilon : \begin{cases} \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^\sigma = \tilde{y}^\sigma\left(x^j, y^\mu\right) \end{cases}$$

Variations as Lie derivatives: $\Xi \in \mathcal{X}(Y)$ - generator of $\{\Phi_\varepsilon\} \Rightarrow$

$$\delta S_D(\gamma) = \int_D J^r\gamma^* \mathfrak{L}_{J^r\Xi}\lambda$$

First variation formula:

$$J^r\gamma^*(\mathfrak{L}_{J^r\Xi}\lambda) = J^{2r}\gamma^* \mathbf{i}_{J^{2r}\Xi}\mathcal{E}(\lambda) - J^{2r-1}\gamma^* d\mathcal{J}^{\Xi} \qquad (1)$$

◇ $\mathcal{E}(\lambda) \in \Omega_{n+1}(J^{2r}Y)$ - **Euler-Lagrange form**:

$$\mathcal{E}(\lambda) = \frac{\delta\mathcal{L}}{\delta y^\sigma}\theta^\sigma \wedge d^n x, \qquad \theta^\sigma := dy^\sigma - y^\sigma_i dx^i$$

◇ $\mathcal{J}^{\Xi} \in \Omega_{n-1}(J^{2r-1}Y)$ - **Noether current**

◇ $\gamma \in \Gamma(Y)$ is an **extremal** of S if: $\forall D \subset X$ piece, $\forall$ compactly supported variation $supp(\Xi \circ \gamma) \subset D$:

$$\delta S_D(\gamma) = 0$$

In coords.: γ - extremal $\Leftrightarrow$ Euler-Lagrange equations:

$$\frac{\delta \mathcal{L}}{\delta y^\sigma} \circ J^{2r}\gamma = 0$$

Noether's first theorem:

$$\mathfrak{L}_{J^r\boldsymbol{\Xi}}\lambda = 0 \Rightarrow \; J^s\gamma^* d\mathcal{J}^{\boldsymbol{\Xi}} \approx 0$$

($\approx$ - equality *along critical sections* γ).

Identification of $\mathcal{E}(\lambda)$, $\mathcal{J}^{\boldsymbol{\Xi}}$:

- integration by parts $\longrightarrow$ coordinates needed!

- via *Lepage forms* (Krupka, 1973)$\longrightarrow$ coordinate-free, diff. forms only (see Sec. 1.4).

Natural bundles and natural (generally covariant) Lagrangians:
$\mathcal{M}_n$ - category of smooth n-dim manifolds, $\mathcal{FB}$ - category of smooth fiber bundles.
Natural bundle functor$:=$ a functor $\mathfrak{F} : \mathcal{M}_n \rightarrow \mathcal{FB}$, such that:
- $\forall\ M \in Ob(\mathcal{M}_n)$: $\mathfrak{F}(M)$ is a fiber bundle over M;
- $\forall\ \alpha_0 : M \rightarrow M' \in Morf(\mathcal{M}_n)\ \Rightarrow$ the fibered manifold morphism $\mathfrak{F}(\alpha_0) : \mathfrak{F}(M) \rightarrow \mathfrak{F}(M')$ covers α_0.

Natural (generally covariant) Lagrangians $=$ globally def. Lagrangians $\lambda \in \Omega_n(J^r\mathfrak{F}(M))$ s.th:

$$J^r\mathfrak{F}(\phi)^*\lambda = \lambda, \quad \forall\phi \in Diff(M)$$

In terms of infinitesimal generators:

$$\mathfrak{L}_{J^r\mathfrak{F}(\xi)}\lambda = 0, \quad \forall\xi \in \mathcal{X}(M) \tag{2}$$

1.2. Variational completion of differential equations

References:

1. N. Voicu, D. Krupka, *Canonical variational completion of differential equations,* Journal of Mathematical Physics 56, 043507 (2015).

2. N. Voicu: S*ource Forms and Their Variational Completions*, in vol. The Inverse Problem of the Calculus of Variations - Local and Global Theory, ed. Dmitri Zenkov, Atlantis Press-Springer (2015).

3. M. Hohmann, C. Pfeifer, N. Voicu, *Canonical variational completion and 4D Gauss–Bonnet gravity*, European Physical Journal Plus 136, 180 (2021).

Aim: Given an arbitrary PDE/ODE system:
- find out whether it is locally variational;
- if not, transform it into a locally variational one, by *adding a meaningful correction term.*

Motivation:

◇ Historically first variant of Einstein field eqs.:

$$R_{ij} = 8\pi\kappa T_{ij} \tag{3}$$

$\longrightarrow$ inconsistent with local energy-momentum conservation.

◇ Corrected version:

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi\kappa T_{ij} \tag{4}$$

$\longrightarrow$ *variational*, with Lagrangian function $=$ "simplest scalar" R.

Q: *Is there any systematic way of finding the "correction term", based on calculus of variations?*

Setting: (Y, π, X) - fibered manifold, $\dim X = n$.

Consider an arbitrary *PDE system of order* r over Y :

$$\varepsilon_\sigma(x^A, y^\sigma, ..., y^\sigma_{A_1 ... A_r}) = 0$$

$\longrightarrow$ a local *source form:*

$$\varepsilon := \varepsilon_\sigma \theta^\sigma \wedge d^n x \in \Omega_{n+1}(J^r Y). \tag{5}$$

Use: **Vainberg-Tonti Lagrangian** (Vainberg 1956, Tonti 1969):

$$\lambda_\varepsilon = \mathcal{L}_\varepsilon d^n x$$

attached to ε and to a given chart:

$$\mathcal{L}_\varepsilon(x^A, y^\sigma, ..., y^\sigma{}_{j_1 ... j_r}) := y^\sigma \int_0^1 \varepsilon_\sigma(x^A, u y^\sigma, ..., u y^\sigma{}_{j_1 ... j_r}) du. \tag{6}$$

Key property: Euler-Lagrange form $\mathcal{E}(\lambda_\varepsilon) = \mathcal{E}_\nu \theta^\nu \wedge d^n x$ of λ_ε :

$$\mathcal{E}_\nu = \varepsilon_\nu \underbrace{- \int_0^1 u\{y^\sigma (H_{\nu\sigma} \circ \chi_u) + ... + y^\sigma{}_{B_1 \dots B_r} (H_{\nu\sigma}{}^{B_1 \dots B_r} \circ \chi_u)\} du}_{\kappa_\nu},$$

where:

◦ $\chi_u : (x^A, y^\sigma, y^\sigma{}_j, ..., y^\sigma{}_{j_1 \dots j_r}) \mapsto (x^A, uy^\sigma, uy^\sigma{}_j, ..., uy^\sigma{}_{j_1 \dots j_r})$, $u \in [0, 1]$.

◦ H - **Helmholtz form** of ε - "obstructions from local variationality" of ε.

Definition 7, [1]: **Canonical variational completion** of ε :

$$\mathcal{E}(\lambda_\varepsilon) = \varepsilon + \kappa \tag{7}$$

$\Rightarrow \kappa = \kappa_\nu \theta^\nu \wedge d^n x \in \Omega_{n+1}(J^r Y)$ - completely expressed in terms of H.

Applications of canonical variational completion:

✓ Vacuum Einstein equations $R_{ij} - \frac{1}{2} R g_{ij} = 0$ - c.v.c. of $R_{ij} = 0$, [1].

✓ Energy-momentum tensors in general relativity (symmetrization [1], Lagrangian for perfect fluid [2]).

✓ Linearly damped oscillations, [1].

✓ "Renormalized" (truncated) Gauss-Bonnet gravity theory - shown to be non-variational, [3].

✓ **Finsler gravity** - see Chapter 3.

1.3. Energy-momentum tensor and energy-momentum balance

Ref.: [1]. N. Voicu, *Energy-momentum tensors in classical field theories – a modern perspective*, International Journal of Geometric Methods in Modern Physics, 13, 1640001 (2016).

Ideas:

1. Use a "Hilbert-type" definition of energy-momentum tensors, in general Lagrangian field theories ($\sim$ Gotay&Marsden 1992, Fernandez&co. 2000) ;
2. Find a general *energy-momentum balance law*, valid in any natural field theory of index 1 in the background variables.
3. Application: energy-momentum balance law in *general metric-tensor/metric-affine theories*.

Setting:

◇ Configuration manifold:

$$Y = Y^{(b)} \times_M Y^{(m)},$$

where $Y^{(b)}$, $Y^{(m)}$ - natural bundles over M (b - "background", m - "matter").

◇ A generally covariant Lagrangian:

$$\lambda = \lambda_b + \lambda_m \in \Omega_n(J^r Y)$$

◇ **Assumption:** Natural lift $l^b : \mathcal{X}(M) \to \mathcal{X}(Y^{(b)}),\ \xi \mapsto \Xi^{(b)}$ - *of order 1*:

$$\Xi^{(b)} = \xi^i \partial_i + (C^\sigma{}_i \xi^i + C^{\sigma j}{}_i \xi^i{}_{,j}) \frac{\partial}{\partial y^\sigma}.$$

Euler-Lagrange form of λ_m :

$$\mathcal{E}\left(\lambda_m\right) = \mathcal{E}^{(b)} + \mathcal{E}^{(m)}.$$

Lemma 8, [1]: There is a unique splitting:

$$h\mathbf{i}_{J^{s+1}\Xi}\mathcal{E}^{(b)} = \mathcal{B}(\xi) + hd(\mathcal{T}(\xi)), \qquad \forall \xi \in \mathcal{X}(M), \tag{8}$$

such that $\mathcal{T} : \mathcal{X}(M) \to \Omega_{n-1}(J^{s+1}Y)$, $\mathcal{B} : \mathcal{X}(M) \to \Omega_n(J^{s+2}Y)$ are $\mathcal{F}(M)$-linear mappings with horizontal values ($h : \Omega(J^{s+1}Y) \to \Omega(J^{s+2}Y)$ - *horizontalization* morphism).

◇ $\mathcal{T}$ - **energy-momentum tensor**, $\mathcal{B}$ - **balance function**.

In fibered coords $\left(x^i, y^\sigma, y^I\right)$ on Y :

$$\mathcal{T} = \mathcal{T}^j_i dx^i \otimes \mathbf{i}_{\partial_j} d^n x, \qquad \mathcal{T}^j_i = C^{\sigma j}_i \frac{\delta \mathcal{L}_m}{\delta y^\sigma}. \tag{9}$$

First variation formula revisited:

$$\int_D J^{s+2}\gamma^*\mathcal{B}(\xi) + \int_{\partial D} J^{s+1}\gamma^*(\mathcal{T}(\xi) - \mathcal{J}^{\Xi}) \approx_{\gamma^{(m)}} 0, \quad (\gamma^{(m)} := proj_{Y^{(m)}} \circ \gamma).$$

Theorem 10, [1] (Coordinate-free energy-momentum balance law): For any piece $D \subset M$ and any $\xi \in \mathcal{X}(M)$ with support contained in D, there holds:

$$\int_D J^{s+2}\gamma^* \mathcal{B}(\xi) \approx_{\gamma(m)} 0. \tag{10}$$

Theorem 11, [1]:

(i): **Energy-momentum balance law** in coordinates**:**

$$\left(d_j \mathcal{T}^{\,j}_{\;i} - (C^{\sigma}_{\;i} - y^{\sigma}_{\;i}) \frac{\delta \mathcal{L}}{\delta y^{\sigma}} \right) \circ J^{s+2}\gamma \approx_{\gamma(m)} 0.$$

(ii) Relation with Noether currents:

$$\int_{\partial D} J^{s+1}\gamma^* \mathcal{T}(\xi) \approx_{\gamma(m)} \int_{\partial D} J^{s+1}\gamma^* \mathcal{J}^{l(\xi)}.$$

Example. General metric-tensor theories:

$$Y^{(b)} = Met(M) \times_M T^p_q(M), \qquad \lambda_m = \mathbb{L}_m \sqrt{|\det g|} d^n x.$$

Denote: $y^\sigma \in \{g^{ij}, y^{i_1 \dots i_p}_{j_1 \dots j_q}\}$ - background variables and

$$\mathfrak{T}_\sigma = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta y^\sigma}, \qquad T^j_{\ i} = \frac{1}{\sqrt{|\det g|}} \mathcal{T}^j_{\ i} = C^{\sigma j}_{\ i} \mathfrak{T}_\sigma. \tag{11}$$

Energy-momentum balance law:

$$(y^\sigma_{\ ;i} \mathfrak{T}_\sigma + T^j_{\ i;j}) \circ J^{s+2}\gamma \approx_{\gamma(m)} 0, \quad i = 1, ..., n. \tag{12}$$

In particular, in **metric-affine theories:** $y^\sigma \in \{g^{ij},\ N^i_{\ jk} := K^i_{\ jk} - \Gamma^i_{\ jk}\}$:

$$(T^j_{\ i;j} + N^j_{\ kh;i} \frac{\delta \mathbb{L}_m}{\delta N^j_{\ kh}}) \circ J^{s+2}\gamma \approx_{\gamma(m)} 0. \tag{13}$$

1.4. A special property of Lepage equivalents of Lagrangians

[1]. N. Voicu, S. Garoiu, B. Vasian, *On the closure property of Lepage equivalents of Lagrangians*, Differential Geometry and its Applications 81, 101852 (2022).

Main idea: For general Lagrangians $\lambda \in \Omega_n(J^r Y)$ of order $r \geq 1$, build *two* local Lepage equivalents with the **closure property**:

$$\mathcal{E}(\lambda) = 0 \quad \Leftrightarrow \quad d\rho_\lambda = 0.$$

Application: Having a well defined Lepage formulation of *Hamiltonian* field theory.

(Only) previously known examples of ρ_λ with closure property:
✓ mechanics ($\dim X = 1$) - Poincaré-Cartan form;
✓ *first order* Lagrangians (Krupka 1977, Betounes 1984).

Setting: (Y, π, X) - fibered manifold, $\lambda \in \Omega_n(J^r Y)$ - Lagrangian

Definition (Krupka, 1973)**:** $\rho_\lambda \in \Omega_n(J^s Y)$ - **Lepage equivalent** of λ, if:
(i) $\int_D J^r\gamma^*\lambda = \int_D J^r\gamma^*\rho_\lambda$, for all γ, D.
(ii) The first contact comp. $p_1 d\rho_\lambda$ is a source form ($\Leftrightarrow \pi^{s+1,0}$-horizontal).

Euler-Lagrange form/Noether currents in terms of ρ_λ**:**

$$\mathcal{E}(\lambda) = p_1 d\rho_\lambda, \qquad \mathcal{J}^{\mathbf{\Xi}} = \mathbf{i}_{J^s\mathbf{\Xi}}\rho_\lambda.$$

Principal Lepage equivalent $\rho_\lambda =: \Theta_\lambda$ (Krupka, 1981) - **no** closure property**:**

$$\Theta_\lambda = \mathcal{L}d^n x + \Big(\sum_{k=0}^{r-1} f_\sigma^{AB_1\dots B_k}\theta^\sigma_{B_1\dots B_k}\Big)\wedge \mathbf{i}_{\partial_A} d^n x, \qquad (14)$$

$$f^{B_1\dots B_{r+1}} = 0, \quad f_\sigma^{B_1\dots B_k} = \frac{\partial \mathcal{L}}{\partial y^\sigma_{B_1\dots B_k}} - d_A f_\sigma^{AB_1\dots B_k}. \qquad (15)$$

★ **Our idea, [1]: Use $\Theta_{\lambda'}$, for a conveniently chosen λ' equivalent to λ.**

Consider $\lambda \in \Omega_n(J^rY)$ - arbitrary Lagrangian.

I. Canonical Lepage equivalent Φ_λ: Decompose λ locally as:

$$\lambda = \lambda_{VT} + hd\alpha, \tag{16}$$

where λ_{VT} - Vainberg-Tonti Lagrangian of $\mathcal{E}(\lambda)$ and set:

$$\Phi_\lambda := \Theta_{\lambda_{VT}} + d\alpha. \tag{17}$$

Properties of canonical Lepage equivalent:

1. Closure property $\mathcal{E}(\lambda) = 0 \;\Leftrightarrow d\Phi_\lambda = 0$.
2. Φ_λ - uniquely defined by λ.
3. Generally Φ_λ - just locally defined. Yet, in *tensor* field theories with second order Euler-Lagrange equations, Φ_λ - globally well defined.

II. Minimal Lepage equivalent ϕ_λ : If λ - order-reducible, then use:

$$\lambda = \lambda' + hd\alpha, \qquad \phi_\lambda := \Theta_{\lambda'} + d\alpha, \tag{18}$$

(where λ' - of minimal order).

Properties of minimal Lepage equivalents:

1. Closure property.
2. If λ - second order, reducible $\Rightarrow \phi_\lambda$ - of order 1.
3. In general, ϕ_λ -not unique.

Example: Hilbert Lagrangian $\lambda \in \Omega_4(J^2Met(M))$, $\lambda = R\sqrt{|\det g|}d^4x$:

$$\Phi_{\lambda_g} = \Theta_{\lambda_g} = \phi_{\lambda_g}. \tag{19}$$

2 Geometry of Finsler spacetimes

2.1. Definitions and basic geometric objects

[1]. M. Hohmann, C. Pfeifer, N. Voicu, *Mathematical foundations for field theories on Finsler spacetimes*, Journal of Mathematical Physics 63, 032503 (2022).

[2] M. Hohmann, C. Pfeifer, N. Voicu, *Finsler gravity action from variational completion*, Physical Review D 100, 064035 (2019).

Aim of the section: Present the notion of **Finsler spacetime** as defined in [1] and a minimal list of related notions, to be used in the sequel.

Setting: M - n-dim. connected, orientable, $\mathcal{C}^\infty$-smooth manifold

$\diamond$ $\overset{\circ}{TM} := TM\backslash\{0\}$ slit tangent bundle.

$\diamond$ An open subset $\mathcal{Q} \subset TM\backslash\{0\}$ is a **conic subbundle** if:

- for $\forall x \in M$, $\mathcal{Q}_x := \mathcal{Q} \cap T_x M$ is non-empty;
- *conic property:* $(x, \dot{x}) \in \mathcal{Q} \Rightarrow (x, \alpha\dot{x}) \in \mathcal{Q}, \quad \forall\alpha > 0$.

$\diamond$ (Bejancu&Farran, 1990): **Pseudo-Finsler space** $= (M, L)$, where:

$L : \mathcal{A} \to \mathbb{R}$ - smooth on a conic subbundle $\mathcal{A} \subset TM$ and:

(i) $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall\alpha > 0$;

(ii) $g_{\mu\nu}(x, \dot{x}) = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial\dot{x}^\mu \partial\dot{x}^\nu}$ is nondegenerate on $\mathcal{A}$.

$\mathcal{A}$ - set of *admissible vectors*.

Definition 25, [1] A 4-dim. pseudo-Finsler space is a **Finsler spacetime** if: $\exists$ a conic subbundle $\mathcal{T} \subset \mathcal{A}$, with connected fibers $\mathcal{T}_x$ on which:

✓ $L > 0$, g has Lorentzian signature $(+,-,-,-)$

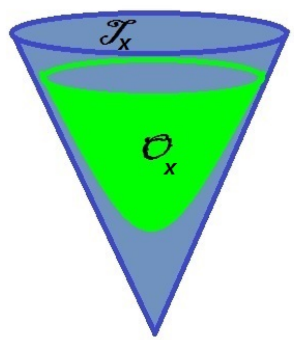
✓ L can be continuously extended as 0 to $\partial\mathcal{T}$.

Physical interpretations:

∘ *Interval*: $ds^2 = L(x, dx) = g_{ij}(x, \dot{x})dx^i dx^j$

∘ $\mathcal{T}_x :=$ **future-pointing timelike cone** at x.

∘ **Observer space** at $x \in M$: $\mathcal{O} := \{(x, \dot{x}) \in \mathcal{T} \mid L(x, \dot{x}) = 1\}$:



◦ *Finslerian metric tensor:*

$$g : \mathcal{A} \to T^0_2 M, (x, \dot{x}) \mapsto g_{(x,\dot{x})} = g_{ij}(x, \dot{x}) dx^i dx^j$$

◦ **Hilbert form** on $\mathcal{A}_0 := \mathcal{A} \setminus L^{-1}(0)$:

$$\omega := F_{\cdot i}(x, \dot{x}) dx^i, F = |L|^{1/2}$$

◦ *Arc length* of a non-null admissible curve $c : [a, b] \to M$ **(**$\sim$ proper time):

$$l(c) = \int_a^b \sqrt{L(c(t), \dot{c}(t))} dt = \int_a^b \sqrt{g_{ij}(x, \dot{x}) dx^i dx^j} dt = \int_{\mathrm{Im}(c,\dot{c})} \omega, \qquad (20)$$

(!) The positive **2-homogeneity** of L ensures that $l(c)$ - well-defined.

Geodesics of (M, L) : $\ddot{x}^i(s) + 2G^i(x(s), \dot{x}(s)) = 0$

Canonical nonlinear connection $T\mathcal{A} = H\mathcal{A} \oplus V\mathcal{A} \to$ coeffs: $G^i{}_j = \dot{\partial}_j G^i$.

Local adapted basis of $T\mathcal{A}$**:** $\{\delta_i = \partial_i - G^j{}_i \dot{\partial}_j, \dot{\partial}_i := \partial_{\dot{x}^i}\}$.

Examples of Finsler spacetime functions L :

✓ **Lorentzian** (quadratic in $\dot{x}$)**:**

$$L(x,\dot{x}) = a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}$$

✓ **Randers** $L = \epsilon F^2$, with $\epsilon = sign(F)$, where:

$$F\left(x,\dot{x}\right) = \sqrt{\left|a_x\left(\dot{x},\dot{x}\right)\right|} + b_x\left(\dot{x}\right).$$

✓ **Bogoslovsky/Kropina** *(VSR,VGR - Cohen&Glashow):*

$$L(x,\dot{x}) = \epsilon|a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}|^{1-q}(b_{\rho}(x)\dot{x}^{\rho})^{2q},$$

where: $\epsilon = sign(a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu})$.

✓ **Quartic metrics** *($\rightarrow$ birefringence - Pfeifer&Wohlfarth, Perlick etc.):*

$$L(x,\dot{x}) = \epsilon\sqrt{\left|\left(a_{\mu\nu}\left(x\right)\dot{x}^{\mu}\dot{x}^{\nu}\right)\left(h_{\rho\sigma}\left(x\right)\dot{x}^{\rho}\dot{x}^{\sigma}\right)\right|},$$

where $\epsilon = sign(...)$.

Homogeneity of Finslerian geometric objects

$\diamond$ L - homog. of degree 2 $\Rightarrow g, G^i{}_j$ etc. $\rightarrow$ all homogeneous of some degree.

Fiber homotheties:

$$\chi_\alpha : \overset{\circ}{TM} \rightarrow \overset{\circ}{TM}, \quad \chi_\alpha(x, \dot{x}) = (x, \alpha\dot{x}) \qquad (\alpha > 0)$$

- generated by the *Liouville vector field*

$$\mathbb{C} = \dot{x}^i \dot{\partial}_i. \tag{21}$$

Definition 30, [1]: A tensor field $T \in \Gamma(T^p_q(\mathcal{A}))$ is k-**homogeneous,** if:

$$\forall \alpha > 0 : \qquad \chi^*_\alpha T = \alpha^k T.$$

$\diamond$ *Anisotropic tensor fields* $\mathbb{T} : \mathcal{A} \rightarrow T^p_q(M) \rightsquigarrow$ d-tensor fields $T \in \Gamma(T^p_q(\mathcal{A}))$ (for which k-homog. is defined).

2.2. The positively projectivized tangent bundle PTM^+ (The projective sphere bundle)

1. On arbitrary manifolds M, $\dim M = n$. Define:

$$PTM^+ := \overset{\circ}{T}M_{/\sim} \tag{22}$$

where: $(x, \dot{x}) \sim (x, u) \Leftrightarrow \exists \alpha > 0 : u = \alpha \dot{x}$.

◦ PTM^+ - smooth, *orientable* $(2n-1)$-dim. manifold, *natural bundle* over M, with *fibers* $\simeq \mathbb{S}^{n-1}$.
◦ $(\overset{\circ}{T}M, \pi^+, PTM^+, \mathbb{R}^*_+)$ - principal bundle, with projection:

$$\pi^+ : \overset{\circ}{T}M \to PTM^+, \ (x, \dot{x}) \mapsto [(x, \dot{x})]. \tag{23}$$

◦ **0-homogeneous objects on** $\overset{\circ}{T}M \leftrightarrows$ **geom. objects on** PTM^+ :
◦ *Homogeneous local coords* of $[(x, \dot{x})]$: $(x^i, \dot{x}^i)$ (unique up to a factor) (see Chern-Chen-Lam 1999).

2. **On Finsler spacetimes (M, L):** The set of *non-null admissible directions:*

$$\mathcal{A}_0^+ = \left\{[(x, \dot{x})] \in \pi^+(\mathcal{A}) |\ L(x, \dot{x}) \neq 0\right\}$$

has a **contact structure** - **the Hilbert form** $\omega^+ = \dot{\partial}_i F dx^i$.

◦ *Canonical volume form:* $d\mathbf{\Sigma}^+ := \frac{\epsilon}{3!}\omega^+ \wedge (d\omega^+)^3 \ \ (\epsilon := sign(\det g))$.

◦ *Reeb vector field on* $\mathcal{A}_0^+$: $\ell^+ = l^i \delta_i, \quad l^i = \frac{\dot{x}^i}{F}$.

Proposition 38, [1] (Set of future pointing timelike directions $\mathcal{T}^+$):
Define $\mathcal{T}^+ := \pi^+(\mathcal{T}) \subset \mathcal{A}_0^+$. Then**:**
1. $\pi^+ : \mathcal{O} \to \mathcal{T}^+$ is a diffeomorphism.
2. If $\rho^+ \in \Omega_7(\mathcal{T}^+)$ - compactly supported and $\rho := \left(\pi^+\right)^* \rho^+$, then:

$$\int_{\mathcal{T}^+} \rho^+ = \int_{\mathcal{O}} \rho. \tag{24}$$

2.3. Finsler spacetimes, Finsler spaces, Lorentzian manifolds: a brief comparison

References:

[1] N. Voicu, *Conformal maps between pseudo-Finsler spaces*, International Journal of Geometric Methods in Modern Physics 15(01), 1850003 (2018).
[2] A. Fuster, S. Heefer, C. Pfeifer, N. Voicu, *On the non metrizability of Berwald Finsler spacetimes*, Universe 6 (5), 64 (2020).

Main aim: Show that:

1. Finsler spacetimes may *strikingly* differ from positive definite Finsler spaces
2. Yet: Finsler spacetimes share with Lorentzian ones some essential features ($\rightarrow$ OK for physics!)

Focus on: *projective* and *conformal* structures.

On the non-metrizability of Berwald-Finsler spacetimes, [2]:

(M, L) is called of **Berwald** type if G^i - quadratic in $\dot{x}$:

$$G^i = G^i{}_{jk}(x)\dot{x}^j\dot{x}^k$$

$\Leftrightarrow G^i_{jk}$ define *a symmetric affine connection on* M, whose autoparallels are geodesics of (M, L).

Theorem (Szábó,'s Metrizability Theorem, 1981): Let (M, F) be a (positive definite, $\overset{\circ}{TM}$-smooth) Finsler space of Berwald type. Then, there exists a Riemannian metric a on M such that the affine connection of the Berwald space is the Levi-Civita connection of a.

Consequence: Parametrized geodesics of (M, F) = same as those of (M, a).

Results in [2]:

Necessary condition for pseudo-Riemann metrizability**:** horizontal Chern-Rund Ricci tensor components $R_{ij} := R_i{}^k{}_{jk}$ must be symmetric:

$$R_{ij} = R_{ji}.$$

Example: Berwald spacetime function on $\mathbb{R}^4$ with $R_{ij} \neq R_{ji}$:

$$L(x, \dot{x}) = a_x(\dot{x}, \dot{x}) s^{-p} (k + m\ s)^{p+1},$$

$a = 2dx^0 \otimes dx^1 + x^1\, \phi(x^2, x^3)\, dx^0 \otimes dx^0 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3,\ b = dx^0,$

where: $s := \dfrac{(b_x(\dot{x}))^2}{a_x(\dot{x}, \dot{x})}) \Rightarrow L$ - non-Lorentz metrizable.

Theorem 42: If (M, L) is Berwald with $\mathcal{A} = \overset{\circ}{TM}$, then:

$$R_{ij} = R_{ji}.$$

Conformal symmetries of a pseudo-Finsler space (M, L), [1].

$\phi \in Diff(M)$ - *conformal symmetry* if $\exists \sigma : M \to \mathbb{R}$ - smooth, s. th.

$$L \circ d\phi = e^{\sigma} L. \tag{25}$$

Particular case: $\sigma = 0 \Rightarrow \phi$ - *isometry* of (M, L).

Remark: $\nexists$ Liouville-type classification of conformal symmetries of (flat) pseudo-Finsler spaces. Examples, [1]:

$$M := \mathbb{R}^k \times \mathbb{R}^{n-k}, \qquad L := L_1^{\alpha} L_2^{1-\alpha}, \tag{26}$$

with $L_1 = \left|\dot{x}^1 \dot{x}^2 ... \dot{x}^k\right|^{2/k}$, L_2 - arbitrary $\to$ infinite-dim. conformal group.

Theorem 44, [1] (Pseudo-Finslerian extension of Weyl Theorem): If a conformal symmetry of a connected pseudo-Finsler space (M, L) preserves unparametrized geodesics of (M, L), then $\sigma = const$.

Other results in Lorentzian geometry which extend to Lorentz-Finsler:
Conformal/Killing vector fields for (M, L) = generators of conf. symmetries/isometries of L.

Proposition 47, [1]: Any *essential* (= non-Killing, for any $e^{\sigma}L$) conformal vector field must be lightlike, i.e., $L \circ \xi = 0$, at least at a point.
(pseudo-Riemannian case - see Kuhnel 2008).

Theorem 48, [1]: Assume a Lorentz-Finsler space (M, L) admits a Killing vector field ξ with the property that $L(x, \xi(x)) \geq 0$, $\forall x \in M$. If $\xi = 0$ at one point $x \in M$, then ξ vanishes identically.
(*pseudo-Riemannian case - Sanchez, 1997*)

Theorem 49, [1]: If ξ is a Killing vector field for a Lorentz-Finsler space (M, L), having an isolated zero at some point $x \in M$, then: $\dim M$ - even and $L \circ \xi$ takes all possible signs on each neighborhood of x.
(*pseudo-Riemannian case - Sanchez, 1997*).

2.4. Inequalities from Finsler and Lorentz-Finsler norms

Reference:

[1] N. Minculete, C. Pfeifer, N. Voicu, *Inequalities from Lorentz-Finsler norms*, Mathematical Inequalities and Applications 24(2), 373–398 (2021).

Main idea:
Finsler geometry is actually behind many notorious inequalities. Such as the arithmetic-geometric mean one...

Consider: (M, L) - pseudo-Finsler space, $x \in M$.
$\mathcal{T} \subset T_xM \simeq \mathbb{R}^{n+1}$ - open, connected conic subset on which $L > 0$.

Pseudo-Finsler norm: $F = \sqrt{L} : \mathcal{T} \to (0, \infty)$.

Cauchy-Schwarz and reverse Cauchy-Schwarz inequalities

(Bao-Chern-Shen, 2000/Minguzzi 2015, Aazami&Javaloyes 2016):

I. L **- positive definite** $\Rightarrow$ **Cauchy-Schwarz (fundamental) inequality**:

$$dF_v(w) \leq F(w) \quad \Leftrightarrow \quad g_v(v,w) \leq F(v)F(w).$$

II. L **- Lorentzian** $\Rightarrow$ **reverse Cauchy-Schwarz inequality:**

$$dF_v(w) \geq F(w) \quad \Leftrightarrow \quad g_v(v,w) \geq F(v)F(w).$$

Remark, [1]: Ineqs. still hold in the pos. semidef./degenerate-Lorentzian case.

Examples of reverse Cauchy-Schwarz inequalities, [1]:

1) **Aczél's inequality:** $a^i, b^i > 0 \Rightarrow$

$$(a^0b^0 - a^1b^1 - .... - a^nb^n)^2 \geq [\left(a^0\right)^2 - \left(a^1\right)^2 ... - (a^n)^2][\left(b^0\right)^2 - \left(b^1\right)^2 ... - (b^n)^2].$$

2) **Popoviciu's inequality**. If $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ a^i, b^i > 0 \Rightarrow$

$$a^0b^0 - a^1b^1 - ... - a^nb^n \geq \left[(a^0)^q - \left(a^1\right)^q - ... - (a^n)^q\right]^{\frac{1}{q}} \left[\left(b^0\right)^p - \left(b^1\right)^p - ... - (b^n)^p\right]^{\frac{1}{p}}.$$

3) **Arithmetic-geometric mean inequality:**

$$\frac{\alpha_0 + ... + \alpha_n}{n+1} \geq (\alpha_0\alpha_1...\alpha_n)^{\frac{1}{n+1}}, \quad \forall \alpha_i \in \mathbb{R}_+^*. \tag{27}$$

4) **Weighted arithmetic-geometric mean inequality**:

$$\sum_{i=0}^{n} a_i v^i \geq (v^0)^{a_0}(v^1)^{a_1}...(v^n)^{a_n}, \qquad a_i \geq 0,\ v^i > 0. \tag{28}$$

Example of (positive definite) CS inequality - Hőlder inequality:

$$a^0 b^0 + a^1 b^1 + ... + a^n b^n \ \leq \ \left[\left(a^0\right)^q + ... + (a^n)^q\right]^{\frac{1}{q}} \left[\left(b^0\right)^p + ... + (b^n)^p\right]^{\frac{1}{p}},$$

Playing to discover new inequalities:

1) Use a **Kropina** deformation of Miknowski metric $\eta \Rightarrow$

$$2\eta(v,w) \ \geq \ \frac{w^0}{v^0}\eta(v,v) + \frac{v^0}{w^0}\eta(w,w).$$

2) **A Finslerian extension of Aczél's inequality:**

$$[\rho(v)\rho(w) - \hat{g}_v(v,w)]^2 \geq [\rho^2(v) - \hat{F}^2(v)][\rho^2(w) - \hat{F}^2(w)],$$

where: $\hat{F}$- pos. def. Finsler, $\rho \in \Omega_1(\mathbb{R}^{n+1})$.

See [1] for more examples (*triangle/reverse triangle ineqs.*)...

3 Finsler-based field theory

3.1. The general framework

Ref.: [1]. M. Hohmann, C. Pfeifer, N. Voicu, *Mathematical foundations for field theories on Finsler spacetimes*, Journal of Mathematical Physics 63, 032503 (2022).

Main results:

1. **Construct general configuration bundles** (Y, Π, X)**, allowing:**
- k-homogeneous Finslerian geometric objects as sections**;**
- well defined fibered automorphisms;
- compactly supported variations;
$\Rightarrow$ best option: $X := PTM^+$.

2. **Analyze the common features of** (Y, Π, PTM^+) **and of Lagrangians built upon them.**

Structure of fibered manifolds over PTM^+

Consider: (M, L) - Finsler spacetime, (Y, Π, PTM^+) - fibered manifold $\Rightarrow$

$$Y \xrightarrow{\Pi} PTM^+ \xrightarrow{\pi_M} M. \tag{29}$$

Fibered automorphisms of (Y, Π, PTM^+) :

$$\begin{array}{ccc}
Y & \xrightarrow{\Phi} & Y \\
\Pi \downarrow & & \downarrow \Pi \\
PTM^+ & \xrightarrow{\phi} & PTM^+ \\
\pi_M \downarrow & & \downarrow \pi_M \\
M & \xrightarrow{\phi_0} & M
\end{array}$$

Bundles having k-homogeneous Finslerian geometric objects as sections:
A *k-homogeneous (Finslerian) geometric object*= a local section:

$$\overset{\circ}{\gamma} : \mathcal{Q} \to \overset{\circ}{Y}, \quad (x, \dot{x}) \mapsto (x, \dot{x}, y(x, \dot{x})),$$

of some fiber bundle $(\overset{\circ}{Y}, \overset{\circ}{\Pi}, T\overset{\circ}{M}, Z)$ obeying:

$$\Gamma(x, \alpha\dot{x}) = \left(x, \alpha\dot{x}, \alpha^k y\right), \quad \forall\alpha > 0.$$

Necessary cond.: $\exists$ an *action* $H : \mathbb{R}^*_+ \times \overset{\circ}{Y} \to \overset{\circ}{Y}$ by fibered automorphisms:

$$H(\alpha, \cdot) = H_\alpha \in Aut(\overset{\circ}{Y}), \qquad H_\alpha(x, \dot{x}, y) = \left(x, \alpha\dot{x}, \alpha^k y\right), \tag{30}$$

Then: k-homogeneity = equivariance:

$$\begin{array}{ccc} \overset{\circ}{Y} & \xrightarrow{H_\alpha} & \overset{\circ}{Y} \\ \overset{\circ}{\gamma}\uparrow & & \uparrow\overset{\circ}{\gamma} \\ T\overset{\circ}{M} & \xrightarrow{\chi_\alpha} & T\overset{\circ}{M} \end{array} \qquad H_\alpha \circ \overset{\circ}{\gamma} = \overset{\circ}{\gamma} \circ \chi_\alpha.$$

★ **Idea: "factor away" the action of $\mathbb{R}^*_+$ from both $\mathring{Y}$ and $T\mathring{M}$.**

Theorem 58 (The orbit space Y): Consider a fiber bundle $(\mathring{Y}, \mathring{\Pi}, T\mathring{M}, Z)$, equipped with action $H : \mathbb{R}^*_+ \times \mathring{Y} \to \mathring{Y}$ as in (30). Then:

1. The orbit space $Y = \mathring{Y}_{/\sim}$ of the action is a fiber bundle over PTM^+, with typical fiber Z and projection:

$$\Pi : Y \to PTM^+, \qquad \Pi\,[x, \dot{x}, y] = [x, \dot{x}]\,.$$

2. *k-homogeneous* sections $\mathring{\gamma} : \mathcal{Q} \to \mathring{Y}$, where $\mathcal{Q} \subset T\mathring{M}$ is a conic subbundle, are in a one-to-one correspondence with local sections $\gamma : \pi^+(\mathcal{Q}) \to Y$.

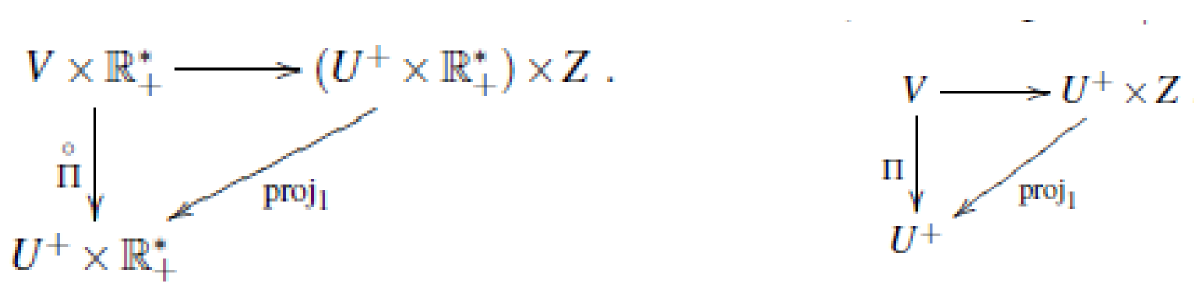


Figure: local trivializations on $\mathring{Y}$ and Y.

Fibered homogeneous coordinates on Y (unique up to positive rescaling):

$$[x, \dot{x}, y] \mapsto \left(x^i, \dot{x}^i, y^\sigma\right) \tag{31}$$

$:=$ local coords of an arbitrarily chosen representative of the class $[x, \dot{x}, y]$.

Examples:

1. **Finsler (2-homogeneous) functions** $L : \mathcal{A} \to \mathbb{R} \Rightarrow$

$$\overset{\circ}{Y} = T\overset{\circ}{M} \times \mathbb{R}, \quad H_\alpha(x, \dot{x}, \hat{L}) = (x, \alpha\dot{x}, \alpha^2 \hat{L}), \quad \forall \alpha > 0. \tag{32}$$

Sections of Y: $\gamma[(x, \dot{x})] = [x, \dot{x}, L(x, \dot{x})]$, that is:

$$L = \hat{L} \circ \gamma \circ \pi^+.$$

2. 0-**homogeneous metric d-tensors** $g : \mathcal{A} \to T^0_2(T\overset{\circ}{M}) \Rightarrow$

$$\overset{\circ}{Y} = T^0_2(T\overset{\circ}{M}), \quad H_\alpha(x, \dot{x}, y) = (x, \alpha\dot{x}, y), \quad \forall \alpha > 0.$$

Other examples: *d-tensors, connections*.

Finsler field Lagrangians, action, extremals:

Finslerian field: $=$ a (local) section $\gamma \in \Gamma(Y)$.
Field Lagrangian of order r:$=$ a Π^r-horizontal 7-form $\lambda \in \Omega_7(J^r Y)$:

$$\lambda^+ = \Lambda d\mathbf{\Sigma}^+, \tag{33}$$

where: $d\mathbf{\Sigma}^+ =$ (any) invariant volume form on PTM^+.

Property (0-homogeneity): $\dot{x}^i \dot{d}_i \Lambda = 0$.

Action attached to λ^+ and to a *piece* $D^+ \subset PTM^+$:

$$S_{D^+} : \Gamma(Y) \to \mathbb{R}, \quad S_{D^+}(\gamma) = \int\limits_{D^+} J^r \gamma^* \lambda^+$$

$\Rightarrow$ tools in Chapter 1 can be consistently applied.

3.2. The energy-momentum distribution tensor

Setting: Use L as the *background variable* (section of: $Y_g = (\overset{\circ}{TM} \times \mathbb{R})_{/\sim}$)

∘ Configuration bundle $\left(Y, \Pi, PTM^+\right)$:

$$Y := Y_g \times_{PTM^+} Y_m$$

where: Y_m - fiber bundle over PTM^+, Y_m - *natural over* M.

∘ *Canonical lifts* Ξ of $\xi_0 \in \mathcal{X}(M)$ = double lifts:

$$\xi_0 \in \mathcal{X}(M) \quad \mapsto \quad \xi \in \mathcal{X}(PTM^+) \quad \mapsto \quad \Xi \in \mathcal{X}(Y). \tag{34}$$

∘ *Natural matter Lagrangians*:

$$\mathfrak{L}_{J^r\Xi}\lambda_m^+ = 0,$$

for all Ξ as in (34).

Theorem 61 (Existence of energy-momentum distribution tensor Θ): Let $\lambda_m^+ \in \Omega_7(J^rY)$ be a natural Finsler Lagrangian and $\mathcal{E}_g(\lambda_m^+) \in \Omega_8(J^{s+1}Y)$ $(s+1 \leq 2r)$, the Y_g-component of its Euler-Lagrange form. Then, there exist unique $\mathcal{F}(M)$-linear mappings $\Theta : \mathcal{X}(M) \to \Omega(J^{s+1}Y)$, $\mathcal{B} : \mathcal{X}(M) \to \Omega(J^{s+2}Y)$, with horizontal values, such that:

$$h\mathbf{i}_{J^{s+1}\underline{\Xi}}\mathcal{E}_g(\lambda_m^+) = \mathcal{B}(\xi_0) + hd\Theta(\xi_0), \quad \forall \xi_0 \in \mathcal{X}(M). \tag{35}$$

Energy-momentum scalar $\mathfrak{T}$:

$$\mathcal{E}_g(\lambda_m^+) =: -\frac{1}{2}\mathfrak{T}\,\hat{L}^{-1}\theta \wedge d\mathbf{\Sigma}^+, \qquad \Theta^j{}_i := \mathfrak{T}\hat{L}^{-1}\dot{x}^j\dot{x}_i, \tag{36}$$

where: $\theta := d\hat{L} - \hat{L}_{,i}dx^i - \hat{L}_{.i}d\dot{x}^i, \;\; \dot{x}_i := \frac{1}{2}\hat{L}_{.ij}\dot{x}^j.$

Energy-momentum distribution tensor:

$$\Theta(\xi_0) = (\Theta^j{}_i\xi^i)\mathbf{i}_{\delta_j}d\mathbf{\Sigma}^+ = \mathfrak{T}\omega^+ \otimes \mathbf{i}_{\ell^+}d\mathbf{\Sigma}^+.$$

Balance function: $\mathcal{B}(\xi_0) = -\Theta^j{}_{i|j}\xi^i d\mathbf{\Sigma}^+.$

Theorem 65: For any local section $\gamma = (L, \gamma_m) \in \Gamma(Y_g \times_{PTM^+} Y_m)$ such that

$$\mathsf{supp}(J^r\gamma^*\lambda_m^+) \subset \mathcal{T}^+. \tag{37}$$

and γ_m - *critical for the action*, there hold:

1. **Averaged energy-momentum conservation law:** At any $x \in M$ and in any corresponding fibered chart:

$$\int_{\mathcal{T}_x^+} (\Theta^j{}_{i|j} \circ J^{s+1}\gamma) d\mathbf{\Sigma}_x^+ \approx_{\gamma(m)} 0, \tag{38}$$

where $d\mathbf{\Sigma}^+ =: d^4x \wedge d\mathbf{\Sigma}_x^+$.

2. **Relation to Noether currents:** For any $\xi_0 \in \mathcal{X}(M)$:

$$\int_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\Theta(\xi_0) \approx_{\gamma(m)} \int_{\partial\mathcal{T}^+(D_0)} J^{s+1}\gamma^*\mathcal{J}^{\Xi}, \tag{39}$$

where Ξ denotes the canonical lift of ξ_0 to Y.

Energy-momentum tensor density on M :

If supp$(J^r\gamma^*\lambda^+_m) \subset \mathcal{T}^+$ (e.g., γ has compact support **supp** $(\gamma) \subset \mathcal{T}^+$), then:

$$\mathcal{T}^i_{\ j}(x) := \int\limits_{\mathcal{O}^+_x} (\Theta^i_{\ j} \circ J^{s+1}\gamma)_{|(x,\dot{x})} d\mathbf{\Sigma}^+_x , \qquad \forall x \in M \, . \tag{40}$$

$\Rightarrow$ *integral is finite*, $\mathcal{T}^i_{\ j}$ - comps. of a tensor density on M.

3.3. Concrete model: Finsler gravity sourced by a kinetic gas

Refs.:

[1] M. Hohmann, C. Pfeifer, N. Voicu, *Finsler gravity action from variational completion*, Physical Review D 100, 064035 (2019).
[2] M. Hohmann, C. Pfeifer, N. Voicu, *Kinetic gases as direct gravity sources*, Physical Review D 101, 024062 (2020).
[3] M. Hohmann, C. Pfeifer, N. Voicu, *The kinetic gas universe*, European Physical Journal C 80, 809 (2020).

Main results:

1. Construct a concrete, correctly defined **vacuum action**, starting from a *physical principle+canonical variational completion*.
2. Construct a **matter action** (kinetic gas) $\Rightarrow$ field eq.&energy-momentum distribution.

Advantage: description of the gravitational field, fully taking the *velocity distribution of sources* into account.

1. Construction of vacuum action:

Geodesics of a Finsler spacetime $\nabla_{\dot{c}}\dot{c} = 0$
($\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ - given by the canonical nonlinear connection)

Geodesic deviation equation: $\nabla_{\dot{c}}\nabla_{\dot{c}}\xi = \mathcal{R}(\dot{c}, \xi)$;

Finslerian Ricci scalar: $R := trace(\mathcal{R}) = R^i{}_{ik}\dot{x}^k$

Postulated vacuum field equation (Rutz 1993):

$$R = 0. \tag{41}$$

Rutz eqn. is **not** variational $\longrightarrow$ build the "closest" variational eq.

Canonical variational completion of Rutz's equation, [1]:

○ Dynamical variable: $L \mapsto \Gamma(Y_g)$, $Y_g := (\overset{\circ}{TM} \times \mathbb{R})_{/\sim}$

○ Canonical volume form on $\mathcal{A}_0^+ \subset PTM^+$: $d\mathbf{\Sigma}^+ = \omega^+ \wedge d\omega^+ \wedge d\omega^+ \wedge d\omega^+$

○ Source form: $\varepsilon = (RL^{-1})\theta \wedge d\mathbf{\Sigma}^+ \in \Omega_8(J^4 Y_g)$

$\Rightarrow$ Vainberg-Tonti Lagrangian:

$$\lambda_g^+ = \hat{L}^{-1} R d\mathbf{\Sigma}^+.$$

Variational completion of Rutz's equation

($=$ same eq. as *Pfeifer&Wohlfarth 2011*)**:**

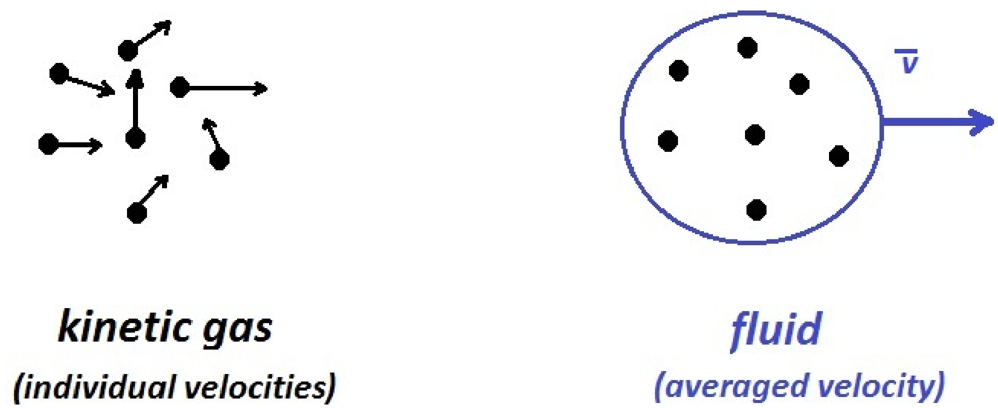
$$\frac{1}{2} g^{ij} R_{\cdot i \cdot j} - 3(L^{-1}R) - g^{ij}(P_{i|j} - P_i P_j + (\nabla P_i)_{\cdot j}) = 0, \qquad (42)$$

where $P = P_i dx^i$ - trace of *Landsberg tensor.*

2. Kinetic gases in general relativity (see Sarbach-Zannias 2014):

Kinetic gas = a large number N of interacting point particles, described by a smooth **1-particle distribution function:**

$$\varphi = \varphi\left(x, \dot{x}\right).$$



Worldlines = piecewise smooth normalized geodesics $\gamma \to (\gamma(s), \dot{\gamma}(s)) \in \mathcal{O}$

✓ $\mathcal{O}$ - **observer space of a Lorentzian metric**

$$\mathcal{O} := \{(x, \dot{x}) \in TM \mid g_x(\dot{x}, \dot{x}) = 1,\ \dot{x}\text{- future pointing}\}$$

✓ **Assumption:** $\varphi\,(x, \cdot) : \mathcal{O}_x \to \mathbb{R}$ has *compact support*, $\forall x \in M$.

✓ **Number N_σ of particle trajectories** $(\gamma, \dot{\gamma})$ crossing a hypersurface $\sigma \subset \mathcal{O}$:

$$N_\sigma = \int_\sigma \varphi d\Omega. \tag{43}$$

✓ Gravitational field - from **Einstein-Vlasov equations**:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}, \qquad T^{\mu\nu}(x) := \int_{\mathcal{O}_x} m\varphi \dot{x}^\mu \dot{x}^\nu d\boldsymbol{\Sigma}.$$

✓ For **collisionless gases** $\Rightarrow$ **Liouville equation:**

$$\ell(\varphi) = 0.$$

Our approach, [2]:

★ **Idea: Couple φ directly to gravity (no $\dot{x}$-averaging!)**
$\longrightarrow$ **this is possible in Finsler geometry.**

∘ Rewrite $\varphi : \mathcal{O} \longrightarrow \mathbb{R}$ as a function on $J^4 Y_g$:

$$\varphi^+ : J^4 Y_g \longrightarrow \mathbb{R}, \ \varphi^+(J^4_{[(x,\dot{x})]}\gamma) := \varphi(x, \dot{x}).$$

∘ Construct **matter action** on $J^4 Y_g$ as:

$$S_{m,D} := -mN\tau = -m \int_D \varphi d\mathbf{\Sigma} = -m \int_{\pi^+(D)} (\varphi^+ \circ J^4\gamma) d\mathbf{\Sigma}^+,$$

with: $D \subset \mathcal{O}$ (piece).

∘ **Matter Lagrangian**: $\lambda_m^+ := -m\varphi^+ d\mathbf{\Sigma}^+$ - generally covariant.

Total Lagrangian: $\lambda^+ = \frac{1}{2\kappa^2}\lambda_g^+ + \lambda_m^+$.

Theorem 68: The Euler-Lagrange equation attached to λ^+ is:

$$\frac{1}{2}g^{ij}(LR_0)_{\cdot i \cdot j} - 3R_0 - g^{ij}(P_{i|j} - P_i P_j + (\nabla P_i)_{\cdot j}) = \kappa^2 m\varphi\,. \tag{44}$$

Energy-momentum distribution tensor comps.:

$$\Theta^i{}_j = m\varphi^+ \hat{L}^{-1}\dot{x}^i\dot{x}_j. \tag{45}$$

Energy-momentum density on M - components (supp$(\varphi(x,\cdot))$ - compact!):

$$\mathcal{T}^i_j(x) := m\int\limits_{\mathcal{O}_x^+}(\varphi^+ l^i l_j)\circ J^6\gamma\; d\boldsymbol{\Sigma}_x^+ = \; m\int\limits_{\mathcal{O}_x}\varphi l^i l_j d\boldsymbol{\Sigma}_x \tag{46}$$

- formally similar to pseudo-Riemannian (GR) approach.

Averaged e.-m. conservation law (38) becomes:

$$\int_{\mathcal{O}_x} \ell(\varphi) l_j d\boldsymbol{\Sigma}_x = 0. \tag{47}$$

Particular cases:

1. Collisionless gases:

- *Pointwise covariant conservation law* of Θ = *Liouville equation:*

$$\ell(\varphi) = 0.$$

2. **Lorentzian spaces** (M, a):

- Averaged energy-momentum conservation law (38) $\Leftrightarrow$

$$T^i{}_{j;i} = 0.$$

3.4. Cosmologically symmetric Finsler spacetimes

Reference:

[1]: M. Hohmann, C. Pfeifer, N. Voicu, *Cosmological Finsler spacetimes*, Universe 6 (5), 65 (2020).

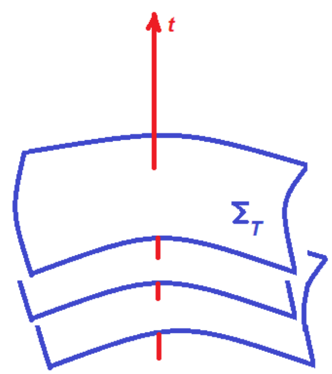
Main results:

1. Use the Copernic principle to identify the Lie algebra of **generators of cosmological symmetry** (& **general form of Finsler functions with cosmological symmetry.**)

2. For cosmologically symmetric **Berwald spacetime functions** $\longrightarrow$ **complete classification.**

Cosmological (Copernic) principle:
At largest scales, the Universe is: **homogeneous ("same at all points"):** *and* **isotropic ("same in each direction"):**

Consider: (M, L) - Finsler spacetime.
Global time function = a smooth $t : M \to \mathbb{R}$ such that
- ◇ $dt(X) > 0, \ \forall X \in \bar{\mathcal{T}}$ and
- ◇ the **spatial slices** $\boldsymbol{\Sigma}_T := \{p \in M | t(p) = T = constant\}$ are connected.



Definition, [1]: (M, L) - **cosmological Finsler spacetime** if:

1. It admits a global time function $t : M \to \mathbb{R}$ and
2. All spatial slices $\mathbf{\Sigma}_T$ obey:

(i) $\mathbf{\Sigma}_T$ - **homogeneous:** $\exists$ a Lie group G of *isometries* of (M, L) acting transitively on each slice $\mathbf{\Sigma}_T$:

$$\forall T \in \mathbb{R}, \forall q_1, q_2 \in \mathbf{\Sigma}_T \exists \varphi \in G : \varphi(q_1) = \varphi(q_2)$$

(ii) $\mathbf{\Sigma}_T$ - **isotropic:** at all $p \in \mathbf{\Sigma}_T$: the *isotropy group* at p :

$$G_p := \{\psi \in G | \psi(p) = p\}$$

acts transitively on the projective space $PT_p\mathbf{\Sigma}_T$:

$$\forall \left[v_1\right], \left[v_2\right] \in PT_p\mathbf{\Sigma}_T : \ \exists \varphi_p \in G_p : d\varphi_p \left(\left[v_1\right]\right) = \left[v_2\right].$$

Theorem 72, [1]: For a cosmologically symmetric Finsler spacetime (M, L) :

$$\dim G = 6, \quad \dim G_p = 3.$$

Proposition 73, [1]: The identity component of G_p is isomorphic to $SO(3)$.

Remark *(Kobayashi&Nomizu, 1963)* **:**
Σ_T - homogeneous, $G_p \simeq SO(3) \overset{!}{\Rightarrow} \Sigma_T$ admits a G-invariant *Riemannian* metric h.

Consequences:

1. h - maximally symmetric $\Rightarrow h$ has constant sectional curvature κ;

2. L and h have the **same Killing vector fields** $X_{(k)}$, $k = 1, ..., 6$.

3. One can use *spherical coords.* (t, r, φ, θ) given by h.

Theorem *(≡ Hohmann&Pfeifer 2016):* If (M, L) - cosmologically symmetric Finsler spacetime, then:

$$L = L(t, \dot{t}, w), \qquad w^2 = \frac{\dot{r}^2}{1 - kr^2} + r^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right).$$

Theorem 75, [1] (Classification of cosmologically symmetric Berwald spacetime functions): If (M, L) - cosmological Berwald spacetime, then L falls into one of the following classes:

a) *pseudo-Riemannian spaces:* $L(x, \dot{x}) = a_{ij}(x) \dot{x}^i \dot{x}^j$:

b) *nontrivially Finslerian:* of the form:

$$L(t, \dot{t}, w) = \dot{t}^2 B^2(t) \Phi \left(\frac{w}{\dot{t} B(t)} \right).$$

where B, Φ - arbitrary real functions and $\kappa \in \{0, \pm 1\}$.

4 Outlook and perspectives

I. A geometric toolkit for the calculus of variations:

1. *Geometric formulation of higher order Hamiltonian field theory* (Hamilton-de Donder equations), based on canonical Lepage equivalents.

2.*Energy-momentum tensors:*
- Extend (if possible) the definition of energy-momentum tensors in Ch.1 to the case when the differential index of $Y^{(b)}$ is greater than 1 (e.g., in purely affine theories).
- Obtaining a general construction of a conserved gravitational *energy-momentum pseudotensor*, in general field theories.

3. *Extending the Vainberg-Tonti Lagrangian construction* - e.g., using other groups of fiber automorphisms.

II. Finsler spacetimes:

Classes of Finsler spacetimes which are relevant for solving the Finsler gravity field equation (44):

- Spacetimes with (α, β)-metric.
- Spacetimes with $\dot{x}$- compactly supported Ricci scalar $R(x, \cdot)$; in particular, Ricci-flat ones $R = 0$.
- Compactly supported deviations from Lorentzian metrics a.
- Weakly Landsberg spacetimes. Weak unicorns.
- Berwald spacetime functions with special properties (e.g., spherical symmetry, $T\overset{\circ}{M}$-smoothness etc.).

III. Finslerian field theory:

1. *Solutions of the Finslerian field equation*:
- Vacuum spatially spherically symmetric solutions.
- Cosmologically symmetric solutions of the (non-vacuum) field equation (44).
- Linearized Finslerian perturbations of Lorentzian metrics.

2. *Comparison of Finslerian equation with the Einstein-Vlasov equations.*
Focus on: cosmologically symmetric case $\stackrel{?}{\Rightarrow}$ *dark energy*.

3. Build models for: *electromagnetic field, ultrarelativistic gas.*

4. *Finsler geometry as the geometry of modified dispersion relations:*
Cotangent bundle formulation of Finsler field theory framework, geometry of curved momentum spaces.

Thank you !