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Studies on positive linear operators

SUMMARY

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Contents

1 Introduction

1.1 Thesis field

The original results which are part of this thesis are concerned with research in the mathematical domain approximation theory. This field is a topic of interest in mathematical research since a lot of domains from mathematics are related to it. For example, in real and complex analysis, the general theory of sequences and series, asymptotic expansion, moduli of smoothness, K-functionals and convexity are fundamental in the study of approximation process, also in approximation theory some aspects from functional analysis and operator theory are present (we mention the abstract theory of positive linear operators together with Korovkin's theorem, C_0-semigroups of operators, etc.). Approximation theory is related with the theory of probabilities through Feller's general theory and with the theory of differential equations through special properties of some classes of operators. Apart from this, approximation theory deals with the possibility of reducing general mathematical objects (such as functions) to simpler classes of objects (such as polynomials). Throughout research in mathematics this approach is fundamental thus making approximation theory a subject of interest with great applicability.

A significant moment in the development of approximation theory as a distinguished research field in the framework of mathematical analysis has its beginning with the famous theorems of Chebyshev's best approximation and of K. Weierstrass who, in nineteenth century proved the approximation of continuous functions on a compact set by polynomials. The theorem proposed by Weierstrass was also proved by S. N. Bernstein who introduced the famous operators which carry his name (these operators were later modified by L. V. Kantorovich and J. L. Durrmeyer to approximate integrable functions as well). Later the basis of approximation theory as a research topic in mathematics was further consolidated by results due to Popoviciu, Bohman and Korovkin through which continuous functions on compact sets can be approximated by positive and linear operators, namely, they found that any positive linear operator which satisfies some conditions can be used instead of Bernstein's operator.

Nowadays, approximation theory is concerned with methods through which positive linear operators are obtained, for which Korovkin's theorem can be used to check whether they approximate functions, estimations of the degree of approximation by the said operators which can be obtained in the form of quantitative estimates (namely, results in terms of moduli of smoothness whose purpose is to measure the smoothness of functions) and Voronovskaya theorem (which will be later disscused). The fundamental results in this directions are due to G. G. Lorenz, R. DeVore, F. Altomare, Z. Ditzian, V. Totik, H. Gonska, P. L. Butzer, U. Abel and many others. For a comprehensive presentation of these results the books of Lorentz and DeVore ([37]), DeVore ([36]), Altomare ([15]) and Ditzian and Totik ([39]) can be consulted.

1.2 Motivations for the choice of the theme

As we have mentioned before approximation theory is concerned with the approximation of difficult processes by much simpler ones which can be easily studied where their properties are similar to the properties of the processes considered. Such a situation can be seen in practice for example in computer science where a calculator has available only the operations of addition and multiplication, therefore, estimations of irrational numbers can be obtained by the calculator only if it uses approximations which imply these two operations, i. e. the calculator can do this by using approximations by polynomials (e.g. Taylor's polynomial, however here difficulties arise since not all functions are smooth, many being continuous, but here positive and linear operators such as Bernstein operators and others which will be studied further in the thesis can be used).

Other than this, approximation theory can be useful in the theory of differential equation as one can determine the solutions of a Cauchy problem if one has theorems which give conditions under which the C_0-semigroup associated with the Cauchy problem can be generated. However, here a drawback exists since these theorems do not give an explicit form of the C_0-semigroup. Approximation theory solves this problem by providing theorems which generate the C_0-semigroup and also give an approximation of it by iterates of positive linear operators who generate the semigroup, hence allowing the properties of the C_0-semigroup to be studied by analyzing the properties of the operator which approximates it.

Other motivations for choosing this field of research are represented by the connection between approximation theory and functional analysis (in the sense that many concepts from functional analysis can be used in approximation theory such as: Banach spaces, uniform boundedness principle, etc.) and more recently there is a connection between the development of Artificial Intelligence and approximation theory as neural networks can be seen as approximation operators.

1.3 Structure of the thesis

This thesis is divided in seven chapters. In the second chapter, with the title *Preliminaries*, we present the notations used throughout the thesis and results from literature, obtained by other researchers in this field, which were the most relevant to developing the original results contained in this thesis, such as operators used in approximation theory and theorems regarding them, moduli of smoothness, Voronovskaya theorems, C_0-smemigroups and a short summary of the results on geometric series of positive and linear operators. These preliminary results are all part of the References and they are cited whenever they are mentioned throughout the thesis.

The third chapter, *Generalized Voronovskaya theorem and the convergence of power series of positive linear operators*, is dedicated to obtaining new approximation operators which are constructed as more general power series of positive linear operators (so it comes as a generalization of the existing results regarding geometric series of positive linear operators from [1, 4, 80] and others mentioned throughout the thesis). Also here we obtained a generalization of Voronvsakya theorem by giving an explicit form of the limit used in such theorems. This results are part of the original article *Generalized*

Voronovskaya theorem and the convergence of power series of positive linear operators, J. Math. Anal. Appl., 531 (2024), Issue 2, Part 2.

In the fourth chapter, *The representation of the limit of power series of positive linear operators by using operators semigroup generated by their iterates*, we obtained a characterization of a general power series of positive linear operators by using the C_0-semigroup generated by the iterates of positive linear operators belonging to a certain class. This result can be found in the original article: *The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterates,* Dolomites Research Notes on Approximation (2023), 16(3), 39-47.

In the fifth chapter, *A Voronovskaya type theorem associated to geometric series of Bernstein - Durrmeyer operators*, we obtained a Voronovskaya theorem for the operators obtained by U. Abel in [1] which are the geometric series associated to Bernstein-Durrmeyer operator. Here the main difficulties arose from the space of functions on which this operator was studied which is unusual in approximation thoery. These results are part of the original article: *A Voronovskaya type theorem associated to geometric series of Bernstein-Durrmeyer operators*, Carpathian Journal of Mathematics(2025), 41(2).

In the sixth and seventh chapter we obtained exponential variants of Kantorovich Stancu operators and of Bernstein-Durrmeyer operators following the construction given in [18]. Regarding these operators we obtained approximation results via Korovkin theorems and by studying the norm on a weighted version of L^p spaces of these operators, then classical asymptotic results and quantitative estimations are obtained. Also quantitative results using the relation between K-functionals and the moduli of smoothness (given in [62]) are obtained. These chapters are part of the original articles: *Exponential Bernstein-Durrmeyer operators*, General Mathematics(2024), Volume 32, no. 2, 84-97 and *Exponential Kantorovich-Stancu operators,* Bull. Univ. Transilvania Brasov, Ser. 3, Math. Comput. Sci., 5(67), 2025, no. 2, 127-144.

1.4 Originals results contained in the thesis

During my doctoral studies the research I did is comprised in the following original articles:

1. *Generalized Voronovskaya theorem and the convergence of power series of positive linear operators*

   Here our aim was first to obtain a generalized version of Voronovskaya's theorem in the form of the limit of $n^s(L_n - I)^s f,\ s \in \mathbb{N}$, when L_n are certain positive linear operators. Equivalently, this is an explicit form of Voronovskaya theorem for Micchelli combinations of operators. Then, we apply this result in order to obtain the limit of certain power series of positive linear operators.

2. *The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterates*

   In this paper our aim was to give a characterization of the limit of power series of the from $\beta_n \sum_{k=0}^{\infty}(L_n)^k,\ n \in \mathbb{N}$, where $\beta_n \in \mathbb{R}$ by using the C_0-semigroup generated by the iterates of the positive and linear operators $(L_n)_n,\ n \in \mathbb{N}$ which belong to a certain class. This result was obtained by using the eigenstructure of both the operators and the C_0-semigroup.

3. *A Voronovskaya type theorem associated to geometric series of Bernstein - Durrmeyer operators*

   In this article we obtained an asymptotic result regarding the convergence of operators $P_n = \frac{1}{n}\sum_{n=0}^{\infty} M_n$, $n \in \mathbb{N}$, where M_n are the well known Bernstein-Durrmeyer operators and P_n are the geometric series associated to this operators. This result was rather difficult to obtain since the space we worked, a subspace of L^∞, generated a lot of existence conditions with a lot of computations. We mention that this space is rarely encountered in approximation theory.

4. *Exponential Kantorovich-Stancu operators*

   Here we introduced an exponential variant of Bernstein-Kantorovich operators modified in Stancu sense. Concerning these operators we prove they verify Korovkin's theorem conditions and also that they approximate functions from a weighted L^p space. Moreover, we will obtain a Voronovskaya type theorem and some quantitative estimates of the approximation using the first order modulus of continuity. Also, we will prove some estimates concerning the approximation of functions from a weighted L^p space using Peetre's K-functional. Finally, we will obtain an estimate which involves the first order modulus of continuity and the second order modulus of smoothness by using the equivalence relation between these moduli and the corresponding K-functionals.

5. *Exponential Bernstein-Durrmeyer operators*

   In this article we introduced an exponential variant of Bernstein-Durrmeyer operators. Regarding these new operators we obtain some convergence results, a Voronovskaya type theorem and some quantitative estimates using the first order modulus of continuity and the second order modulus of smoothness and then the relation between them and K-functionals. Also we study their simultaneous approximation properties.

1.5 Dissemination of the results

The results enumerated in the previous section were published in various mathematical journals and some of them were presented in the framework of international conferences on approximation theory.

1. During the 2022 edition of "Functional Analysis, Approximation Theory and Numerical Analysis", held in Matera, Italy, 5-8 July, 2022 the talk: *On the convergence of series of powers of positive linear operators* was held.

   Also, in the proceedings of this conference we published the second article:

   Ș. Garoiu, R. Paltanea, *The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterate,* Dolomites Research Notes on Approximation (2023), 16(3), 39-47.

This article was co-authored with my doctoral advisor Prof. Dr. Radu Păltănea.

2. During the fourteenth edition of "International Conference on Approximation Theory and Applications", held in Sibiu, Romania, 12-14 September 2022, I delivered the talk: *Voronovskaya type results for geometric series of Durrmeyer operators*, related to third article from the previous section.

3. During the fourth edition of the International Conference on Mathematics and Computer Science, held in Brașov, Romania, 15-17 September, 2022, I delivered the talk: *On the convergence of power series of positive linear operators*

   Also, I published the following article:

   Ş. Garoiu, R. Păltănea, *Generalized Voronovskaya theorem and the convergence of power series of positive linear operators*, J. Math. Anal. Appl., 531 (2024), Issue 2, Part 2.

4. During the fifth edition of the International Conference on Mathematics and Computer Science, held in Brașov, Romania, 13-15 June, 2024, I delivered the talk: *A Voronovskaya type theorem associated to geometric series of Bernstein - Durrmeyer operator*

   Also, I published the article:

   Ș. Garoiu, *A Voronovskaya type theorem associated to geometric series of Bernstein-Durrmeyer operators*, Carpathian Journal of Mathematics(2025), 41(2),

5. Also, I published the following papers:

   Ș. Garoiu, *Exponential Bernstein-Durrmeyer operators*, General Mathematics(2024), Volume 32, no. 2, 84-97,

   Ș. Garoiu, *Exponential Kantorovich-Stancu operators,* Bull. Univ. Transilvania Brasov, Ser. 3, Math. Comput. Sci., 5(67), 2025, no. 2, 127-144.

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2 Preliminaries

2.1 Some notations and basic results

First, we will proceed with specifying some of the notations which are common through the entire thesis.

We denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By $C([a,b])$ we mean the Banach space of real continuous functions $f : [a,b] \to \mathbb{R}$, endowed with the usual sup-norm $\|f\| = \max_{x\in[a,b]} |f(x)|$. Also, by $C^k([a,b])$, $k \in \mathbb{N}$, we mean the space of real continuous functions which admit a continuous k^{th} derivative and by $B([a,b])$ we denote the space of bounded functions on $[a,b]$.

Next, by $L^p([0,1])$, $1 \le p \le \infty$ we mean the space of p-integrable functions, endowed with the norm $\|f\|_p = \left\{\int_0^1 |f(x)dx|^p\right\}^{\frac{1}{p}}$. We say that $f \in L^p([0,1])$ if $\|f\|_p < \infty$.

Let Π be the space of polynomials and for $j \in \mathbb{N}_0$, let Π_j be the space of polynomials of degree at most j. The monomial functions are given by $e_j(x) = x^j$, $j \in \mathbb{N}_0$, $x \in [0,1]$. From these sets of polynomials we will need throughout the thesis the function:

$$\psi(x) = x(1-x), \ x \in [0,1]. \tag{2.1}$$

Next, we will present some of the main results from current literature in Approximation Theory which are in the same topic as the thesis.

Definition 2.1.1. *Let X be a space of functions. By positive and linear operators we mean operators $L : X \to X$ which satisfy:*

1. *$Lf > 0$ for $f \in X$ and $f > 0$,*
2. *$L(\alpha f + \beta g) = \alpha(Lf) + \beta(Lg)$, for all $f,\ g \in X$ and $\alpha,\ \beta \in \mathbb{R}$.*

If L is a positive and linear operator we will denote by $L^k = \underbrace{L \circ \cdots \circ L}_{k\text{-times}}$, $k \in \mathbb{N}_0$,with $L_0 = I$ where I is the identity operator, the k-times iterates of L.

Next, we will make the following notation for the moments and for the absolute moments of operators L:

$$m_n^j(x) = (L(e_1 - xe_0)^j)(x), \tag{2.2}$$
$$M_n^j(x) = (L|e_1 - xe_0|^j)(x). \tag{2.3}$$

Also we denote by D^jL, $j = 0, 1, 2 \dots$, the j^{th}, order derivative of the operator L.

It is well known that positive and linear operators play an important role in Approximation Theory since S. N. Bernstein proved that continuous functions on compact intervals can be approximated by polynomials (Weierstrass Theorem) using such operators. Another reason for which positive linear operators are intensively studied is the famous Korovkin Theorem (see [66], [67]).

Theorem 2.1.2 (Korovkin). *Let Y be a linear subspace of X and let $L_n : Y \to X$ be a sequence of positive linear operators. If functions $\varphi_0, \varphi_1, \varphi_2 \in Y \cap C([0,1])$ form a Chebyshev system on $[0,1]$ and if*

$$\lim_{n\to\infty} L_n \varphi_i = \varphi_i, \text{ uniformly for } i = 0, 1, 2, \tag{2.4}$$

then

$$\lim_{n\to\infty} L_n f = f, \text{ uniformly for any } f \in Y \cap C([0,1]). \tag{2.5}$$

Here, by a Chebyshev system of order $l+1$ we mean a set of functions $\varphi_0, \dots, \varphi_l \in C([0,1])$ for which their linear combination $\varphi = \sum_{j=0}^{p} a_j \varphi_j$, where $p \le l$ and $a_0, \dots, a_p \in \mathbb{R}$, has at most l roots on $[0,1]$. For more results on Korovkin approximation we refer the reader to the following papers: [13], [14], [23] and [83].

In literature, there exist a lot of particular positive linear operators which verify Korovkin's Theorem. We will mention only those who will appear in this thesis. A first example is the operator obtained by Bernstein (see [28]) when he proved Weierstrass's Theorem ([93]) regarding approximation on compact sets, namely the operators B_n which are named after him:

$$\left(B_n f\right)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \; n \in \mathbb{N}, \; x \in [0,1], \; f \in C([0,1]), \tag{2.6}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } 0 \le k \le n, \tag{2.7}$$

and $p_{n,k}(x) = 0$ for $k > n$. These operators were intensively studied, see: [30], [68], [38], etc. Also, there are present a lot of generalizations of these operators. We mention Stancu operators (see [89]) $B_n^{\alpha,\beta}$, obtained for functions $f \in C([0,1])$

$$\left(B_n^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \; n \in \mathbb{N}, \; x \in [0,1], \; f \in C([0,1]), \tag{2.8}$$

where $0 < \alpha < \beta$. Further, operators B_n can be modified to approximate functions $f \in L^1([0,1])$, thus obtaining Kantorovich operators ([64]):

$$(\mathcal{K}_n f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \; x \in [0,1], \; f \in L^1([0,1]). \tag{2.9}$$

Here, since they are relevant to our thesis we will mention the exponential variant of operators $\mathcal{K}_n$ introduced by Angeloni and Costarelli in [18]:

$$(K_n f)(x) = \sum_{k=0}^{n} e^{\mu x} p_{n,k}(a_{n+1}(x))(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) e^{-\mu t} dt, \tag{2.10}$$

where $\mu > 0$, $f \in C([0,1])$ $x \in [0,1]$, $n \in \mathbb{N}$, and $a_n(x) := \frac{e^{\frac{\mu x}{n}} - 1}{e^{\frac{\mu}{n}} - 1}$ are increasing, continuous and convex functions on $[0,1]$ such that $a_n(0) = 0$ and $a_n(1) = 1$.

2.2 Moduli of continuity and smoothness

2.3 Voronovskaya theorems

2.4 C_0-semigroups of operators and approximation of C_0-semigroups

This section is dedicated to the study of C_0-semigroups and how they can be approximated by iterates of approximation processes. Here, we recall some of the works done in [15] and others.

2.4.1 C_0-semigroups

Let K be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. We denote by $(E, \|\cdot\|)$ be a Banach space and by $\mathcal{L}(E)$ the space of bounded linear operators defined on E. If one endows $\mathcal{L}(E)$ with the supremum norm:

$$\|B\| := \sup_{\substack{f\in E \\ \|f\|\leq 1}} \|B(f)\|, \; B \in \mathcal{L}(E),$$

then $(\mathcal{L}(E), \|\cdot\|)$ is a Banach space as well.

Definition 2.4.1. *Let $(T(t))_{t\geq 0} \in \mathcal{L}(E)$. The family $(T(t))_{t\geq 0}$ is a semigroup of bounded operators in E if:*

1. *$T(0) = I$, where I is the identity operator on E,*
2. *$T(t+s) = T(t)T(s)$ for every $s, t \geq 0$, and $T(t)T(s) = T(t)\circ T(s)$.*

A semigroup $(T(t))_{t\geq 0}$ is a C_0-semigroup (strongly continuous semigroup) if, for every $t_0 \geq 0$ and for a function f from E, the following limit holds on $(E, \|\cdot\|)$:

$$\lim_{t\to t_0^+} T(t)(f) = T(t_0)(f).$$

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Banach space $(E, \|\cdot\|)$ and let $(A, D(A))$ be a linear operator on E, where:

$$A(f) := \lim_{t\to 0^+} \frac{T(t)(f) - f}{t}, \text{ for every } f \in E. \tag{2.11}$$

and $D(A)$ is the domain of the operator A and is given by

$$D(A) := \left\{ f \in E \mid \exists \lim_{t\to 0^+} \frac{T(t)(f) - f}{t} \in E \right\}. \tag{2.12}$$

Then $(A, D(A))$ is called the generator of the C_0-semigroup $(T(t))_{t\geq 0}$.

2.4.2 Approximation of C_0-semigroups

In the book by Altomare and his co-authors ([15]) there is an extensive review regarding some theorems which give necessary and sufficient conditions under which a linear operator $(A, D(A))$ on a Banach space $(E, \|\cdot\|)$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. The disadvantage of these generation theorems is the fact that they don't give an explicit form of the C_0-semigroup, hence not giving any information about the semigroup. Therefore, in approximation theory there are results which not only give conditions under which a suitable operator is the generator of a C_0-semigroup, but also, they provide means to approximate the semigroup by using the iterates of linear operators, hence allowing one to study the properties of the semigroups by studying the properties of the operator which approximates them.

To this purpose, a first result is given by Trotter (see [91])

Theorem 2.4.2 (Trotter)**.** *On the Banach space $(E, \|\cdot\|)$ let $(L_n)_{n\geq 1}$ be a sequence of bounded linear operator and let $(\rho(n))_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \rho(n) = 0$. Assume there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|L_n^k\| \leq M e^{\omega \rho(n) k},\ \forall\ k, n \geq 1, \tag{2.13}$$

where L_n^k is the k^{th} iterate of L_n. Let $(A, D(A))$ be a linear operator on the Banach space $(E, \|\cdot\|)$ given by:

$$A(f) := \lim_{n\to\infty} \frac{L_n f - f}{\rho(n)},\ f \in D(A), \tag{2.14}$$

with domain:

$$D(A) := \left\{ g \in E \mid \exists \lim_{n\to\infty} \frac{L_n g - g}{\rho(n)} \right\}. \tag{2.15}$$

If the following assumptions hold:

(i) $D(A)$ is a dense subset of E;

(ii) $(\lambda I - A)(D(A))$ is a dense subset of E for some $\lambda > \omega$;

then the closure of $(A, D(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, with the property, that for each $t \geq 0$ and for each sequence of positive integers $(k(n))_{n\geq 1}$ with $\lim_{n\to\infty} k(n)\rho(n) = t$ the following limit holds:

$$\lim_{n\to\infty} L_n^{k(n)} f = T(t)(f),\ f \in E. \tag{2.16}$$

Moreover, for every $t \geq 0$, $\|T(t)\| \leq M e^{\omega t}$.

2.5 Geometric series of positive linear operators

Since one of the main concerns of this thesis is the study of general power series of positive linear operators we will recall some of the existent literature in this direction, namely we will give a brief presentation of the results from [3], [4], [78] and [80] which are relevant to the main content of the thesis.

One of the first studies considering the geometric series of some positive and linear operators is due to Păltănea, see [78]. Namely, he studied the properties of geometric series associated to Bernstein operators B_n.

There the author introduced the operators

$$A_n = \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k, \tag{2.17}$$

where B_n^k are the k-times iterates of B_n. Concerning the iterates B_n^k (studied in [2, 5, 31, 32, 55, 65, 74, 75, 85]) they can always be defined, however, there are cases when operators A_n aren't well defined, for example if one considers the eigenvalues of B_n (given in [33]). Therefore a careful selection of the space which can be the domain of definition of these operators is needed. In this sense, in paper [78] the author proved that operators A_n from (2.17) are well defined from the space:

$$\psi C([0,1]) := \{f | \exists g \in C([0,1]),\ f = \psi g\}, \tag{2.18}$$

onto itself, where $\psi(x) = x(1-x),\ x \in [0,1]$. Moreover, space $\psi C([0,1])$ can be endowed with the norm:

$$\|f\|_\psi = \|g\|,\ f \in \psi C([0,1]), \tag{2.19}$$

and $(\psi C([0,1]), \|\cdot\|)_\psi$ is a Banach space (note that convergence with respect to norm $\|\cdot\|_\psi$ implies convergence with respect the usual sup-norm $\|\cdot\|$).

For any $f \in B([0,1]) \cap C((0,1))$ and $x \in [0,1]$, in [4], the following operator was defined:

$$F(f)(x) := (1-x)\int_0^x t f(t)\,dt + x\int_0^x (1-t)\,f(t)dt, \tag{2.20}$$

and it was proven that $F(f) \in \psi C([0,1]) \cap C^2(0,1)$ and:

$$(F(f)(x))'' = -f(x),\ f \in B([0,1]) \cap C((0,1)),\ x \in [0,1]. \tag{2.21}$$

In paper [78], R. Păltănea proved that the following limit holds:

$$\lim_{n\to\infty} ||A_n(\psi f) - 2F(f)||_\psi = 0,\ f \in \psi C([0,1]), \tag{2.22}$$

however, here a slightly modified version of $F(f)$ was considered in the sense that $f \in C([0,1])$ instead of $f \in B([0,1]) \cap C((0,1))$. Later, in [3] the convergence of operators A_n was studied on a more general subspace of $C([0,1])$, namely, on the space

$$C_0([0,1]) := \{f \in C([0,1])|\ f(0) = 0,\ f(1) = 0\}. \tag{2.23}$$

and a similar result to the one in (2.22) was obtained using the eigenvalues of B_n.

A generalization of the operators A_n was introduced by Abel et al. ([4]), namely, in formula (6.1) operators B_n were replaced by positive linear operators L_n belonging to a general class of operators. If we denote by G_{L_n} the geometric series attached to these operators L_n, then the following result was proved: $\lim\limits_{n\to\infty} ||G_{L_n}(f) - 2G(f/\psi)||_\psi = 0$, which holds for functions f belonging to the space $C_\psi\left[0,1\right]=\{f:C([0,1])\to C([0,1]):\exists\, g\in B[0,1]\cap C\left(0,1\right),\ f=\psi g\}$ which together with the norm $||\cdot||_\psi$ is a Banach space. The operators A_n were also studied on the space $C_0[0,1]=\{f\in C([0,1]):\ f(0)=f(1)=0\}$ endowed with the usual sup-norm, in paper [80]. There a Voronovskaya theorem was obtained. Further studies on approximation operators defined as geometric series of positive and linear operators were done in [4]. There the authors considered the operator

$$G_L=\sum_{k=0}^{\infty}L^k, \tag{2.24}$$

which is the geometric series associated to a positive linear operator $L:X\to X$, where X is a linear subspace of $C([0,1])$ and L^k are the k-times iterates of L, for $k\geq 1$ and with $L^0=I$, I being the identity operator. As before the iterates of L can always be defined (for more studies regarding the iterates of positive linear operators see [48, 49, 51, 63, 94]), however for operator G_L to be well defined a suitable choice of the domain of definition must be made. The domain, considered by authors in [4] for this purpose, was the space:

$$C_\psi([0,1]):=\{f:[0,1]\to\mathbb{R}|\exists g\in B([0,1])\cap C((0,1)):f=\psi g\}, \tag{2.25}$$

endowed with the norm:

$$\|f\|_\psi:=\sup_{x\in(0,1)}\frac{|f(x)|}{\psi(x)},\ \ f\in C_\psi([0,1]). \tag{2.26}$$

Note that one can also write

$$C_\psi([0,1])=\{f\in C([0,1])|\ \exists M>0:|f(x)|\leq M\psi(x),x\in[0,1]\}. \tag{2.27}$$

Note that $(C_\psi([0,1]),\|\cdot\|_\psi)$ is a Banach space and convergence with respect to norm $\|\cdot\|_\psi$ implies convergence with respect to the usual sup-norm $\|\cdot\|$. Also $C_\psi([0,1])$ is an extension of the space $\psi C([0,1])$.

Next, Abel et al. proved in [4] that if $L:C([0,1])\to C([0,1])$ are operators belonging to a certain class Λ, of linear and positive operators (see below the definition of this class of operators), then operator G_L is well defined from $C_\psi([0,1])$ onto itself.

One has $L\in\Lambda$ if L are positive and linear operators which satisfy the following conditions (see Definition 1 from [4]):

1. L preserve linear functions;
2. $\|L\|_\psi<1$;
3. $L\neq B_1$, where B_1 is the Bernstein operator for $n=1$.

Next, it was proved that if $(L_n)_{n\in\mathbb{N}}$ is a sequence of positive and linear operators such that $L_n\in\Lambda$, $n\in\mathbb{N}$ then for any $f\in B([0,1])\cap C((0,1))$ the following limit holds under some conditions (see Theorem 2, [4]):

$$\lim_{n\to\infty}\|\alpha_n G_n(\psi f)-2F(f)\|_\psi=0, \tag{2.28}$$

where $G_n = G_{L_n}$, α_n is a normalization factor (given in relation (11) from [4]) and $F(f)$ is expressed as in (2.20), $F(f) \in \psi C([0,1]) \cap C^2((0,1))$.

Later, in [80], it was proved that operator G_L is well defined on the space $C_0([0,1])$, given in (2.23). Moreover, on this space a convergence result (but with regard to the norm on the space $C_0([0,1])$) similar to the one in (2.28) holds and a Voronovskaya theorem was obtained.

Remark that between the three spaces considered as the domain of definition of operators G_L the following relation is true with respect to the sup-norm $\|\cdot\|$:

$$\overline{\psi C([0,1])} = \overline{C_\psi([0,1])} = C_0([0,1]), \tag{2.29}$$

where by $\overline{A}$ is meant the closure of a set A.

3 Generalized Voronovskaya theorem and the convergence of power series of positive linear operators

In this chapter we will obtain more general power series of positive linear operators than those introduced by Abel, Ivan and Păltănea in paper [4], which as we mentioned in the previous chapter is a geometric series of positive linear operators. In this sense we obtained a generalized Voronovskaya theorem and some convergence results regarding the power series operator mentioned above. These results were all published in paper [46]: **Ş. Garoiu**, R. Păltănea, *Generalized Voronovskaya theorem and the convergence of power series of positive linear operators*, J. Math. Anal. Appl., 531 (2024), Issue 2, Part 2.

Let $B_n : C([0,1]) \to C([0,1])$ be the Bernstein operators:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ n \in \mathbb{N},\ x \in [0,1],\ f \in C([0,1]),$$

with polynomials $p_{n,k}$ defined as:

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \le k \le n,\ x \in [0,1].$$

The Voronovskaya theorem in supremum norm version says that

$$\lim_{n\to\infty} \left\| n(B_n f - f) - \frac{1}{2}\psi f'' \right\| = 0,\ \text{for } f \in C^2([0,1]). \tag{3.1}$$

There exists a vast literature regarding Voronovskaya type results for various operators, from which we mention only paper [52], in which the limit is given in a stronger form, using a weighted norm.

One of the objectives of this chapter is to give a generalization of Voronovskaya theorem by giving an explicit form of the limit $\lim_{n\to\infty} n^s (L_n - I)^s$, $s \in \mathbb{N}$, where operators L_n belong to a general class of positive linear operators. This result is equivalent with the explicit Voronovskaya theorem for Micchelli combinations of operators L_n, given by $I - (I - L_n)^s$, defined in [71]. In the case of Bernstein operators a partial representation of the limit was given by Agrawal [6].

On the other hand, this result will play an essential role in the study of the limit of power series of operators. A particular case of the power series is given by the geometric series, which was the only one considered until now. The geometric series of a sequence of operators L_n, $n = 0, 1 \ldots$ is given by $\sum_{k=0}^{\infty} (L_n)^k$. As seen in Section 2.5 this geometric series is not defined for all functions of $C([0,1])$, as for instance, for the eigenfunctions of L_n. Therefore it is convenient to select some adequate subspaces of $C([0,1])$. From

Section 2.5 a first space that can be taken in consideration is the subspace $\psi C([0,1])$ defined in (2.18), endowed with the norm $\|\cdot\|_\psi$ given in (2.19).

Recall that in paper [78], it was proved that, operator $A_n = (1/n)\sum_{k=0}^{\infty}(B_n)^k$, is well defined from the space $\psi C([0,1])$ into itself and

$$\lim_{n\to\infty} \|A_n(f) - 2F(f/\psi)\|_\psi = 0,\ f \in \psi C([0,1]), \tag{3.2}$$

where $F(f)$ is given in (2.20).

A different proof of relation (3.2) based on the eigenstructure of operators B_n was given in [3].

In [4] an analogous result was obtained for operators A_n, constructed starting from more general operators than B_n and for the space $C_\psi([0,1])$. Other versions and generalizations of the geometric series of operators were given in [1], [4], [56], [84], [10] and [80].

We mention that in [4] an inverse Voronovakaya theorem was obtained. A previous inverse Voronovakaya theorem was obtained by other method in [16].

Another objective of this chapter is to consider certain more general power series of operators having the form

$$\sum_{k=0}^{\infty} \beta_{n,k}(L_n)^k,\ \beta_{n,k} \in \mathbb{R}, \tag{3.3}$$

where operators L_n are defined on the space $C_\psi([0,1])$ and then to study the convergence of these power series.

3.1 Generalized Voronovskaya theorem

For $j \in \mathbb{N}_0$, we denote

$$\sigma_j = \left[\frac{j+1}{2}\right], \tag{3.4}$$

where $[a]$ is the greatest integer less than or equal to the real number a. Next $C^\infty([0,1])$ denotes the space of functions $f : [0,1] \to \mathbb{R}$ which admits derivatives of any order on $[0,1]$.

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators. For $n \in \mathbb{N}$, $j \in \mathbb{N}_0$, $x \in [0,1]$, we consider

$$\begin{aligned} m_n^j(x) &= (L_n(e_1 - xe_0)^j)(x), \\ M_n^j(x) &= (L_n|e_1 - xe_0|^j)(x). \end{aligned}$$

In the sequel $(L_n)_{n\in\mathbb{N}}$ will be a sequence of positive linear operators $L_n : C([0,1]) \to C([0,1])$ which satisfy the following conditions, for $n \in \mathbb{N}$:

L1) $L_n(e_j) = e_j$, $j = 0, 1$;

L2) $m_n^j(x) = \psi(x)n^{-\sigma_j}Q_{n,j}(x)$, $j \geq 2$, $n \in \mathbb{N}$, $x \in [0,1]$, where $Q_{n,j} \in C^\infty([0,1])$ are such that for each $j, p \in \mathbb{N}_0$, there exists a constant $C_{j,p} > 0$ with the property that $Q_{n,j}^{(p)}(x) \leq C_{j,p}$ for all $x \in [0,1]$ and $n \in \mathbb{N}$.

L3) $m_n^2(x) = \alpha_n \psi(x)$, with $\alpha_n \in (0,1)$, for $n \geq n_0$, $n_0 \in \mathbb{N}$ and there is $\alpha > 0$ such that $\lim_{n\to\infty} n\alpha_n = \alpha$.

Lemma 3.1.1. *(see Lemma 1 from [46]) For any integer $j \geq 0$:*

$$M_n^j(x) = \mathrm{O}\left(\frac{\psi(x)}{n^{j/2}}\right), \text{ uniformly for } n \in \mathbb{N}, \text{ and } x \in [0,1]. \tag{3.5}$$

Further, the first order modulus of continuity of a bounded function $g : [0,1] \to \mathbb{R}$, is

$$\omega_1(g,h) = \sup\{|g(u) - g(v)|,\ u, v \in [0,1],\ |u-v| \leq h\},\ h > 0.$$

Lemma 3.1.2. *(see Lemma 2 from [46]) Let $k \geq 1$ and let $(g_n)_{n\in\mathbb{N}}$, $g_n \in C^k([0,1])$, ($n \in \mathbb{N}$), be a sequence of functions such that*

$$\lim_{h\to 0} \omega_1(g_n^{(k)}, h) = 0, \text{ uniformly for } n \in \mathbb{N}. \tag{3.6}$$

Then,

$$(L_n g_n)(x) = \sum_{j=0}^{k} \frac{1}{j!} m_n^j(x)(D^j g_n)(x) + \mathrm{o}\left(\frac{\psi(x)}{n^{k/2}}\right), \text{ uniformly for } x \in [0,1]. \tag{3.7}$$

For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we consider the operator $P_n^j : C^j([0,1]) \to C([0,1])$ given by

$$(P_n^j f)(x) = \frac{1}{j!} m_n^j(x)(D^j f)(x),\ f \in C^j([0,1]),\ x \in [0,1]. \tag{3.8}$$

Also, for any $n \in \mathbb{N}$, $p \in \mathbb{N}_0$ and any integers $j_1, \ldots, j_p \geq 0$, denote

$$P_n^{j_p,\ldots,j_1} = P_n^{j_p} \circ \cdots \circ P_n^{j_1}; \text{ if } p \geq 1 \text{ and } P_n = I, \text{ if } p = 0. \tag{3.9}$$

Lemma 3.1.3. *(see Lemma 3 from [46]) Let $j_1, \ldots j_p \geq 0$ and $p \in \mathbb{N}$ be integers such that at least one of them is greater or equal to 2. Then, for any integer $0 \leq k \leq j_1 + \ldots + j_p$, there are certain functions $\tau_{n,k}^{j_1,\ldots,j_p} \in C^\infty([0,1])$, $n \in \mathbb{N}$, with the following properties*

i) for any integer $s \geq 0$ there is a constant C depending only on $j_1, \ldots, j_p$, k and s, for which $|(\tau_{n,k}^{j_1,\ldots,j_p})^{(s)}(x)| \leq C n^{j_1+\ldots+j_p-2}$, for $n \in \mathbb{N}$ and $x \in [0,1]$;

ii) for any $f \in C^{j_1+\ldots+j_p}([0,1])$ there holds:

$$(P_n^{j_p,\ldots,j_1} f)(x) = \frac{\psi(x)}{n^{\sigma_{j_1}+\ldots+\sigma_{j_p}}} \sum_{k=0}^{j_1+\ldots+j_p} f^{(k)}(x)\tau_{n,k}^{j_1,\ldots,j_p}(x),\ x \in [0,1]. \tag{3.10}$$

Lemma 3.1.4. *(see Lemma 4 from [46]) Let $s \geq 2$ be an integer and $f \in C^{2s}([0,1])$ be a function. Let $j_1, j_2, \ldots, j_p \geq 0$ be integers, $p \in \mathbb{N}$ and $r = j_1 + \ldots + j_p$ such that $r < 2s$. For $n \in \mathbb{N}$, we denote $g_n = n^q P_n^{j_p,\ldots,j_1} f$, where $q = \sigma_{j_1} + \ldots + \sigma_{j_p}$. Then $g_n \in C^{2s-r}([0,1])$ and*

$$\lim_{h\to 0} \omega_1(g_n^{(2s-r)}, h) = 0, \text{ uniformly with regard to } n \in \mathbb{N}. \tag{3.11}$$

For integers $s \geq 2$ and $0 \leq p \leq s$, we define

$$\begin{aligned} \Lambda_p^s &= \{(j_1, \ldots, j_p) : \ j_1, \ldots, j_p \in \mathbb{N}_0, \ j_1 + \cdots + j_p \leq 2s\}, \text{ if } p \geq 1 \\ \Lambda_0^s &= \emptyset. \end{aligned} \tag{3.12}$$

Lemma 3.1.5. *(see Lemma 5 from [46]) Let s and p be integers such that $s \geq 2$ and $0 \leq p \leq s$. For any $f \in C^{2s}([0,1])$, $n \in \mathbb{N}$ and $x \in [0,1]$ there holds*

$$((L_n)^p f)(x) = \sum_{(j_1,\ldots,j_p)\in\Lambda_p^s} (P_n^{j_p,\ldots,j_1} f)(x) + \mathrm{o}\left(\frac{\psi(x)}{n^s}\right), \ x \in [0,1], \tag{3.13}$$

where $\mathrm{o}\left(\frac{\psi(x)}{n^s}\right)$ $(n \to \infty)$ *is uniform with regard to* $x \in [0,1]$.

We present now the following result which is a slight improvement of Theorem 6 from [46].

Theorem 3.1.6. *For any integer $s \geq 1$ and for every function $f \in C^{2s}([0,1])$ we have*

$$((L_n - I)^s f)(x) = \frac{1}{n^s}\left(\frac{\alpha}{2}\psi D^2\right)^s f(x) + \mathrm{o}\left(\frac{\psi(x)}{n^s}\right), \text{ uniformly for } x \in [0,1]. \tag{3.14}$$

Consequently there holds:

$$\lim_{n\to\infty}\left\|n^s((L_n - I)^s f) - \left(\frac{1}{2}\psi D^2\right)^s f\right\|_\psi = 0 \tag{3.15}$$

Remark 3.1.7. *(see Remark 7 from [46])* In the case $s = 1$ the Voronovskaya result given in (3.14) was obtained as a particular case, but with other conditions on operators L_n, in [52].

The next Corollary is an improvement of Corollary 8 from [46].

Corollary 3.1.8. *For any integer $s \geq 1$ and for every function $f \in C^{2s}([0,1])$ we have*

$$\lim_{n\to\infty}\left\|n^s((L_n - I)^s f) - \left(\frac{\alpha}{2}\psi D^2\right)^s f\right\| = 0. \tag{3.16}$$

The following Remark is a slight improvement of Remark 9 from [46].

Remark 3.1.9. Let Z_n^s be the Micchelli type combinations of operators $(L_n)_n$, given by

$$Z_n^s = I - (I - L_n)^s, \ n, s \in \mathbb{N}. \tag{3.17}$$

Relation (3.14) can be written equivalently as

$$\lim_{n\to\infty}\left\|n^s(Z_n^s f - f) + \left(-\frac{\alpha}{2}\psi D^2\right)^s f\right\|_\psi = 0, \ f \in C^{2s}([0,1]). \tag{3.18}$$

This follows from equality $Z_n^s - I = (-1)^{s+1}(L_n - I)^s$.

3.2 Generalized power series of positive linear operators

In this section we consider positive linear operators $(L_n)_{n\in\mathbb{N}}$ which satisfy conditions L1), L2) and L3) from the previous section.

Remark 3.2.1. *(see Remark 10 from [46])* From conditions L1)-L3) one can deduce that $L_n\psi = (1-\alpha_n)\psi$, since $(L_n\psi)(x) = (L_ne_1)(x) - m_2(x) - 2x(L_ne_1)(x) + x^2(L_ne_0)(x) = \psi(x) - m_n^2(x)$.

For a bounded linear operator $T : C_\psi([0,1]) \to C_\psi([0,1])$, for simplicity, we will write $\|T\|_\psi$ instead of $\|T\|_{\mathcal{L}(C_\psi([0,1]),C_\psi([0,1]))}$. Since L_n satisfies condition L3) one obtains that L_n is a bounded linear operator on $C_\psi([0,1])$ and that $\|L_n\|_\psi = 1-\alpha_n$. Indeed, if $f \in C_\psi([0,1])$ then, using Remark 3.2.1 we have, for $x \in (0,1)$:

$$|(L_nf)(x)| \le (L_n|f|)(x) = \left(L_n\frac{|f|}{\psi}\psi\right)(x) \le \|f\|_\psi(L_n\psi)(x) = \|f\|_\psi(1-\alpha_n)\psi(x).$$

Then

$$\|L_n\|_\psi = \sup_{\|f\|_\psi\le 1}\sup_{x\in(0,1)}\frac{|(L_nf)(x)|}{\psi(x)} \le \sup_{\|f\|_\psi\le 1}\sup_{x\in(0,1)}\frac{(L_n\|f\|_\psi\psi)(x)}{\psi(x)} = 1-\alpha_n.$$

Let $s \ge 0$ be an integer. We define $A_n^s : C_\psi([0,1]) \to C_\psi([0,1])$ by

$$A_n^s = \frac{\alpha_n^{s+1}}{s!}\sum_{k=s}^{\infty}(k)_s(L_n)^{k-s} \tag{3.19}$$

where $(k)_s = k(k-1)\dots(k-s+1)$ and the convergence of the series is considered with respect to the norm $\|\cdot\|_\psi$.

The following lemma proves the existence of our operator.

Lemma 3.2.2. *(see Lemma 11 from [46]) Operator $A_n^s : C_\psi([0,1]) \to C_\psi([0,1])$ is well defined and $\|A_n^s\|_\psi = 1$.*

Lemma 3.2.3. *(see Lemma 12 from [46]) In the space $C_\psi([0,1])$ the following equalities hold*

i) $(I-L_n)^{s+1}\sum_{k=s}^{\infty}(k)_s(L_n)^{k-s} = s!I$,

ii) $\sum_{k=s}^{\infty}(k)_s(L_n)^{k-s}(I-L_n)^{s+1} = s!I$,

where by I we denoted the identity operator.

Let us define the following operators: $\tilde{F} : C_\psi([0,1]) \to C_\psi([0,1])$, $\tilde{F}(f) = F(f/\psi)$, $H_s : C_\psi([0,1]) \to C_\psi([0,1])$, $H_s = \left(\frac{2}{\alpha}\right)^{s+1}\tilde{F}^{s+1}$ and $\tilde{D} : C^2([0,1]) \to \psi C([0,1])$, $\tilde{D}f = -\frac{\alpha}{2}\psi D^2 f$, where $s \in \mathbb{N}_0$ and operator F is given in (2.20) and since we have (2.27) then $F(f) \in C_\psi([0,1])$ therefore $\tilde{F}$ is well defined from $C_\psi([0,1])$ into itself.

Lemma 3.2.4. *(see Lemma 13 from [46]) For $f \in C_\psi([0,1])$, $s \in \mathbb{N}_0$ and $x \in (0,1)$ we have that:*

$$(\tilde{D}^{s+1} H_s f)(x) = f(x). \tag{3.20}$$

Finally, we can prove our main result of this section

Theorem 3.2.5. *(see Theorem 14 from [46]) For $f \in C_\psi([0,1])$ and $s \in \mathbb{N}$ there holds*

$$\lim_{n\to\infty} \|A_n^s f - \alpha^{s+1} H_s f\|_\psi = 0. \tag{3.21}$$

3.3 Applications

For these applications we refer the reader to Section 4 from [46].

1. Bernstein operators

A first example of operators which verify conditions L1), L2) and L3) are Bernstein operators B_n given in (2.6) and (2.7). Their moments $m_n^j(x) = B_n(e_1 - xe_0)^j(x)$, $j \in \mathbb{N}_0$ satisfy the following relations [68], [37]:

$$m_n^0(x) = 1, \quad m_n^1(x) = 0, \quad m_n^2(x) = \frac{x(1-x)}{n},\ n \in \mathbb{N},\ x \in [0,1] \tag{3.22}$$

and

$$m_n^{j+1}(x) = \frac{x(1-x)}{n}((m_n^j)'(x) + j m_n^{j-1}(x)),\ j \in \mathbb{N},\ n \in \mathbb{N},\ x \in [0,1]. \tag{3.23}$$

From (3.22) one deduces immediately that operators B_n satisfy conditions L1) and L3) with $\alpha_n = \frac{1}{n}$ and $\alpha = 1$. From (3.23) one can deduce by induction that, for $j \geq 2$ we have

$$m_n^j(x) = \psi(x) n^{-\sigma_j} \cdot Q_{n,j}(x), \tag{3.24}$$

where $Q_{n,j}$ is a polynomial of degree $j-2$ with bounded coefficients with regard to n. This implies that condition L2) is also satisfied. Consequently we can apply Theorem 3.1.6 and Theorem 3.2.5 to Bernstein operators.

2. Modified Durrmeyer operators

For a parameter $\rho \geq 1$, consider operators $U_n^\rho : C([0,1]) \to C([0,1])$, see [79], [53], given by

$$(U_n^\rho f)(x) = \sum_{k=0}^{n} F_{n,k}^\rho(f) p_{n,k}(x),\ n \in \mathbb{N},\ x \in [0,1],\ f \in C([0,1]),$$

where $F_{n,0}(f) = f(0)$, $F_{n,n}(f) = f(1)$ and

$$F_{n,k}(f) = \int_0^1 f(t) \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} dt,\ 1 \leq k \leq n-1,$$

B denoting the beta function. These operators become the genuine Durrmeyer operators, for $\rho = 1$ and $\lim_{\rho\to\infty} U_n^\rho f = B_n f$, for any $n \in \mathbb{N}$ and $f \in C([0,1])$, [53]. It is known that $U_n^\rho e_j = e_j$, $j = 0, 1$ and hence condition L1) holds. With the notation $m_n^j(x) = (U_n^\rho(e_1 - xe_0)^j)(x)$, for $n \in \mathbb{N}$, $j \in \mathbb{N}_0$, $x \in [0,1]$ we have, for $j \geq 1$:

$$m_n^{j+1}(x) = \frac{1}{n\rho + j}\Big(\rho\psi(x)(m_n^j)'(x) + j(1-2x)m_n^j(x) + j(\rho+1)\psi(x)m_n^{j-1}(x)\Big). \quad (3.25)$$

From this, one obtains that $m_n^2(x) = \frac{\rho+1}{n\rho+1}\psi(x)$. This means that condition L3) holds with $\alpha_n = \frac{\rho+1}{n\rho+1}$, for $n \geq 2$ and $\alpha = \frac{\rho+1}{\rho}$. Also, from relation (3.25) condition L2) follows by induction. Therefore Theorem 3.1.6 and Theorem 3.2.5 can be applied to operators U_n^ρ.

4 The representation of the limit of power series of positive linear operators by using the operators semigroup generated by their iterates

In this Chapter we obtain a characterization of the limit of power series of positive linear operators using the C_0-semigroup generated by the iterates of these operators. The results introduced here were published in paper [47]: **Ș. Garoiu**, R. Paltanea, *The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterate,* Dolomites Research Notes on Approximation (2023), 16(3), 39-47.

Let L be a positive linear operator and L^k its k-times iterates, if $k \geq 1$ with $L^0 = I$, where I is the identity operator.

If $(L_n)_n,\ L_n : C([0,1]) \to C([0,1])$ is a sequence of positive linear operators, the geometric series of L_n is of the form

$$\beta_n \sum_{k=0}^{\infty} (L_n)^k,\ n \in \mathbb{N} \tag{4.1}$$

where $\beta_n \in \mathbb{R}$ is a normalization factor. As stated in Chapter 2, Section 5 the geometric series of linear and positive operators is not defined for every function in $C([0,1])$. For instance, with the hypothesis that L_n preserve constant functions, then the operators in (4.1) are not defined for such functions. In order to define this geometric series of operators it is necessary to restrict the domain of definition of operators. A space that can be taken in consideration is the space $\psi C([0,1])$, defined in (2.18), which can be endowed with the norm $\|\cdot\|_\psi$ defined in (2.19).

Let us recall that a first study of the convergence of geometric series attached to a sequence of operators $(L_n)_n$ was made in the paper [78], namely the case when $L_n = B_n$, where B_n are Bernstein operators (given in (2.6) and (2.7)), was considered. There it is shown that one can define operators $A_n : \psi C([0,1]) \to \psi C([0,1])$,

$$A_n = \frac{1}{n} \sum_{k=0}^{\infty} (B_n)^k,\ n \in \mathbb{N}$$

and this sequence has a limit when $n \to \infty$ in the space $(\psi C([0,1]), \|\cdot\|_\psi)$, which can be explicitly described.

In this direction several papers extended this study for diverse classes of positive linear operators and for other spaces of functions, see [1], [3], [4], [56], [78], [80], [84].

Recently, in the paper by Acar, Aral and Raşa ([10]) it was given a new way to describe the uniform limit of geometric series of form (4.1) using the C_0-semigroup of operators generated by the iterates of L_n. For more details see Chapter 2, Section 5 which contains some of the results obtained by the authors of the mentioned paper.

Our aim is to study the convergence of more general power series of the form:

$$\sum_{k=0}^{\infty} \beta_{n,k}(L_n)^k \tag{4.2}$$

using the C_0-semigroup generated by the iterates of operators L_n. The framework of our approach differs from the study made in [10] in the sense that we consider another type of operators, another space of functions and a stronger type of convergence.

To do this beside the space $\psi C([0,1])$ we consider the space $C_\psi([0,1])$ defined in (2.25) with its respective norm $\|\cdot\|_\psi$ given in (2.26). We recall that this space is an extension of space $\psi C([0,1])$ and one can also write (see [4]):

$$C_\psi([0,1]) = \{f \in C([0,1]),\ \|f\|_\psi < \infty\}.$$

Also $C_\psi([0,1])$ endowed with the norm $\|\cdot\|_\psi$ is a Banach space, but it is not a Banach space with regard the sup-norm $\|\cdot\|$, since

$$\overline{\psi C([0,1])} = \overline{C_\psi([0,1])} = C_0([0,1]),$$

where $C_0([0,1]) = \{f \in C([0,1]) | f(0) = 0, f(1) = 0\}$.

Note that if $f, f_n \in C_\psi([0,1])$, $n \in \mathbb{N}$ and $\|f - f_n\|_\psi \to 0$, $(n \to \infty)$ then $\|f - f_n\| \to 0$, $(n \to \infty)$. For this reason, we can name $\|\cdot\|_\psi$ the strong norm on the space $C_\psi([0,1])$.

If $L : C_\psi([0,1]) \to C_\psi([0,1])$ is a linear bounded operator we will use the notation

$$\|L\|_\psi = \sup_{\|f\|_\psi \le 1} \|Lf\|_\psi. \tag{4.3}$$

4.1 Auxiliary results

Throughout this chapter we will consider a sequence $(L_n)_n$ of positive linear operators $L_n : C([0,1]) \to C([0,1])$, $L_n \neq I$, satisfying the following conditions.

A1) There exist $\alpha \in (0,1)$ and $\alpha_n \in (0,1)$, $n \in \mathbb{N}$ such that $L_n(\psi) = (1 - \alpha_n)\,\psi$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} n\alpha_n = \alpha$.

A2) The operators L_n admit the eigenvalues $a_{n,j}$ associated to eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with $deg\ p_{n,j} = j$, where, for a polynomial p we denote by $deg\ p$, the degree of p.

A3) There exist the polynomials p_j, $j \ge 0$, such that $\lim_{n\to\infty} p_{n,j} = p_j$, $j = 0, 1, \ldots$.

A4) For any $j \ge 0$ there exists $l_j \in (0,1]$, such that

$$\lim_{n\to\infty} (a_{n,j})^n = l_j$$

and moreover if $l_j = 1$, then $a_{n,j} = 1$, for all $n \in \mathbb{N}$.

A5) We have $L_n(\psi\Pi) \subset \psi\Pi$.

A6) There exists a C_0 - semigroup of operators $(T(t))_{t>0}$, such that

$$T(t)f = \lim_{n\to\infty} (L_n)^{k_n} f, \text{ uniformly for } f \in C([0,1]),\ t \geq 0, \tag{4.4}$$

if $k_n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{k_n}{n} = t$.

From conditions A1)-A6) one can deduce the following consequences.

Remark 4.1.1. *(see Remark 1, [47])* Because L_n is a positive linear operator and $L_n \neq I$, from condition A4) there are at most two values of $j \geq 0$, for which $l_j = 1$.

Remark 4.1.2. *(see Remark 2, [47])* Condition A4) implies that

$$\lim_{n\to\infty} a_{n,j} = 1, \text{ and } \lim_{n\to\infty} n(1 - a_{n,j}) = -\ln l_j,\ j = 0, 1, \ldots. \tag{4.5}$$

Remark 4.1.3. *(see Remark 3, [47])* Conditions A3), A4) and A6) imply

$$T(t)p_j = l_j^t p_j,\ j \in \mathbb{N}_0,\ t \geq 0. \tag{4.6}$$

Note that condition A6) is assured in certain hypothesis by Trotter's theorem 2.4.2 ([91]).

Remark 4.1.4. *(see Remark 4, [47])* For $r \geq 0$, because the polynomials $p_{n,j}$, $0 \leq j \leq r$ have the property $deg\ p_{n,j} = j$, they form a basis of Π_r and consequently $L_n(\Pi_r) \subset \Pi_r$. Then, by induction, $L_n^k(\Pi_r) \subset \Pi_r$, $k \in \mathbb{N}$, for any $n, k \in \mathbb{N}$. From condition A6) it results that $T(t)(\Pi_r) \subset \Pi_r$, $r \in \mathbb{N}_0$.

Remark 4.1.5. *(see Remark 5, [47])* We mention that the first part of condition A1) is a consequence of the following conditions: $L_n(e_j) = e_j$, $j = 0, 1$ and $L_n(\Pi_2) \subset \Pi_2$. Indeed, it is proved in [4] that if $L : C([0,1]) \to C([0,1])$ is a positive linear operator such that $L_n(e_j) = e_j$, $j = 0, 1$ and $L_n(\Pi_2) \subset \Pi_2$, then there exists $\beta \in [0,1)$ such that $L\psi = \beta\psi$.

Also, condition A5) is a consequence of the following conditions: $L_n(C([0,1])) \subset \Pi$ and $L_n(e_j) = e_j$, $j = 0, 1$. Indeed, in this case we have $L_n f(0) = f(0)$ and $L_n f(1) = f(1)$, for any $f \in C([0,1])$. Consequently, for $f \in \psi C([0,1])$ it follows that $L_n f(0) = L_n f(1) = 0$ and hence $L_n(\psi C([0,1])) \subset \psi\Pi$.

Finally we need the following lemmas.

Lemma 4.1.6. *(see Lemma 2.1 from [47]) For any $t \geq 0$ one has $T(t)(C_\psi([0,1])) \subset C_\psi([0,1])$ and*

$$\|T(t)\|_\psi = e^{-\alpha t}. \tag{4.7}$$

Lemma 4.1.7. *(see Lemma 2.2 from [47]) Let $r \in \mathbb{N}$. If a sequence of polynomials $(\sigma_n)_n$, $\sigma_n \in \psi\Pi_r$ is uniformly convergent to a polynomial $\sigma^\star \in \psi\Pi_r$, then sequence $(\sigma_n)_n$ converges to $\sigma^\star$ in the norm $\|\cdot\|_\psi$ as well.*

4.2 Main results

A main tool for our purpose is the following lemma.

Lemma 4.2.1. *(see Lemma 3.1 from [47]) For $p \in \Pi$, $s \in \mathbb{N}_0$, $t \geq 0$ and a sequence of positive integers $(k_n)_n$ such that $k_n/n \to t$, $(n \to \infty)$ there exists the limit:*

$$\lim_{n\to\infty} \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p = \frac{1}{t^{s+1}} \int_0^t u^s T(u) p du. \tag{4.8}$$

uniformly on $[0,1]$, where $(i)_s = i(i-1)\dots(i-s+1)$.

Corollary 4.2.2. *(see Corollary 3.2 from [47]) For any $p \in \psi\Pi$, $s \in \mathbb{N}_0$, $t > 0$ and a sequence of positive integers $(k_n)_n$ such that $k_n/n \to t$, $(n \to \infty)$ there holds*

$$\lim_{n\to\infty} \left\| \frac{1}{k_n^{s+1}} \sum_{i=0}^{k_n} (i)_s (L_n)^i p - \frac{1}{t^{s+1}} \int_0^t u^s T(u) p du \right\|_\psi = 0. \tag{4.9}$$

Now we need the following theorem which, with modified notations, follows from a result proved in the book of Nachbin [73], see Lemma 2, pg. 95.

Theorem A *Let $b > 0$. For any function $f \in C([0,\infty))$, such that $f(x)e^{-bx} \to 0$, $(x \to \infty)$, and any $\varepsilon > 0$ there exist a polynomial p such that*

$$\sup_{x\in[0,\infty)} e^{-bx} |f(x) - p(x)| < \varepsilon.$$

In the terminology from [73], the function e^{-bx}, $x \geq 0$ is a fundamental weight.

Define the space:

$$\tilde{C}_\alpha([0,\infty)) = \{g \in C([0,\infty)) |\ \exists b \in (0,\alpha),\ \lim_{x\to\infty} f(x)e^{-bx} = 0\}. \tag{4.10}$$

Our main result is the following:

Theorem 4.2.3. *(see Theorem 3.3 from [47]) If $g \in \tilde{C}_\alpha([0,\infty))$ and $f \in \psi C([0,1])$ then*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=0}^{\infty} g\left(\frac{i}{n}\right) (L_n)^i f - \int_0^\infty g(t) T(t) f dt \right\|_\psi = 0. \tag{4.11}$$

4.3 Applications

For this section we refer the reader to the Section 4 of [46].

1. Bernstein operators

Let $B_n : C([0,1]) \to C([0,1])$ be the Bernstein operators defined as:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \leq k \leq n,\ x \in [0,1].$$

Operators B_n satisfy conditions A1)-A6), see [30], [33] and [68]. More exactly, we have $B_n\psi = \frac{n-1}{n}\psi$ and hence $\alpha_n = 1/n$ and $\alpha = 1$. B_n admits the eigenvalues $a_{n,j}$ corresponding to the eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with *deg* $p_{n,j} = j$ and

$$l_j := \lim_{n\to\infty} a_{n,j}^n = e^{-j(j-1)/2},\ j \in \mathbb{N}_0.$$

For $j = 0, 1$, we have $p_{n,j}(t) = e_j$ and $B_n(e_j) = e_j$. The existence of the polynomials $p_j = \lim_{n\to\infty} p_{n,j}$ is proved in [33]. Finally the existence of the semigroup of operators generated by the iterates of B_n is given, for instance in [15].

2. Operators U_n^ρ

For $\rho > 0$ and $n \in \mathbb{N}$, $n \ge 2$, operators U_n^ρ are defined, (see [53], [79]), as follows:

$$(U_n^\rho f)(x) = \sum_{k=0}^{n} p_{n,k}(x) F_{n,k}^\rho(f),\ f \in C([0,1]),\ x \in [0,1],$$

where

$$F_{n,0}(f) = f(0),\ F_{n,n}(f) = f(1);$$
$$F_{n,k}(f) = \int_0^1 f(t)\frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}dt,\ 1 \le k \le n-1.$$

The eigenstructure of these operators was investigated in [57].

These operators also satisfy the conditions A1)-A6). More precisely we have the following: $U_n^\rho\psi = \frac{n-1}{n\rho+1}\psi$, thus we can take $\alpha_n = \frac{\rho+1}{n\rho+1}$ and $\alpha = 1$. Then, U_n^ρ admits eigenpolynomials $p_{n,j}$, $0 \le j \le n$, with deg $p_{n,j} = j$. Moreover $p_{n,j} = e_j$ and $U_n^\rho(e_j) = e_j$, for $j = 0, 1$. The eigenvalues are, see [57]:

$$a_{n,j} = \rho^j \frac{n(n-1)\dots(n-j+1)}{(n\rho)(n\rho+1)\dots(n\rho+j-1)},\ 0 \le j \le n.$$

Therefore we have

$$l_j := \lim_{n\to\infty} a_{n,j}^n = e^{-\frac{j(j-1)}{2}\cdot\frac{\rho+1}{\rho}},\ j \ge 0.$$

The existence of limit polynomials $p_j = \lim_{n\to\infty} p_{n,j}$ is also shown in [57]. For proving the existence of the semigroup of operators generated by the iterates of operators U_n^ρ one can apply Corollary 2.2.11 from [15].

5 A Voronovskaya type theorem associated to geometric series of Bernstein - Durrmeyer operators

In this chapter we aim to give a Voronovskaya type theorem for the geometric series operators associated to Bernstein-Durrmeyer operators, introduced by Abel.

This result was published in paper [43]: **Ș. Garoiu**, *A Voronovskaya type theorem associated to geometric series of Bernstein-Durrmeyer operators*, Carpathian Journal of Mathematics(2025), 41(2).

5.1 Geometric series of Bernstein-Durrmeyer operators

U. Abel, in paper [1], introduced the geometric series associated to Bernstein-Durrmeyer operators (which first appeared in paper [41] and independently in [70]; their properties were later studied in [34], [37], [77] etc.)

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\, f(t)\, dt, \qquad f \in L^{\infty}([0,1]). \tag{5.1}$$

Namely, the operators he studied are defined as follows:

$$P_n = \frac{1}{n}\sum_{k=0}^{\infty} (M_n)^k .$$

These operators are well defined on the space V, which is

$$V = \{ f \in L^{\infty}([0,1]) : ||f||_* < \infty \}, \tag{5.2}$$

where $\| \cdot \|_*$ is the norm:

$$||f||_* = \sup_{y\in(0,1)} \left| (\psi(y))^{-1} \int_0^y f(x)\, dx \right| .$$

Also, V endowed with the norm $||\cdot||_*$ is a Banach space.

For $f \in V$, define the function F on $(0,1)$ by

$$F(y) = (\psi(y))^{-1}\int_0^y f(x)\, dx, \qquad y \in (0,1). \tag{5.3}$$

Then, $f = (\psi F)'$ a. e. on $[0,1]$ and $||f||_* = ||F||_\infty$.

Further, the operator $P : V \to V$, was defined as

$$P(f)(x) = \int_0^1 \int_x^t F(u)\, dudt, \qquad x \in [0,1],\ f \in V, \tag{5.4}$$

Integrating by parts in (5.4) it can be seen that

$$P(f)(x) = -\int_0^x tF(t)\, dt + \int_x^1 (1-t) F(t) dt, \qquad x \in [0,1],\ f \in V, \tag{5.5}$$

and here, by differentiation one finds

$$P'(f)(x) = -F(x). \tag{5.6}$$

In his paper, Abel proved that operators P_n satisfy the following convergence result

Theorem 5.1.1. *If $f \in V$, then, in $(V, ||\cdot||_*)$, the convergence*

$$\lim_{n\to\infty} ||P_n(f) - P(f)||_* = 0, \tag{5.7}$$

holds.

Also, Abel obtained the following two results concerning the norm of operators M_n and P_n on the space V.

Proposition 5.1.2. *For each $n \in \mathbb{N}$, operators M_n map V to V, that is, $M_n(V) \subset V$, and*

$$||M_n||_{\mathcal{L}(V,V)} = \frac{n}{n+2}. \tag{5.8}$$

Proposition 5.1.3. *For each $n \in \mathbb{N}$ operators P_n map V to V, that is, $P_n(V) \subset V$, and*

$$||P_n||_{\mathcal{L}(V,V)} = \frac{1}{2} + \frac{1}{n}. \tag{5.9}$$

More recent results concerning the power series of approximation operators can be seen in [10], [46], [47], [56] and [84]. Also a small summary of the main results on this topic can be seen in Chapter 2 Section 5 as well.

5.2 A Voronovskaya type result

In this section we will provide our main result, namely we will prove our Voronovskaya type theorem associated to operators P_n. First, we will denote by $G_{M_n} = \sum_{k=0}^{\infty} (M_n)^k$ the geometric series associated to Bernstein - Durrmeyer operators M_n, where the convergence holds on V. Next, we will prove that the following identities hold.

Lemma 5.2.1. *(see Lemma 2.1 from [43]) Operator $G_{M_n} \in V$ and it verifies the identities:*

$$(I - M_n) \circ G_{M_n} = I, \tag{5.10}$$

and

$$G_{M_n} \circ (I - M_n) = I, \tag{5.11}$$

where I denotes the identity operator.

In the following we will work on space $V_1 = V \cap C([0,1])$. Note that condition $f \in V_1$ is equivalent with conditions $f \in C([0,1])$ and the relation below holds

$$\int_0^1 f(t)dt = 0. \tag{5.12}$$

On this space, we define the operator $U : V_1 \to C([0,1])$ through

$$U(f)(y) = \begin{cases} (\psi(y))^{-1} \int\limits_0^y f(x)\,dx, & y \in (0,1) \\ f(0), & y = 0 \\ -f(1), & y = 1 \end{cases} \qquad f \in V_1, \tag{5.13}$$

and norm $\|\cdot\|_*$ as:

$$\|f\|_* = \sup_{y \in [0,1]} |U(f)(y)|. \tag{5.14}$$

Here, we have $U(f)(1) = \lim_{y\to 1} \frac{\int_0^y f(t)dt}{\psi(y)}$ and $U(f)(0) = \lim_{y\to 0} \frac{\int_0^y f(t)dt}{\psi(y)}$, so, since $\psi(0) = \psi(1) = 0$, using l' Hospital's rule we will have that $U(f)(1) = -f(1)$ and $U(f)(0) = f(0)$.

Next, we will define the operator

$$\Theta(h)(x) = -\int_0^x th(t)dt + \int_x^1 (1-t)h(t)dt, \tag{5.15}$$

where $h \in C([0,1])$ and $x \in [0,1]$. This operator has the following property.

Proposition 5.2.2. *(see Proposition 2.3 from [43]) The operator Θ maps $C([0,1])$ to V_1, i. e.*

$$\Theta(C([0,1])) \subset V_1.$$

From above and from (5.15), we have that

$$P(f) := \Theta(U(f)), \qquad f \in V_1. \tag{5.16}$$

Next, on the space V_1, the following result concerning the norm $\|\cdot\|_*$ holds.

Lemma 5.2.3. *(see Lemma 2.2 from [43]) For any function $f \in V_1$ we have that:*

$$\|f\|_* \leq 2\|f\|_\infty. \tag{5.17}$$

Now, we can prove our main result.

Theorem 5.2.4. *(see Theorem 2.2 from [43]) Let* $f \in V_1$ *be a ten times differentiable function on* $[0,1]$ *and which satisfies the following conditions* $\int_0^1 f(y)\log\psi(y)dy = 0$, $f(0)+f(1)=0$, $f'(0)-f'(1)=0$, $f''(0)+f''(1)=0$ *and* $f'''(0)-f'''(1)=0$. *Then:*

$$\lim_{n\to\infty} n(P_n(f) - P(f)) = 2P(f) - \Theta(T'\psi') - \frac{1}{2}\Theta(T''\psi), \tag{5.18}$$

with regard to the norm $\|\cdot\|_*$.

6 Exponential Kantorovich Stancu operators

In this chapter we introduce a new class of operators of exponential type, obtained as a modification in Stancu sense (similar to the one for operators B_n, see operators $B_n^{\alpha,\beta}$ ([89]), with their expression as in (2.8)), of Bernstein-Kantorovich exponential operators ([18]), K_n given in (2.10). Then we will prove that these operators satisfy Korovkin's theorem, we will obtain a convergence result in a weighted version of the norm on L^p spaces, a Voronovskaya asymptotic result and some evaluations of the order of approximation using K-functionals and moduli of smoothness.

These results were published in [45]:

Ș. Garoiu, *Exponential Kantorovich-Stancu operators,* Bull. Univ. Transilvania Brasov, Ser. 3, Math. Comput. Sci., 5(67), 2025, no. 2, 127-144.

6.1 Definition and convergence results

Let us introduce the following operators:

$$K_n^{\alpha,\beta,\mu}f(x) = (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t}f(t)dt,\ \ f\in C([0,1]), \tag{6.1}$$

where $x\in[0,1]$ and $0\le\alpha\le\beta$, $\mu>0$, $a_n(x)=\frac{e^{\frac{\mu x}{n+\beta}}-1}{e^{\frac{\mu}{n+\beta}}-1}$, $n\in\mathbb{N}$. These operators are a Stancu modification of the exponential Bernstein-Kantorovich operators, K_n given in (2.10), obtained by Angeloni and Costarelli in [18].

In order to show that these operatos verify Korovkin's theorem we will check their convergence for test functions $e_0(x)=1$, $exp_\mu(x)=e^{\mu x}$ and $exp_\mu^2(x)=e^{2\mu x}$, for $x\in[0,1]$, which form a Chebyshev set. First, it is obvious that

$$K_n^{\alpha,\beta,\mu}exp_\mu(x) = exp_\mu(x),\ \ x\in[0,1]. \tag{6.2}$$

In order to obtain our approximation results we will need the following Lemma.

Lemma 6.1.1. *(see Lemma 1 from [45]) For $x\in[0,1]$ we have that:*

$$K_n^{\alpha,\beta,\mu}e_0(x) = \frac{n+\beta+1}{\mu}e^{\mu x}(1-e^{-\frac{\mu}{n+\beta+1}})e^{-\frac{\mu(\alpha+n)}{n+\beta+1}}(1-e^{\frac{\mu x}{n+\beta+1}}+e^{\frac{\mu}{n+\beta+1}})^n, \tag{6.3}$$

$$K_n^{\alpha,\beta,\mu}exp_\mu^2(x) = \frac{n+\beta+1}{\mu}e^{\mu x}e^{\frac{\mu\alpha}{n+\beta+1}}(e^{\frac{\mu}{n+\beta+1}}-1)e^{\frac{\mu n x}{n+\beta+1}}, \tag{6.4}$$

$$\begin{aligned} K_n^{\alpha,\beta,\mu}exp_\mu^3(x) = {} & \frac{n+\beta+1}{2\mu}e^{\mu x}e^{\frac{2\mu\alpha}{n+\beta+1}}(e^{\frac{2\mu}{n+\beta+1}}-1) \\ & \times\left(e^{\frac{\mu(x+1)}{n+\beta+1}}+e^{\frac{\mu x}{n+\beta+1}}-e^{\frac{\mu}{n+\beta+1}}\right)^n, \end{aligned} \tag{6.5}$$

$$K_n^{\alpha,\beta,\mu} exp_\mu^4(x) = \frac{n+\beta+1}{3\mu} e^{\mu x} e^{\frac{3\mu\alpha}{n+\beta+1}} (e^{\frac{3\mu}{n+\beta+1}} - 1) \tag{6.6}$$
$$\times \left(e^{\frac{\mu(x+2)}{n+\beta+1}} + e^{\frac{\mu(x+1)}{n+\beta+1}} + e^{\frac{\mu x}{n+\beta+1}} - e^{\frac{2\mu}{n+\beta+1}} - e^{\frac{\mu}{n+\beta+1}}\right)^n.$$

Having in mind Lemma 6.1.1 and the fact that e_0, exp_μ and exp_μ^2 form a Chebyshev set we can prove that operators $K_n^{\alpha,\beta,\mu} f$ uniformly converge to functions $f \in C([0,1])$.

Theorem 6.1.2. *(see Theorem 1 from [45]) For $f \in C([0,1])$ we have that $K_n^{\alpha,\beta,\mu} f$ converges uniformly to f on $[0,1]$.*

Now, we will provide an approximation result for functions belonging to a weighted version of L^p spaces.

In the following, by the space $L_\mu^p([0,1])$ we mean the space of all functions f that satisfy:

$$\|f\|_{p,\mu} = \left\{\int_0^1 |e^{-\mu x} f(x) dx|^p\right\}^{\frac{1}{p}} < \infty.$$

Also, it can be seen that if $f \in L_\mu^p([0,1])$, then $f \in L^p([0,1])$ and reciprocally.

Theorem 6.1.3. *(see Theorem 2 from [45]) For $f \in L_\mu^p([0,1])$ and $n \in \mathbb{N}$, we have that:*

$$\|K_n^{\alpha,\beta,\mu} f\|_{p,\mu} \le \left(\left(1 + \frac{\beta}{n+1}\right) \frac{e^{\frac{\mu}{n+\beta+1}} - 1}{\frac{\mu}{n+\beta+1}}\right)^{\frac{1}{p}} \|f\|_{p,\mu}, \tag{6.7}$$

and consequently,

$$\|K_n^{\alpha,\beta,\mu} f\|_{p,\mu} \le \Theta_{\mu,\beta} \|f\|_{p,\mu}, \tag{6.8}$$

where $\Theta_{\mu,\beta} = \left(\left(1 + \frac{\beta}{2}\right) \frac{e^\mu - 1}{\mu}\right)^{\frac{1}{p}}$. Moreover,

$$\|K_n^{\alpha,\beta,\mu} f - f\|_{p,\mu} \to 0, \quad \text{as } n \to \infty. \tag{6.9}$$

Remark 6.1.4. *(see Remark 1 from [45]) The inequalities in Theorem 6.1.3 can be rewritten using norm $\|\cdot\|_p$, and also we have that:*

$$\|K_n^{\alpha,\beta,\mu}\|_p \le e^\mu \Theta_{\mu,\beta}. \tag{6.10}$$

6.2 Voronovskaya Theorem

In this section we will prove a Voronovskaya type theorem in order to get the rate of approximation by our operators. In what follows, because the Chebyshev set considered is $\{e_0, exp_\mu, exp_\mu^2\}$, we will write our function $f \in C^2([0,1])$ as $f(x) = (f \circ \ln_\mu)(exp_\mu)$, $x \in [0,1]$ where $\ln_\mu(x) = \log_{e^\mu}(x)$ is the inverse of $exp_\mu(x)$.

For such functions, the following Voronovskaya formula holds.

Theorem 6.2.1. *(see Theorem 3 from [45]) Let $f \in C^2([0,1])$. The following limit*

$$\lim_{n\to\infty} n(K_n^{\alpha,\beta,\mu} f - f)(x) = \left[-\frac{1}{2} - \alpha + (1+\beta+\mu)x - \mu x^2\right](\mu f(x) - f'(x)) + \frac{x(1-x)}{2}(f''(x) + \mu f(x)), \tag{6.11}$$

holds uniformly for $x \in [0,1]$.

6.3 Quantitative estimates

In what follows we will provide some characterizations of the rate of convergence of our operators to functions from $C([0,1])$. The results are obtained in terms of certain K-functionals, which will be defined along the way, first order modulus of continuity and second order modulus of smoothness. Some of these results are obtained using the equivalence between K-functionals and the moduli presented.

To obtain the estimates mentioned in the beginning of this section we will need the following auxiliary result.

Lemma 6.3.1. *(see Lemma 2 from [45]) For $y \in [0,1]$ we have that*

$$\sum_{k=0}^{n} p_{n,k}(y) \left| \frac{k+\alpha}{n+\beta+1} - y \right| \le \Omega_{n,\beta}, \tag{6.12}$$

where $\Omega_{n,\beta} = \frac{\sqrt{(\beta+1)^2 + n/4}}{n+\beta+1}$.

Now, we can state our first quantitative result which involves the first order modulus of continuity defined as:

$$\omega_1(f,\delta) = \sup\{|f(t) - f(x)|,\ t,x \in [0,1],\ |t-x| < \delta\},\ f \in C([0,1]),\ \delta > 0.$$

To this purpose, the following theorem holds.

Theorem 6.3.2. *(see Theorem 4 from [45]) Let $f \in C([0,1])$. Then, for $n \in \mathbb{N}$ we have that:*

$$|K_n^{\alpha,\beta,\mu} f(x) - f(x)| \le |f(x)| \frac{C^1_{\alpha,\beta,\mu}}{n} + e^{\mu}\omega_1(exp_\mu^{-1} f, \tau_n) + e^{\mu}\omega_1\left(exp_\mu^{-1} f, \frac{1}{\sqrt{n+\beta+1}}\right)\left\{1 + \frac{1}{2\sqrt{n+\beta+1}} + \Omega_{n,\beta}\right\}, \tag{6.13}$$

where

$$\tau_n = \max_{x\in[0,1]} |a_{n+1}(x) - x|, \tag{6.14}$$

and $C^1_{\alpha,\beta,\mu}$ is a constant depending on α, β, μ.

Further, we will provide an estimate for the approximation of functions $f \in L^p([0,1])$, $1 \leq p < \infty$ using Peetre's K-functional:

$$\mathcal{K}_1(f,\delta)_p = \inf_{g \in C^1[0,1]} \{\|f-g\|_p + \delta \|g'\|_\infty\},\ \delta > 0,\ 1 \leq p < \infty. \tag{6.15}$$

Theorem 6.3.3. *(see Theorem 5 from [45]) Let $f \in L^p([0,1])$, $1 \leq p < \infty$ and $n \in \mathbb{N}$. Then:*

$$\|K_n^{\alpha,\beta,\mu} f - f\|_p \leq \frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + e^\mu(\Theta_{\beta,\mu}+1)\mathcal{K}_1\left(f, \frac{\delta_n^{\alpha,\beta}}{\Theta_{\beta,\mu}+1}\right)_p, \tag{6.16}$$

where $\delta_n^{\alpha,\beta} = \frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n$ and $C^1_{\alpha,\beta,\mu}$ is a constant depending on α, β and μ.

Now, to proceed with our last result we will need the following definitions of K-functionals and of the second order smoothness modulus ω_2:

$$\mathcal{K}_j(f,\delta) = \inf_{g \in C^j([0,1])} \{\|f-g\|_\infty + \delta^j \|g^{(j)}\|_\infty\},\ f \in C([0,1]),\ \delta > 0,\ j = 1,2,$$

and

$$\omega_2(f,\delta) = \sup_{h \in [0,\delta]} \sup_{x \in [0,1-\frac{h}{2}]} \left|\Delta_h^2(f,x)\right|,$$

where $\Delta_h^2(f,x) = f(x) - 2f(x+h) + f(x+2h)$.

It is well-known that between these K-functionals and ω_1 and ω_2 the following relations exist (see [62]): $\mathcal{K}_j(f,\delta) \leq C_j\omega_j(f,\delta),\ f \in C([0,1]),\ \delta > 0,\ j = 1,2$, where C_j are constants depending only on j.

Theorem 6.3.4. *(see Theorem 6 from [45]) Let $f \in C([0,1])$. Then:*

$$\|K_n^{\alpha,\beta,\mu} f - f\|_\infty \leq 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + C_1^*\omega_1\left(f, \frac{1}{n}\right) + C_2^*\omega_2\left(f, \frac{1}{\sqrt{n}}\right), \tag{6.17}$$

for every $n \in \mathbb{N}$, where C_1^* and C_2^* are constants depending on α, β, μ.

7 Exponential Bernstein Durrmeyer operators

In this chapter, our aim is to give an exponential variant of Bernstein-Durrmeyer operators M_n from (5.1) similar to the variant proposed by Angeloni and Costarelli for Bernstein-Kantorovich exponential operators ([18]), K_n from (2.10). Then we will prove that these operators satisfy Korovkin's theorem, we will obtain a convergence result in a weighted version of the norm on L^p spaces, a Voronovskaya asymptotic result, some evaluations of the order of approximation using K-functionals and moduli of smoothness and finally a simultaneous approximation result is proved.

These results were published in [44]:

Ș. Garoiu, *Exponential Bernstein-Durrmeyer operators*, General Mathematics(2024), Volume 32, no. 2, 84-97.

7.1 Definition and some remarks

Let us recall the well known Bernstein-Durrmeyer operators:

$$M_n f(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t) f(t)\,dt,\ n \in \mathbb{N},\ f \in L^\infty([0,1]).$$

As we mentioned in the beginning, we will introduce an exponential variant of operators M_n as follows:

$$M_n^* f(x) = (n+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)e^{-\mu t}dt,\ f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}, \tag{7.1}$$

where $\mu > 0$ is a real parameter.

First, we have the following simple remark, which states that our operators preserve exponential function $\exp_\mu(x) = e^{\mu x}$, $x \in [0,1]$.

Remark 7.1.1. *(see Remark 1 from [44]) For $x \in [0,1]$ and $n \in \mathbb{N}$ we have that:*

$$M_n^* exp_\mu(x) = (n+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)dt = e^{\mu x}\sum_{k=0}^{n} p_{n,k}(x) = e^{\mu x}.$$

Next, the following identity holds.

Remark 7.1.2. *(see Remark 2 from [44]) For $f \in C([0,1])$, $n \in \mathbb{N}$, $x \in [0,1]$ we have*

$$M_n^\star f(x) = exp_\mu(x) M_n(f \exp_{-\mu})(x). \tag{7.2}$$

Note that our operators are constructed in a similar fashion as the operators obtained by Angeloni and Costarelli ([18]) from (2.10), however here we won't need the modification of the argument of the Bernstein basis polynomials $p_{n,k}$.

Let us recall the definition of the first order modulus of continuity

$$\omega_1(f,\delta) = \sup\{|f(t)-f(x)|,\ |t-x|<\delta\ x,t\in[0,1],\ \delta>0\}.$$

For $f \in C^1([0,1])$ we recall the definition of the second order modulus of smoothness

$$\omega_2(f,\delta) = \sup_{h\in[0,\delta]} \sup_{x\in[0,1-\frac{h}{2}]} \left|\Delta_h^2(f,x)\right|,$$

where $\Delta_h^2(f,x) = f(x) - 2f(x+h) + f(x+2h)$ is the second order finite difference of f, with step h.

Using the above moduli we will provide some quantitative estimates regarding the degree of approximation by our operators. Another such estimation will be given using the well-known Peetre's K- functionals

$$\mathcal{K}_j(f,\delta) = \inf_{g\in C^j([0,1])}\{\|f-g\|_\infty + \delta^j\|g^{(j)}\|_\infty\},\ f\in C([0,1]),\ \delta>0,\ j=1,2. \tag{7.3}$$

and the following relation between $\mathcal{K}_j$ and ω_j:

$$\mathcal{K}_j(f,\delta) \le C_j\omega_j(f,\delta),\ j=1,\ 2$$

where C_j is a constant depending only on j (for more details see [62]). Here by $C^j([0,1])$ we mean the space of functions having continuous j^{th} derivative on $[0,1]$ for $j=1,\ 2$.

7.2 Convergence results

Let $\gamma_i(x) = x^i \exp_\mu(x),\ i=0,\ 1,\ 2,\ x\in[0,1]$. Then in view of the following Lemma we can consider functions $\gamma_i,\ i=0,\ 1,\ 2$, as the test functions which will be used when verifying the conditions of Korovkin theorem for operators M_n^*.

Lemma 7.2.1. *(see Lemma 1 from [44]) The set* $\{\gamma_i(x)|\ i=0,\ 1,\ 2,\dots,\ x\in[0,1]\}$ *is a Chebyshev system.*

For test functions $\gamma_0,\ \gamma_1,\ \gamma_2$ the following result holds.

Lemma 7.2.2. *(see Lemma 2 from [44]) The following identities are true:*

$$M_n^*\gamma_0(x) = \gamma_0(x) = exp_\mu(x), \tag{7.4}$$

$$M_n^*\gamma_1(x) = \frac{n}{n+2}\gamma_1(x) + \frac{1}{n+2}\gamma_0(x) \tag{7.5}$$

$$M_n^*\gamma_2(x) = \frac{n(n-1)}{(n+2)(n+3)}\gamma_2(x) + \frac{n}{(n+2)(n+3)}\gamma_1(x) + \frac{2}{(n+2)(n+3)}\gamma_0(x) \tag{7.6}$$

Now, we can prove our convergence result.

Theorem 7.2.3. *(see Theorem 1 from [44]) Let $f \in C([0,1])$. Then $M_n^* f$ converges uniformly to f on $[0,1]$.*

Further, we will prove that our operators approximate functions belonging to L^p spaces. However, since our operators are defined as in (7.1) it will be in hand to work with a weighted version of L^p spaces, namely we will prove the convergence of operators M_n^* in $L_\mu^p([0,1])$. Having this in mind, we say that a function $f:[0,1]\to\mathbb{R}$ belongs to $L_\mu^p([0,1])$ if $f \in L^p([0,1])$ and

$$\|f\|_{p,\mu} = \left\{ \int_0^1 \left| e^{-\mu t} f(t) \right|^p dt \right\}^{\frac{1}{p}} < \infty. \tag{7.7}$$

Even though, if $f \in L_\mu^p([0,1])$ then $f \in L^p([0,1])$ (and conversely), our choice of the space is motivated by the definition of operators M_n^*.

Theorem 7.2.4. *(see Theorem 2 from [44]) Let $f \in L_\mu^p([0,1])$. Then:*

$$\|M_n^* f\|_{p,\mu} \le \|f\|_{p,\mu}. \tag{7.8}$$

Moreover,

$$\lim_{n\to\infty} \|M_n^* f - f\|_{p,\mu} = 0. \tag{7.9}$$

7.3 Voronovskaya theorem

In this section we will give a Voronovskaya result regarding our operators M_n^*. In what follows by $C^2([0,1])$ we mean the space of continuous functions on $[0,1]$ which admit a second derivative at $x \in (0,1)$.

Theorem 7.3.1. *(see Theorem 3 from [44]) If $f \in C^2([0,1])$ then, for $x \in [0,1]$:*

$$\begin{aligned} \lim_{n\to\infty} n(M_n^* f - f)(x) &= [\mu^2 x(1-x) - \mu(1-2x)] f(x) \\ &+ [1 - 2x - 2\mu x(1-x)] f'(x) + x(1-x) f''(x). \end{aligned} \tag{7.10}$$

7.4 Quantitative estimates

In this section we will provide some quantitative estimates of the approximation by operators M_n^*. Namely, we have the following result.

Theorem 7.4.1. *(see Theorem 4 from [44]) For $f \in C([0,1])$, $n \in \mathbb{N}$, $x \in [0,1]$ and $\delta > 0$, we have*

$$|f(x) - M_n^\star f(x)| \le e^{\mu x} \left[1 + \frac{(2n-6)x(1-x)+2}{\delta^2 (n+2)(n+3)} \right] \omega_1 \left(f \cdot exp_{-\mu}, \delta \right), \tag{7.11}$$

and

$$|f(x) - M_n^\star f(x)| \le e^{\mu x}\left[\frac{|1-2x|}{\delta(n+2)}\omega_1\left(f\cdot exp_{-\mu},\delta\right)\right.$$
$$\left.+\left(1+\frac{(2n-6)x(1-x)+2}{2\delta^2(n+2)(n+3)}\right)\omega_2\left(f\cdot exp_{-\mu},\delta\right)\right], \tag{7.12}$$

Consequently, we have

$$\|f - M_n^\star f\| \le \frac{3}{2}e^{\mu}\cdot\omega_1\left(f\cdot exp_{-\mu},\frac{1}{\sqrt{n}}\right), \tag{7.13}$$

and

$$\|f - M_n^\star f\| \le e^{\mu}\cdot\left[\frac{1}{\sqrt{n}}\cdot\omega_1\left(f\cdot exp_{-\mu},\frac{1}{\sqrt{n}}\right)+\frac{5}{4}\omega_2\left(f\cdot exp_{-\mu},\frac{1}{\sqrt{n}}\right)\right]. \tag{7.14}$$

Further, another estimate using ω_1 can be given in the following theorem.

Theorem 7.4.2. *(see Theorem 5 from [44]) Let $f \in C([0,1])$. For $x \in [0,1]$ and $n \in \mathbb{N}$, we have that:*

$$|f(x) - M_n^* f(x)| \le \left(e^{2\mu x}+\frac{1}{2}e^{\mu x}\right)\omega_1\left(f\cdot e^{-\mu\cdot},\frac{1}{\sqrt{n}}\right). \tag{7.15}$$

In what follows, we will proceed with giving an estimate of the degree of approximation, of functions belonging to $C([0,1])$, in terms of the moduli ω_1 and ω_2 using Peetre's K-functionals from (7.3) and the relation between them and these moduli.

Theorem 7.4.3. *(see Theorem 6 from [44]) If $f \in C([0,1])$ and $n \in \mathbb{N}$ we have that:*

$$\|M_n^* f - f\|_\infty \le \frac{\mu^2 e^{\mu}+4\mu}{4n}\|f\|_\infty + K_{n,\mu}^1\omega_1\left(f,\frac{e^{\mu}(2\mu+1)+4}{e^{\mu}(8n+\mu^2)+4\mu}\right) \tag{7.16}$$
$$+K_{n,\mu}^2\omega_2\left(f,\sqrt{\frac{e^{\mu}(2\mu+1)+4}{e^{\mu}(8n+\mu^2)+4\mu}}\right),$$

where $K_{n,\mu}^1 = C_1\frac{\left[e^{\mu}(8n+\mu^2)+4\mu\right](\mu e^{\mu}+2)}{2n[e^{\mu}(2\mu+1)+4]}$ and $K_{n,\mu}^2 = C_2\frac{e^{2\mu}(8n+\mu^2)+4\mu e^{\mu}}{4n[e^{\mu}(2\mu+1)+4]}$, with C_1 and C_2 constants.

7.5 Simultaneous approximation

In what follows we will provide a simultaneous approximation result concerning our operators. In order to do this we will need the r^{th} order derivative of M_n^*. First let us recall a result which is due to Derriennic (see [34]):

Lemma 7.5.1. *Let $f \in C^r([0,1])$. Then, for $x \in [0,1]$, $n \in \mathbb{N}$:*

$$\frac{d^r}{dx^r}M_n f(x) = \frac{(n+1)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^1 p_{n+r,k+r}(t)f^{(r)}(t)dt. \tag{7.17}$$

Next, we will make the following notation

$$Q_{n,j}f(x) = \frac{(n+1)!n!}{(n-j)!(n+j)!} \sum_{k=0}^{n-j} p_{n-j,k}(x) \int_0^1 p_{n+j,k+j}(t) f(t) dt. \tag{7.18}$$

Remark 7.5.2. *We have that:*

$$\frac{d^j}{dx^j}(M_n f)(x) = Q_{n,j} f^{(j)}(x). \tag{7.19}$$

From (7.18) we can see that operators $Q_{n,j}$ are linear and positive and the following Lemma, due to Derriennic (see [34]), holds.

Lemma 7.5.3. *Let $f \in C([0,1])$ be a function. Then $Q_{n,j}f$ converges uniformly to f.*

Now, we can proceed with the main result of this section.

Theorem 7.5.4. *(see Theorem 7 from [44]) Let $f \in C^r([0,1])$ and $s = 0, 1, \ldots, r$. Then we have that:*

$$\lim_{n\to\infty} \frac{d^s}{dx^s}(M_n^* f) = f^{(s)}, \tag{7.20}$$

uniformly on $[0,1]$.

8 Conclusions

A great part of this thesis deals with the study of problems in approximation theory by using power series of operators. In this sense, we obtained convergence theorems of the power series of positive linear operators sequences in two different contexts, one based on Voronovskaya theorems and the other one using C_0-semigroups of operators. In the first context we also obtained a result concerning the iterates of positive linear operators which proved to be an explicit representation of the Voronovskaya Theorem for Micchelli combinations of positive linear operators. So far, to our knowledge, there is no such result except a partial representation of the limit in the Voronovskaya Theorem mentioned above, given only for Bernstein operators. Apart from this, we also studied Voronovskaya theorems concerning a class of operators obtained through geometric series of Bernstein-Durrmeyer operators.

Another research direction approached is the study of some operators obtained as modifications of the exponential kind of Kantorovich and Durrmeyer type operators. This direction is connected to recent advances in this sense from the current literature on the subject.

The subjects addressed in this thesis can be continued by more studies of the power series constructed with positive linear operators and the thesis opens further research concerning the connection between C_0-semigroups and approximation problems and also further generalizations of some classes of operators obtained using different modifications.

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1. **Ș. Garoiu**, *A Voronovskaya type theorem associated to geometric series of Bernstein-Durrmeyer operators*, Carpathian Journal of Mathematics, 41(2), (2025).

2. **Ș. Garoiu**, *Exponential Bernstein-Durrmeyer operators*, General Mathematics, 32(2), (2024), 84-97.

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4. **Ş. Garoiu**, R. Păltănea, *Generalized Voronovskaya theorem and the convergence of power series of positive linear operators*, J. Math. Anal. Appl., 531(2)(2), (2024).

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