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Contributions to constructive approximation theory

SUMMARY

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Contents

1 Introduction

1.1 Considerations on constructive approximation theory

The topics studied in this doctoral thesis are part of the mathematical field of approximation theory. Approximation theory can be seen as a link between pure and applied mathematics. The primary concern of the topic is approximation of real-valued continuous functions by some simpler, more manageable functions. Another point of interest is the quantitative approximation as well as the error of approximation.

The fundamentals of this field were established by the work of some mathematicians among which we can mention K. Weierstrass, S. N. Bernstein, D. Jackson, P. P. Korovkin, G. G. Lorentz and many others.

Moreover, this research field has history in our country. This field was intensively studied by great mathematicians such as T. Popoviciu, D. D. Stancu and A. Lupaş.

One of the most important results, known in literature as the *First Wierstrass approximation theorem*, stated by K. Weierstrass in [116] states that for any continuous function $f \in C([a,b])$ and any $\varepsilon > 0$, there is a real coefficients polynomial function $p(x)$, such that $|f(x) - p(x)| < \varepsilon$, for any $x \in [a,b]$. The most famous proof of Weierstrass's approximation theorem was proposed by S. N. Bernstein in [18], where the author provided a constructive method which led to the well-known Bernstein operators. These operators are defined as follows:

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \; x \in [0,1], \; f \in C([0,1]),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } 0 \le k \le n,$$

and $p_{n,k}(x) = 0$ for $k > n$.

After these operators were introduced, many researchers found a lot of properties concerning them. For example see the following [14, 19, 26, 30, 81, 82, 100, 115].

Also, for extending the family of functions to be approximated, one can find a lot of other operators, such as Kantorovich's, where the function to be approximated should be integrable on $[0,1]$ ($f \in L_1([0,1])$),

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \; x \in [0,1], \; f \in L_1([0,1]),$$

where $p_{n,k}$ is defined above. Kantorovich operators were intensively studied and some of their modification represent an ongoing research topic, for example see: [59, 61, 94, 106, 107].

Another generalization, also for integrable functions on $[0,1]$, was given by J. L. Durrmeyer in [43] and independently by A. Lupaş in [72]:

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)dt, \; f \in L_1([0,1]), \; x \in [0,1],$$

with $p_{n,k}$ defined above. For more results concerning Durrmeyer's operators, see [38, 72, 79, 85].

Other operators that are very important in literature and also relevant for the results obtained in this thesis are mentioned in preliminaries chapter, where we will recall some of their properties which helped in the proofs of our results.

Some comprehensive and useful expositions on the constructive approximation topic can be consulted in the following references [9, 12, 35, 37, 86, 93, 103].

1.2 Motivations for choosing the theme

Approximation theory, as a branch of mathematics, represent a bridge between pure and applied mathematics. One can mention various applications in significant areas of research, such as:

- constructive approximation;
- interpolation;
- probability theory;
- functional analysis;
- operator theory;
- numerical analysis;
- computer aided geometrical design;
- machine learning.

1.3 Structure of the thesis

In this thesis, we present new contributions to approximation by linear positive or non-positive operators and constructive approximation theory. The results are structured in five chapters. First chapter is dedicated to establishing the notations and terminology as well as mentioning the mathematical objects that are used in this thesis. In the second chapter we present some non-positive Durrmeyer type operators. Here are discussed two classes of Durrmeyer type operators which are linear but not positive operators on their entire domain of definition. Third chapter contains new classes of Stancu-Kantorovich type operators modified in King sense. These operators are presented in a different chapter because they posses the positivity property. Here we discuss three new methods of obtaining operators of Stancu-Kantorovich type. Fourth chapter is dedicated to non-positive Kantorovich type operators attached to some linear differential operators. In the first section of this chapter we discuss only the generalization of Bernstein operators in Kantorovich's sense and in the second section we propose a method to obtain a generalization of Kantorovich operators that posses some properties such as simultaneous approximation. The fifth chapter is dedicated to introducing a double weighted second order modulus of continuity. In this chapter we propose a new second order modulus depending on two weight functions. An application of this modulus is presented for Szász-Mirakjan operators.

The first chapter, **Preliminaries**, contains the notations and terminology as well as some key concepts that are essential for the results presented in this thesis. Here we also present some particular operators that are later generalized or used in examples.

Chapter two, **Non-positive Durrmeyer type operators**, is dedicated to introducing some new Durrmeyer type operators that are not positive on the entire interval $[0, 1]$. The main results in this chapter are part of the papers **"On approximation properties of some non positive Bernstein-Durrmeyer type operators"**, An. Şt. Univ. Ovidius Constanţa, Vol. 21(1), 2023; and **"On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense"**, published in Dolomites Research Notes on Approximation, 16(3), 104-117, 2023. The results presented in this chapter, concerning the two classes of operators mentioned above, are proved using some new methods that are not always straightforward.

Chapter three, **Kantorovich type operators modified in King sense** contains three new classes of positive linear operators of Stancu-Kantorovich type. The results in this chapter are part of the paper **"On New Classes of Stancu-Kantorovich-Type Operators"**, Mathematics 2021, which is published in collaboration with Ştefan Lucian Garoiu and Cristina Maria Păcurar. Here, we propose some operators that preserve two of the test functions $e_i,\ i \in \{0, 1, 2\}$ and prove that they are approximation operators. We also study the rate of convergence in each case using the first order modulus of continuity.

Chapter four, **Non-positive Kantorovich type operators attached to some linear differential operators**, is structured in two sections. First section is dedicated to some general Bernstein-Kantorovich operators, where we propose a modification of Bernstein operators using a linear differential operator with constant coefficients. For this class of operators we prove an approximation result, a Voronovskaja type result and also a simultaneous approximation result. We conclude the first section by providing a counterexample that proves the non-positivity of these operator. The results presented in this section are part of two papers **"Approximation Properties of Some Non-positive Kantorovich Type Operators"**, 2022 Proceedings of International E-Conference on Mathematical and Statistical Sciences: A Selçuk Meeting (2022), 188-194, and **Voronovskaja type theorem for some non-positive Kantorovich type operators**, Carpathian Journal of Mathematics Vol. 40, No. 1 (2024), 187-194. The second section of this chapter presents some generalized Kantorovich operators. Here, we propose a Kantorovich modification using a linear differential operator with non constant coefficients. The results in this section are presented for an arbitrary sequence of positive linear operators L_n possessing the simultaneous approximation property. For these operators we provide an approximation result and a Voronovskaja type result. In the case where the coefficients of the differential operator are constant, we were also able to prove a simultaneous approximation result. The section ends with generalizations of some classical operators. The results can be found in the paper **Generalized Kantorovich operators**, General Mathematics, Vol. 32, No.2 (2025), 67-83.

Chapter five, **Double weighted second order modulus**, is dedicated to introducing a new second order modulus depending on two weight functions. This new modulus is useful in order to obtain estimates of the degree of approximation of functions with fast growth to infinity, by general positive linear operators which preserve polynomials of degree one. The chapter concludes with an example for Szász-Mirakjan operators. The results in this chapter can be found in the paper **"Double weighted modulus"**, submitted for publishing, which was obtained in collaboration with professor Radu Păltănea.

In summary, this thesis presents a comprehensive study and research of approximation by positive linear operators, linear operators which are not positive and of constructive approximation theory. This thesis proposes new methods and frameworks and also covers fundamental concepts of the theory.

1.4 Original results contained in the thesis

The original results contained in this thesis are the following:

A *On approximation properties of some non-positive Bernstein-Durrmeyer type operators*

The novelty brought with this paper is related to introducing a new class of Bernstein-Durrmeyer operators that are not positive on the entire interval $[0, 1]$. The lack of positivity led to proposing new methods of proving the approximation result without using the classical Korovkin theorem on the part of the interval on which the operators are not positive.

B *On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense*

Here, we introduce a new class of Bernstein-Durrmeyer operators modified in Bezier-King sense. These operators are not positive on the entire interval $[0, 1]$. Here, in order to prove the approximation of continuous functions on all $[0, 1]$ we proceed in two different manners on the part of the interval where the operators are positive and on the part where they are not. In this paper we prove some results concerning the rate of approximation using the first and second order moduli of smoothness. Also, we provide a Voronovskaja type result.

C *On New Classes of Stancu-Kantorovich-Type Operators*

We introduce three classes of approximation operators that preserve two of the test functions $e_0,\ e_1,\ e_2$ at a time. We prove that the approximation holds on a interval $I \subset [0, 1]$ in each case. The operators studied here are positive linear operators.

D *Approximation Properties of Some Non-positive Kantorovich Type Operators*

We propose a new class of Bernstein-Kantorovich type operators constructed using a linear differential operator with constant coefficients. We prove that the finite differences of order k of a function F on equidistant knots uniformly approximate the k-th derivative of the function F. This result helps us prove that these new operators of Bernstein-Kantorovich type are approximation operators for all continuous functions on $[0, 1]$. We conclude by stating the non positivity of these operators.

E *Voronovskaja type theorem for some non-positive Kantorovich type operators*

In this paper we prove a Voronovskaja type theorem and a simultaneous approximation result for the operators introduced in the paper above.

F *Generalized Kantorovich operators*

We introduce a sequence of more general Kantorovich type operators defined using a linear differential operator with non-constant coefficients. The approximation result and the Voronovskaja type theorem proved here are given for a general sequence of operators $L_n \in C([0,1])$. We conclude this paper by providing examples for some classical operators and a counterexample for each, which proves that their Kantorovich variant is not positive.

G *Double weighted modulus*

We introduce a new second order modulus depending on two weight functions. This new modulus is useful in order to obtain estimates of the degree of approximation of functions with fast growth to infinity, by general positive linear operators which preserve polynomials of degree one. The results conclude with an example for Szász-Mirakjan operators.

1.5 Dissemination of the results

The results mentioned in the previous section were disseminated in the mathematical community in form of published papers in international journals and also as communications at conferences and workshops as follows:

A In the framework of "44th summer symposium in real analysis" held between 20-24 Jun 2022 Paris and Orsay, France, I presented a talk entitled *"On Approximation Properties of some non-positive Bernstein-Durrmeyer Type Operators"*.

Also, at the conference "Functional Analysis, Approximation Theory and Numerical Analysis" held between 5-8 July 2022, Matera Italy, I held the talk entitled *"On Approximation Properties of some non-positive Bernstein-Durrmeyer Type Operators"*.

I published the paper: **B. I. Vasian**, "On Approximation Properties of some non-positive Bernstein-Durrmeyer Type Operators", An. Şt. Univ. Ovidius Constanţa, Vol 31(1), 2023.

B In the framework of the 14th edition of "International conference on approximation theory and its applications", Alexandru Lupaş, Sibiu, September 12-14 2022, I presented a talk entitled "Approximation properties of some non-positive Kantorovich type operators".

C In the framework of "International E-Conference on Mathematical and Statistical Sciences: A Selcuk Meeting" 2022 (ICOMSS'22), I presented a talk entitled "Approximation properties of some non-positive Kantorovich type operators".

I published the paper: **B. I. Vasian**, Approximation Properties of Some Non-positive Kantorovich Type Operators, 2022 Proceedings of International E-Conference on Mathematical and Statistical Sciences: A Selçuk Meeting (2022), 188-194.

D In the framework of 4th Edition of MACOS 2022, "International Conference on Mathematics and Computer Science" held between 15-17 September, 2022, Braşov, Romania, I presented the talk entitled "On approximation properties of some non-positive linear operators".

E In the framework of 5th Edition of MACOS 2024, "International Conference on Mathematics and Computer Science" held between 13-15 June, 2024, Braşov, Romania, I presented the talk entitled "Generalized Kantorovich operators".

I published the paper **B. I. Vasian**, Generalized Kantorovich operators, General Mathematics, Vol. 32, No. 2 (2025), 67-83.

F Other published papers:

B. I. Vasian, Voronovskaja type theorem for some non-positive Kantorovich type operators, Carpathian Journal of Mathematics Vol. 40, No. 1 (2024), 187-194.

B. I. Vasian, On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense, Dolomites Research Notes on Approximation, 16(3) (2023), 104-117.

B. I. Vasian, Ş. L. Garoiu, C. M. Păcurar, On New Classes of Stancu-Kantorovich-Type Operators, Mathematics (2021).

R. Păltănea, **B. I. Vasian**, *Double weighted modulus*, submitted for publishing.

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2 Preliminaries

2.1 Notations and terminology

Let $I \subset \mathbb{R}$ be an interval.

We denote by $\mathcal{F}(I) = \{f : I \to \mathbb{R}\}$ the set of all real functions defined on I. With $\mathcal{B}(I) = \{f : I \to \mathbb{R} \ : \ f \text{ bounded}\}$ we denote the set of all bounded functions defined on I. By $C(I) = \{f : I \to \mathbb{R} \ : \ f \text{ continuous}\}$ we denote the set of all continuous real functions defined on I.

For $k \in \{0, 1, \dots\}$, by $C^k(I)$ we understand the set of all continuously differentiable functions of order k. In particular, by $C^0(I)$ we mean $C(I)$.

We denote by $\mathcal{P}(I)$ the set of all polynomials on I and with Π_k the set of polynomials of degree at most k.

By test function or monomial function we mean $e_j(t) = t^j$, $j \in \{0, 1, \dots\}$.

Let X be a Banach space. We will denote by $\|\cdot\|_X$ the norm on X.

If $X = \mathcal{B}(I)$, then by the norm of a function $\|f(\cdot)\|$, $f \in \mathcal{B}(I)$, on I we mean the supremum norm:

$$\|f\| = \sup_{x \in I} |f(x)|, \ \ f \in \mathcal{B}(I). \tag{2.1}$$

Further we will enlist some operators that are used in the thesis:

- The identity operator Id which satisfy $Id(f) = f$, $f \in \mathcal{F}(I)$;
- Big-O Landau symbol : $O(f(x)) = \{g(x) \ : \ \exists\, c, x_0 > 0 \text{ such that } 0 \leq f(x) \leq cg(x), \ \forall\, x \geq x_0\}$;
- Little-o Landau symbol: $o(f(x)) = \{g(x) \ : \ \forall\, c > 0, \ \exists\, x_0 > 0 \text{ such that } 0 \leq f(x) < cg(x), \ \forall\, x \geq x_0\}$.

Another useful operator is the first *finite difference* operator with step k of a function f:

$$\Delta_k f(x) = f(x + k) - f(x). \tag{2.2}$$

The l-th iterate of Δ_k is denoted by Δ_k^l and is defined as follows

$$\Delta_k^l f(x) = \Delta_k \left[\Delta_k^{l-1} f(x)\right], \tag{2.3}$$

and from the identity above, one can obtain the following formula:

$$\Delta_k^l f(x) = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} f(x + ik). \tag{2.4}$$

Proposition 2.1.1. *[35] If f is a polynomial of degree $l-1$ then $\Delta_k^l f(x) = 0$.*

Let I be an interval and $x_0, x_1, \dots, x_n \in I$, $n+1$ distinct points of I. Let f be a function defined on I. The divided differences of f on $x_0, x_1, \dots, x_n$ are given by

$$\begin{aligned} f[x_k] \ &: = f(x_k), \ k = \overline{0, n}, \qquad (2.5)\\ f[x_k, x_{k+1}, \dots, x_{k+p}] \ &: = \frac{f[x_{k+1}, \dots, x_{k+p}] - f[x_k, x_{k+1}, \dots, x_{k+p-1}]}{x_{k+p} - x_k}, \end{aligned}$$

for $k = \overline{0, n-p},\ j = \overline{0, n}.$

For the divided differences we mention the following:

Proposition 2.1.2. *[35] If f is a polynomial of degree $< n$, then*

$$f\left[x_0, x_1, \ldots, x_n\right] = 0. \tag{2.6}$$

Proposition 2.1.3. *[35] (Mean value theorem for divided differences) If f is n times differentiable, then*

$$f\left[x_0, x_1, \ldots, x_n\right] = \frac{f^{(n)}(\xi)}{n!}, \tag{2.7}$$

for $\xi \in \left(\min_{k\in\{0,1,\ldots n\}} x_k, \max_{k\in\{0,1,\ldots n\}} x_k\right).$

Proposition 2.1.4. *[35] The following relation between finite differences and divided differences holds:*

$$f\left[x, x+h, \ldots, x+lh\right] = \frac{1}{l!h^l}\Delta_h^l f(x). \tag{2.8}$$

In the theory of approximation by operators, moduli of continuity were proved to be very useful. These mathematical objects can be used for measuring the smoothness of a function in a more elegant way.

Definition 2.1.5. *[35] Let $I \subset \mathbb{R}$ be an interval and $f \in \mathcal{F}(I)$. The modulus of continuity of a function f is given by*

$$\omega_1(f,h) = \sup\{|f(x) - f(y)| \ :\ |x-y| \le h;\ x, y \in I\},\ h \ge 0. \tag{2.9}$$

In the case of a continuous function f on an interval $I \subset \mathbb{R}$ satisfying $\omega_1(f,h) = o(h)$, one obtains that f is constant, therefore the modulus of continuity is not useful for measuring higher smoothness. In this case, the moduli of smoothness are needed. These moduli are connected with higher order finite differences.

Definition 2.1.6. *[35] Let $I \subset \mathbb{R}$ and $f \in C(I)$, for I compact. The l-th order modulus of smoothness of f is defined as:*

$$\omega_l(f,h) = \sup_{0<k\le h} \|\Delta_k^l(f,x)\|,\ h \ge 0. \tag{2.10}$$

The moduli of smoothness presented above represent a very useful tool for approximation problems. However, in the recent research, the classical moduli of smoothness proved to be inefficient. To answer these shortcomings, the following moduli were introduced, which will be called in this thesis weighted moduli of smoothness for a function f.

Definition 2.1.7. *[37] Let $I \subset \mathbb{R}$ and $f \in \mathcal{F}(I)$. The weighted modulus of smoothness for f is given by:*

$$\omega_l^{\varphi}(f,h) = \sup_{0<k\le h} \|\Delta_{k\varphi(\cdot)}^l(f,\cdot)\|,\ h \ge 0, \tag{2.11}$$

where the function $\varphi(x)$ is chosen in relation to the problem which arises.

Remark 2.1.8. *The step* $k\varphi(x)$ *in the definition above varies with* $x \in I$.

Remark 2.1.9. *If* $\varphi(x) \equiv 1$, *then* (2.11) *is the classical modulus of smoothness defined in* (2.10).

The function $\varphi(x)$ from the definition of the weighted modulus of smoothness, also called the *weight function*, is defined for $x \in I$ (where $I = (a, b)$ with $a \in \{-\infty, 0\}$ and $b \in \{1, \infty\}$), should satisfy the following:

Proposition 2.1.10. *[37]*

A $\varphi = 1$ *locally;*

B *there are two values* $\gamma(a)$ *and* $\gamma(b)$ *such that* $\gamma(0) \geq 0$, $\gamma(1) \geq 0$ *and* $\gamma(\pm\infty) \leq 1$ *for which*

$$\varphi(x) \simeq \begin{cases} |x|^{\gamma(a)}, & x \to a+ \text{ for } a \in \{-\infty, 0\} \\ x^{\gamma(\infty)}, & x \to \infty \text{ for } b = \infty \\ (1-x)^{\gamma(1)}, & x \to 1- \text{ for } b = 1 \end{cases}; \tag{2.12}$$

C $\varphi(x)$ *is a measurable function (with respect to a measure* μ*) and there exists* M_0, k_0 *constants, such that for each* $0 < k \leq k_0$ *and every finite interval* $J \subset I$, *the following measure inequality holds:*

$$\mu\left(\{x \in I : x \pm h\varphi(x) \in J\}\right) \leq M_0 \mu(E). \tag{2.13}$$

Next, we will recall the definitions of K-functionals, K-functionals of order l and weighted K-functionals.

Definition 2.1.11. *[35] Let* X, Y *be two Banach spaces such that* $Y \subset X$ *continuously embedded. The* K*-functional of* $f \in X$ *is defined by*

$$K(f,t) := K(f,t;X,Y) := \inf_{g \in Y}\{\|f-g\|_X + t\|g\|_Y\}, \ t \geq 0. \tag{2.14}$$

Definition 2.1.12. *The* l*-th order* K*-functional of a function* $f \in \mathcal{F}(\mathcal{I})$ *is given by*

$$K_l(f,t^l) = \inf_{g \in C^l(I)}\{\|f-g\| + t^l\|g^{(l)}\|\}, t \geq 0. \tag{2.15}$$

Definition 2.1.13. *The* l*-th order weighted* K*-functional of* $f \in \mathcal{F}(\mathcal{I})$ *is defined as*

$$K_l^{\varphi}(f,t^l) = \inf_{g \in C^l(I)}\{\|f-g\| + t^l\|\varphi^l g^{(l)}\|\}, t \geq 0. \tag{2.16}$$

One of the most important results concerning K-functionals and moduli of smoothness is the following equivalence theorem.

Theorem 2.1.14. *[35] Suppose* φ *satisfies the conditions in Proposition 2.1.10,* $l \in \{0, 1, \dots\}$ *and* $f \in C(I)$, *where* $I = (0, 1)$, $I = (0, \infty)$ *or* $I = \mathbb{R}$, *then*

$$C_1\omega_l^{\varphi}(f,t) \leq K_l^{\varphi}(f,t^l) \leq C_2\omega_l^{\varphi}(f,t), \ 0 < t \leq t_0, \tag{2.17}$$

where $C_1, C_2, t_0 > 0$ *are constants.*

2.2 Moduli of continuity

2.3 Moduli of smoothness

2.4 Weighted moduli of smoothness

2.5 K-functionals

2.6 Operators

2.7 Function approximation by sequences of positive and linear operators

2.8 Some particular operators

2.9 King type operators

2.10 Bézier curves

3 Non-positive Durrmeyer type operators

In this chapter, we study some Durrmeyer type operators. For these operators we have proved some approximation results along with error estimation and Voronovskaja type theorems. The results in this chapter are based on the work published in two papers: **Vasian B. I.**, *On approximation properties of some non-positive Bernstein-Durrmeyer type operators*, An. Șt. Univ. Ovidius Constanța, Vol. 21(1), 2023; and **Vasian B. I.**, *On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense*, Dolomites Research Notes on Approximation, 16(3), 104-117, 2023.

3.1 On approximation properties of some non-positive Bernstein-Durrmeyer type operators

In this first section we shall present the results on a new type of Bernstein Durrmeyer operators which are not positive on the entire interval $[0,1]$. For these operators we will prove a uniform convergence result on all continuous functions on $[0,1]$ as well as a result given in terms of modulus of continuity ω_1. A Voronovskaja type theorem will be proved as well.

These results have been published in the paper **B.I. Vasian**,*On approximation properties of some non-positive Bernstein-Durrmeyer type operators*, An. Șt. Univ. Ovidius Consstanța, Vol. 21(1), 2023.

As we have seen, Durrmeyer operators presented in (1.1) posses some powerful approximation properties.

Let us introduce the Durrmeyer type modification we will further study.

Definition 3.1.1. *[108]*

Let $\alpha \geq 0$. For every $f \in C([0,1])$, we define:

$$D_n^{\alpha}\left(f,x\right) = (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}\left(x\right)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}\left(t\right) f\left(t\right) dt,\ \ x \in [0,1]\,, \tag{3.1}$$

where $p_{n,k}^{\alpha}\left(x\right) = \left(\frac{n+\alpha}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+\alpha} - x\right)^{n-k}$, $n,k \in \mathbb{N},\ k \leq n$.

Remark 3.1.2. *[108] For $\alpha = 0$ we obtain the classical Bernstein-Durrmeyer operators, and for $\alpha = 1$ we get the operators studied by Deo N. et al. in paper [33].*

Remark 3.1.3. *[108] $D_n^{\alpha}\left(f,x\right)$ defined in (3.1) is a linear operator which is positive for $x \in \left[0, \frac{n}{n+\alpha}\right]$ and non-positive on $\left(\frac{n}{n+\alpha}, 1\right]$.*

In order to prove our results, we need the following.

Lemma 3.1.4. *[74] We have the following recurrence relation:*

$$x\left(\frac{n}{n+\alpha}-x\right)\left(p_{n,k}^{\alpha}(x)\right)'=n\left(\frac{k}{n+\alpha}-x\right)p_{n,k}^{\alpha}(x)\,,\;\; x\in[0,1] \tag{3.2}$$

Further, we will need some results concerning the operator D_n^{α}.

Lemma 3.1.5. *[108] We have the following:*

$$\int_0^{\frac{n}{n+\alpha}} t^{k+s}\left(\frac{n}{n+\alpha}-t\right)^{n-k}dt=\left(\frac{n}{n+\alpha}\right)^{n+s+1}B\left(k+s+1,n-k+1\right), \tag{3.3}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Proposition 3.1.6. *[108] Operators D_n^{α} satisfy the following relations:*
i) $D_n^{\alpha}(e_0,x)=1$;
ii)$D_n^{\alpha}(e_1,x)=\frac{n}{n+2}x+\frac{n}{(n+\alpha)(n+2)}$;
iii)$D_n^{\alpha}(e_2,x)=\frac{n(n-1)}{(n+2)(n+3)}x^2+\frac{4n^2}{(n+2)(n+3)(n+\alpha)}x+\frac{2n^2}{(n+2)(n+3)(n+\alpha)^2}$,
where $x\in[0,1]$.

Now, we denote by $M_{n,m}(x)$ the m-th order moments for operators D_n^{α}, which have the following expression:

$$M_{n,m}(x)=D_n^{\alpha}\left((t-x)^m,x\right) \tag{3.4}$$

$$=(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}}(t-x)^m\cdot p_{n,k}^{\alpha}(t)\,dt.$$

Theorem 3.1.7. *(See Theorem 6 in [108]) The following recurrence relation holds:*

$$(m+n+2)\,M_{n,m+1}(x)=x\left(\frac{n}{n+\alpha}-x\right)\left[2mM_{n,m-1}(x)+M_{n,m}'(x)\right] \tag{3.5}$$

$$+(m+1)\left(\frac{n}{n+\alpha}-2x\right)M_{n,m}(x).$$

With all of the above, we are able to state our first result concerning approximation properties of D_n^{α}.

Theorem 3.1.8. *(See Theorem 8 in [108]) For all $\alpha\geq 0$, $f\in C([0,1])$, and for all $\varepsilon\in(0,1)$, the following holds:*

$$\lim_{n\to\infty}D_n^{\alpha}(f)=f,\;\textit{uniformly on}\;[0,1-\varepsilon]. \tag{3.6}$$

As for the above result, we were able to prove the uniform convergence only on the interval where operators D_n^{α} are positive. Our next aim is to prove that the operators D_n^{α} can approximate all continuous functions on entire $[0,1]$, even though they are not positive operators on the entire interval.

Proposition 3.1.9. *(See Proposition 9 in [108]) For $l\in\{0,1,\dots\}$ we have:*

$$D_n^{\alpha}(e_l,x)=(n+1)\frac{(n!)^2}{(n+l+1)!}\sum_{i=0}^{\min\{n,l\}}\binom{l}{i}\frac{l!}{i!}\frac{1}{(n-i)!}\left(\frac{n}{n+\alpha}\right)^{l-i}x^i. \tag{3.7}$$

With the above result, we can state the following:

Proposition 3.1.10. *(See Proposition 10 in [108]) For all $l \in \{0, 1, \dots\}$, we have*

$$D_n^\alpha (e_l) \to e_l \text{ uniformly on } [0,1]. \tag{3.8}$$

Remark 3.1.11. *[108] From Proposition 3.1.10 and the linearity of the operators D_n^α, we conclude that for all polynomials $P \in \mathcal{P}([0,1])$, the convergence $D_n^\alpha (P, x) \to P(x)$ holds uniformly for $x \in [0,1]$.*

With all of the above results, we are closer to prove the uniform convergence for all continuous functions on $[0,1]$. We also need to prove that the norm of the operators D_n^α is bounded. For this, we have the following result.

Proposition 3.1.12. *(See Proposition 12 in [108]) We have:*

$$\|D_n^\alpha\| \leq e^{2\alpha}, \tag{3.9}$$

for all $n \in \{1, 2, \dots\}$, and $\alpha \geq 0$.

Theorem 3.1.13. *(See Theorem 13 in [108]) For all $f \in C([0,1])$, we have:*

$$\lim_{n\to\infty} D_n^\alpha (f) = f \text{ uniformly on } [0,1]. \tag{3.10}$$

The following results are dedicated to the estimation of the error of approximation using the first modulus of continuity $\omega_1(f, \delta)$.

Theorem 3.1.14. *(See Theorem 14 in [108]) For $f \in C([0,1])$ and $x \in [0,1]$ we have:*

$$|D_n^\alpha (f, x) - f(x)| \leq \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{2}{n+2} \left[x \left(\frac{n}{n+\alpha} \right) + \frac{1}{n+3} \right]} \right\} \omega_1 (f, \delta), \tag{3.11}$$

for $x \in \left[0, \frac{n}{n+\alpha}\right]$, and

$$|D_n^\alpha (f, x) - f(x)| \leq \left\{ e^{2\alpha} + \frac{e^{2\alpha}}{\delta'} \left[\frac{2\alpha}{n} + \frac{n}{(n+\alpha)(n+2)} \right] \right\} \omega_1 (f, \delta'), \tag{3.12}$$

for $x \in \left(\frac{n}{n+\alpha}, 1 \right]$.

Remark 3.1.15. *[108] If we take*

$$\delta = \delta' = \max \left\{ \sqrt{\frac{2}{n+2} \left[\frac{n}{n+\alpha} + \frac{1}{n+3} \right]}, \frac{2\alpha}{n} + \frac{n}{(n+\alpha)(n+2)} \right\}, \tag{3.13}$$

then

$$\| D_n^\alpha(f) - f \| \leq 2e^{2\alpha} \omega(f, \delta). \tag{3.14}$$

3.1.1 Voronovskaja type result

In this section, we will prove another result measuring the error of approximation.

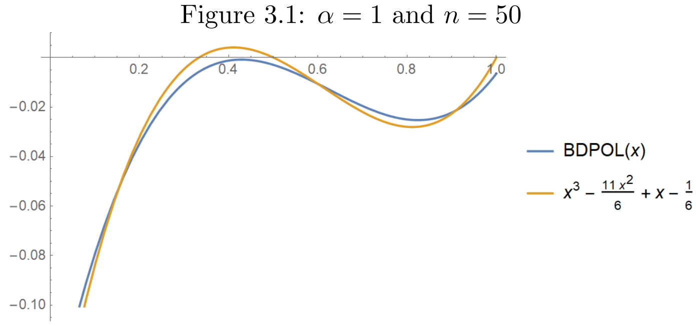
Theorem 3.1.16. *(See Theorem 16 in [108]) Let* $f \in C([0,1))$ *be a bounded, two times differentiable function at the point* $x \in (0,1)$. *Then, the following limit holds:*

$$\lim_{n\to\infty} n\left[D_n^{\alpha}(f,x) - f(x)\right] = (1-2x)\, f'(x) + x(1-x)\, f''(x)\,. \tag{3.15}$$

3.1.2 Some graphs

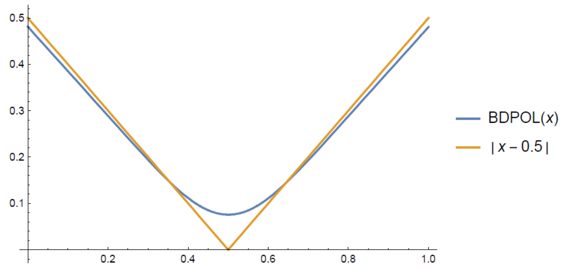
For the first example we considered the function $f(x) = x^3 - (11/6)x^2 + x - 1/6$ for $x \in [0,1]$. In this case we have obtained Figure 3.1.

Figure 3.1: $\alpha = 1$ and $n = 50$



Secondly we took the function $f(x) = |x - 0.5|$ for $x \in [0,1]$ and we got Figure 3.2.

Figure 3.2: $\alpha = 1$ and $n = 50$



As it can be seen from the figures and from the proved result, the operators D_n^{α} have good properties of approximation even though they are not positive operators on the entire $[0,1]$.

3.2 On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense

This section is dedicated to some results concerning Bernstein-Durrmeyer type operators which are defined using the methods proposed by King and Bezier. The operators obtained here are also not positive on the entire $[0,1]$ interval, but they are approximation operators. Some of the results are obtained in a straightforward manner using the first modulus of continuity. For the results concerning the second order modulus of smoothness, we use the appropriate K-functional. In the later, we prove a Voronovskaja type result in order to see the error of approximation.

These results were published in paper **B. I. Vasian**, *On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense*, Dolomites Research Notes on Approximation, Vol. 16, 2023.

Let us consider $\tau : [0,1] \to [0,1]$ such that τ is a differentiable and increasing function satisfying $\tau(0) = 0$ and $\tau(1) = 1$.

For proving our results, we consider f to be a bounded function on $[0,1]$, and we will need the first order modulus of continuity ω_1, the second order modulus of smoothness ω_2 and the appropriate K-functionals.

Having in mind the operators introduced and studied in the previous section, we will introduce the following:

Definition 3.2.1. *[109] Let $\alpha \geq 0$. For every $f \in C([0,1])$, we define:*

$$D_{n,\tau}^{\alpha,\theta}(f,x) = \tag{3.16}$$
$$(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\left(f\circ\tau^{-1}\right)(t)\,dt,\ x\in[0,1],$$

where

$$Q_{n,k}^{\alpha,\tau,\theta}(x) = \left[J_{n,k}^{\alpha,\tau}(x)\right]^{\theta} - \left[J_{n,k+1}^{\alpha,\tau}(x)\right]^{\theta}, \tag{3.17}$$

with $\theta \geq 1$ an integer and $J_{n,k}^{\alpha,\tau}(x) = \sum_{j=k}^{n} p_{n,k}^{\alpha,\tau}(x)$, where

$$p_{n,k}^{\alpha,\tau}(x) = \left(\frac{n+\alpha}{n}\right)^{n}\binom{n}{k}\tau^{k}(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)^{n-k}. \tag{3.18}$$

Remark 3.2.2. *[109] We will mention the following remarks concerning notations:*

A *If index θ is missing, we assumed that $\theta = 1$;*

B *If index τ is missing, then we considered $\tau(x) = x$.*

For the operators $D_{n,\tau}^{\alpha,\theta}$ defined in (3.16) we can mention the following:

Remark 3.2.3. *[109] From the definition, it can be seen that the operators $D_{n,\tau}^{\alpha,\theta}$ are linear operators on $C([0,1])$.*

Remark 3.2.4. *[109] There is $\xi_n \in (0,1)$ having the property $\tau(\xi_n) = \frac{n}{n+\alpha}$, such that $\tau(x) > \frac{n}{n+\alpha}$ for $x \in (\xi_n, 1]$ and $\tau(x) \leq \frac{n}{n+\alpha}$ for $x \in [0,\xi_n]$, therefore the operators $D_{n,\tau}^{\alpha,\theta}$ are not positive on the entire interval $[0,1]$.*

3.2.1 Auxiliary results

In order to prove our results concerning these operators, we will need the following results concerning operators $D_{n,\tau}^{\alpha}$, i.e. when $\theta = 1$.

The following result provides a recurrence formula for $p_{n,k}^{\alpha,\tau}(x)$.

Lemma 3.2.5. *(See Lemma 3.2 in [109]) For the functions $p_{n,k}^{\alpha,\tau}(x)$ in* (3.18), *we have the following:*

$$\tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)\left(p_{n,k}^{\alpha,\tau}(x)\right)' = n\tau'(x)\left(\frac{k}{n+\alpha}-\tau(x)\right)p_{n,k}^{\alpha,\tau}(x),\ \ x\in[0,1]. \tag{3.19}$$

Lemma 3.2.6. *(See Lemma 3.3 in [109]) The operators $D_{n,\tau}^{\alpha}$ satisfy the following relations:*

A $D_{n,\tau}^{\alpha}(e_0,x) = 1$;

B $D_{n,\tau}^{\alpha}(\tau,x) = \frac{1}{n+2}\left(n\tau(x)+\frac{n}{n+\alpha}\right)$;

C $D_{n,\tau}^{\alpha}\left(\tau^2,x\right) = \frac{1}{(n+2)(n+3)}\left(n(n-1)\tau^2(x)+\frac{4n^2}{n+\alpha}\tau(x)+\frac{2n^2}{(n+\alpha)^2}\right)$;

where τ is defined above, and $x\in[0,1]$.

Denote by $M_{n,m}^{\tau,\alpha}(x)$ the central moment of order $m\in\{0,1,2,\dots\}$ of the operators $D_{n,\tau}^{\alpha}$, which is defined as follows

$$M_{n,m}^{\tau,\alpha}(x) = D_{n,\tau}^{\alpha}\left((\tau(t)-\tau(x))^m, x\right),\ \ x\in[0,1].$$

Lemma 3.2.7. *(See Lemma 3.4 in [109]) The following recurrence relation holds:*

$$\begin{aligned}&(m+n+2)\,\tau'(x)M_{n,m+1}^{\tau,\alpha}(x) \\ &= \tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)\left[2m\tau'(x)\,M_{n,m-1}^{\tau,\alpha}(x)+\left(M_{n,m}^{\tau,\alpha}\right)'(x)\right] \\ &\quad +(m+1)\,\tau'(x)\left(\frac{n}{n+\alpha}-2\tau(x)\right)M_{n,m}^{\tau,\alpha}(x).\end{aligned} \tag{3.20}$$

Remark 3.2.8. *[109] To simplify the notations, we denote*

$$\phi_\tau(x) := \tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right).$$

Remark 3.2.9. *[109] The function $\phi_\tau(x)$ attains its maximum for $\tau(x) = \frac{n}{2(n+\alpha)}$ and its maximum value is $\max\phi_\tau = \frac{1}{4}\left(\frac{n}{n+\alpha}\right)^2$.*

Proposition 3.2.10. *(See Proposition 3.6 in [109]) The following norm inequality holds:*

$$\left\|D_{n,\tau}^{\alpha}f\right\| \le e^{2\alpha}\left\|f\right\|, \tag{3.21}$$

for all $f\in C([0,1))$.

The following inequalities are useful in our main results.

Remark 3.2.11. *[109] For $a, b \in [-1, 1]$ and $\theta \geq 1$ integer, the inequality*

$$\left|a^{\theta} - b^{\theta}\right| \leq \theta \left|a - b\right| \tag{3.22}$$

holds.

Remark 3.2.12. *[109] We have the following inequality*

$$\begin{aligned} |Q_{n,k}^{\alpha,\tau,\theta}(x)| &= \left| \left[J_{n,k}^{\alpha,\tau}(x) \right]^{\theta} - \left[J_{n,k+1}^{\alpha,\tau}(x) \right]^{\theta} \right| \\ &\leq \theta \left| J_{n,k}^{\alpha,\tau}(x) - J_{n,k+1}^{\alpha,\tau}(x) \right| = \theta |p_{n,k}^{\alpha,\tau}(x)|, \end{aligned} \tag{3.23}$$

obtained as a consequence of Remark 3.2.11, where $\theta \geq 1$ is an integer.

Using the results stated above, we get the following results concerning the operators $D_{n,\tau}^{\alpha,\theta}$.

Proposition 3.2.13. *(See Proposition 3.7 in [109]) We have the following:*

$$\left\| D_{n,\tau}^{\alpha,\theta} f \right\| \leq \theta e^{2\alpha} \left\| f \right\|, \tag{3.24}$$

for all $f \in C([0, 1])$.

Remark 3.2.14. *[109] We have $D_{n,\tau}^{\alpha,\theta}(e_0, x) = 1$ for all $x \in [0, 1]$. Indeed, computing $D_{n,\tau}^{\alpha,\theta}(e_0, x)$, we get*

$$\begin{aligned} D_{n,\tau}^{\alpha,\theta}(e_0, x) &= \sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x) = \sum_{k=0}^{n} \left\{ \left[J_{n,k}^{\alpha,\tau}(x) \right]^{\theta} - \left[J_{n,k+1}^{\alpha,\tau}(x) \right]^{\theta} \right\} \\ &= \left[J_{n,0}^{\alpha,\tau}(x) \right]^{\theta} = \left[\sum_{k=0}^{n} p_{n,k}^{\alpha,\tau}(x) \right]^{\theta} = 1, \end{aligned}$$

for all $x \in [0, 1]$.

3.2.2 Quantitative approximation

In the following we will establish some quantitative results using different types of moduli of continuity: the classical modulus of continuity ω_1 and a combination of ω_1 and the modulus of smoothness ω_2.

Theorem 3.2.15. *(See Theorem 4.1 in [109]) For $f \in C([0, 1])$ we have*

$$\left| D_{n,\tau}^{\alpha,\theta}(f, x) - f(x) \right| \leq \left\{ 1 + \frac{1}{\delta} \frac{n}{n+\alpha} \sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}} \right\} \omega_1\left(f \circ \tau^{-1}, \delta\right), \tag{3.25}$$

for $x \in [0, \xi_n]$, $\delta > 0$, and

$$\left| D_{n,\tau}^{\alpha,\theta}(f, x) - f(x) \right| \leq \theta e^{2\alpha} \left\{ 1 + \frac{1}{\delta'} \left[\frac{2\alpha}{n} + \frac{n}{(n+2)(n+\alpha)} \right] \right\} \omega_1\left(f \circ \tau^{-1}, \delta'\right), \tag{3.26}$$

for $x \in (\xi_n, 1]$, $\delta' > 0$.

Corollary 3.2.16. *(See Corollary 4.2 in [109]) Let $f \in C([0,1])$. We have:*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \leq 2\omega_1\left(f\circ\tau^{-1}|_{[0,\xi_n]}, \frac{n}{n+\alpha}\sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}}\right), \quad (3.27)$$

for $x \in [0,\xi_n]$, and

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \leq 2\theta e^{2\alpha}\omega_1\left(f\circ\tau^{-1}|_{[\xi_n,1]}, \frac{2\alpha}{n} + \frac{n}{(n+2)(n+\alpha)}\right), \quad (3.28)$$

for $x \in (\xi_n, 1]$.

Lemma 3.2.17. *(See Lemma 4.3 in [109]) For $x \in [0,1]$, we have the following:*

$$D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2, x\right) \leq \theta\frac{n+1}{2(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2, \ for\ x \in [0,\xi_n], \quad (3.29)$$

and

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2, x\right)\right| \leq \theta e^{2\alpha}\frac{2n\left[n^2 - n(2-\alpha) - 3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}, \ for\ x \in (\xi_n, 1]. \quad (3.30)$$

The following result is expressed in terms of ω_1 and ω_2. In order to obtain this result we have to impose some restrictions to function $\tau(x)$ as follows:

- $\tau(x) \in C^2([0,1])$;
- $\inf_{x\in[0,1]} \tau'(x) \geq l, \ l \in \mathbb{R}_+$.
- $\sup_{x\in[0,1]} |\tau''(x)| \leq \beta, \ \beta \in \mathbb{R}_+$.

Theorem 3.2.18. *(See Theorem 4.4 in [109]) For $f \in C([0,1])$, we have:*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \quad (3.31)$$

$$\leq \frac{\theta e^{2\alpha}+1}{2}\left[C_1\omega_1\left(f, \frac{2\zeta_1\left(1+\frac{\beta}{2l}\zeta_1\right)}{\theta e^{2\alpha}+1}\right) + C_2\omega_2\left(f, \sqrt{\frac{\zeta_1^2}{\theta e^{2\alpha}+1}}\right)\right], \ x \in [0,\xi_n],$$

where $\zeta_1 = \frac{\sqrt{\theta}}{l}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}$ and $C_1, \ C_2$ are constants not depending on n, and

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \quad (3.32)$$

$$\leq \frac{\theta e^{2\alpha}+1}{2}\left[C_1^*\omega_1\left(f, \frac{2\zeta_2\left(1+\frac{\beta}{2l}\zeta_2\right)}{\theta e^{2\alpha}+1}\right) + C_2^*\omega_2\left(f, \sqrt{\frac{\zeta_2^2}{\theta e^{2\alpha}+1}}\right)\right], \ x \in (\xi_n, 1],$$

where $\zeta_2 = \frac{1}{l}\sqrt{\theta e^{2\alpha}\frac{2n[n^2-n(2-\alpha)-3\alpha]}{(n+2)(n+3)(n+\alpha)^2}}$, and $C_1^*, \ C_2^*$ are constants not depending on n.

3.2.3 Voronovskaja type result

In this section we will prove a Voronovskaja type result for the operators $D_{n,\tau}^{\alpha,\theta}$.

Lemma 3.2.19. *(See Lemma 5.1 in [109]) Let $f \in C^2[0,1]$. Then:*

$$\left|n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right| \tag{3.33}$$

$$\leq \theta\frac{2n}{n+2}\left|\frac{f'(x)}{\tau'(x)}\right|\left(\tau(x)+\frac{n}{n+\alpha}\right)$$

$$+\theta\frac{n}{(n+2)(n+3)}\left|\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right|\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right] \tag{3.34}$$

$$+\Lambda_n(x);\; x\in[0,\xi_n],$$

where $\Lambda_n(x)\to 0$ as $n\to\infty$.

Theorem 3.2.20. *(See Theorem 5.2 in [109]) For $f \in C^2[0,1]$ and $x \in [0,1)$, we have:*

$$\limsup_{n\to\infty}\left(n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right) \tag{3.35}$$

$$=2\theta\left|\frac{f'(x)}{\tau'(x)}\right|(\tau(x)+1)+\theta\left|\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right|\tau(x)(1-\tau(x));\; x\in[0,1).$$

4 Kantorovich type operators modified in King sense

In this chapter we will present some results concerning Kantorovich type operators. Namely, we will treat some classes of Kantorovich type operators modified using Stancu and King's methods. The classes obtained this way are consisting only of linear positive operators, and for these classes the uniform approximation of all continuous functions on specific intervals is proven.

4.1 On New Classes of Stancu-Kantorovich-Type Operators

This section is dedicated to presenting the results obtained in the paper **Vasian B. I.**, Garoiu Ş.L., Păcurar C.M., *On New Classes of Stancu-Kantorovich-Type Operators*. Mathematics 2021.

Here, we introduced new classes of Stancu-Kantorovich type operators constructed with the method introduced by King in paper [66]. Each class is constructed such that the operators preserve two test functions at a time. Firstly, we will study the operators which preserve e_0 and e_1. Secondly, e_0 and e_2, and lastly e_1 and e_2. For each class we have studied the uniform approximation of continuous functions on some intervals on which operators remain positive. Also, some error estimation results are provided in each case.

Let us introduce the classes we announced.

Definition 4.1.1. *(See Definition 4 in [110]) Let $I \subset \mathbb{R}$ be an interval and $c_n, d_n : I \to \mathbb{R}$ be some functions satisfying $c_n(x) \geq 0$, $d_n(x) \geq 0$ for all $x \in I$, $0 \leq \alpha \leq \beta$ and $n \in \{1, 2, \dots\}$. We define the following operators of Stancu and Kantorovich type:*

$$S_n^{(\alpha,\beta)*}(f,x) = (n+\beta+1)\sum_{k=0}^{n}\binom{n}{k}(c_n(x))^k (d_n(x))^{n-k}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt. \tag{4.1}$$

for any $x \in I$, $m \in \{1, 2, \dots\}$ and $f \in L_1([0,1])$.

For these operators, we can state the following lemma.

Lemma 4.1.2. *(See Lemma 1 in [110]) The operators* (4.1) *satisfy the following*

$$S_n^{(\alpha,\beta)*}(e_0,x) = (c_n(x)+d_n(x))^n, \tag{4.2}$$

$$S_n^{(\alpha,\beta)*}(e_1,x) = \frac{n}{n+\beta+1}c_n(x)(c_n(x)+d_n(x))^{n-1} + \frac{2\alpha+1}{2(n+\beta+1)}(c_n(x)+d_n(x))^n, \tag{4.3}$$

$$S_n^{(\alpha,\beta)*}(e_2,x) = \frac{n(n-1)}{(n+\beta+1)^2} c_n^2(x)(c_n(x)+d_n(x))^{n-2} \tag{4.4}$$
$$+\frac{n(2\alpha+2)}{(n+\beta+1)^2} c_n(x)(c_n(x)+d_n(x))^{n-1}$$
$$+\frac{3\alpha(\alpha+1)+1}{3(n+\beta+1)^2}(c_n(x)+d_n(x))^n$$

for any $x \in I,\ n \in \{0,1,\dots\}$, *where* $e_i(t) = t^i,\ i \in \{0,1,2\}$.

Definition 4.1.3. *We denote by* $M_{n,k}L_n(x)$, *the k-th order moment of the operators* L_n, *having the expression:*

$$M_{n,k}(L_n(x)) = L_n\left((e_1-x)^k, x\right). \tag{4.5}$$

4.1.1 Stancu-Kantorovich type operators which preserve functions e_0 and e_1

Now, we shall impose that Stancu-Kantorovich type operators introduced in (4.1), preserve test functions e_0 and e_1. In other words, we shall construct some operators that satisfy

$$\begin{aligned} &\bullet_1\ S_n^{(\alpha,\beta)*}(e_0,x) = 1;\\ &\bullet_2\ S_n^{(\alpha,\beta)*}(e_1,x) = x;\\ &\bullet_3\ \lim_{n\to\infty} S_n^{(\alpha,\beta)*}(e_2,x) = x^2 \text{ uniformly on some interval.}\end{aligned} \tag{4.6}$$

Now, from the conditions imposed in (4.6) and identities (4.2), (4.3), we obtain

$$c_n(x) = \frac{n+\beta+1}{n}x - \frac{2\alpha+1}{2n}, \tag{4.7}$$

and

$$d_n(x) = \frac{-(n+\beta+1)x}{n} + \frac{2n+2\alpha+1}{2n}, \tag{4.8}$$

for any $n \in \{1,2,\dots\}$ and $x \in I$.

If we need our operators to be positive, we shall assume that the functions $c_n(x)$ and $d_n(x)$ are positive. This assumption yields to:

$$\frac{2\alpha+1}{2(n+\beta+1)} \le x \le \frac{2n+2\alpha+1}{2(n+\beta+1)}, \text{ for all } x \in I, \text{ and } n \in \{1,2,\dots\}.$$

Lemma 4.1.4. *(See Lemma 2 in [110]) For* $0 \le \alpha \le \beta$ *and any integers* $n_0 < n$, *the following inclusion holds:*

$$\left[\frac{2\alpha+1}{2(n_0+\beta+1)}; \frac{2n_0+2\alpha+1}{2(n_0+\beta+1)}\right] \subset \left[\frac{2\alpha+1}{2(n+\beta+1)}; \frac{2n+2\alpha+1}{2(n+\beta+1)}\right].$$

Remark 4.1.5. *Further in this section we will consider that* $0 \le \alpha \le \beta$.

Remark 4.1.6. *Since on the interval* $\left[\frac{2\alpha+1}{2(n_0+\beta+1)}; \frac{2n_0+2\alpha+1}{2(n_0+\beta+1)}\right]$ *we have that* $c_n(x),\ d_n(x) \ge 0$, *for every* $n \in \{1,2,\dots\}$, *we will consider the interval* $I = \left[\frac{2\alpha+1}{2(n_0+\beta+1)}; \frac{2n_0+2\alpha+1}{2(n_0+\beta+1)}\right]$, *where* n_0 *is a positive integer which is arbitrarily fixed. Note that for any* $\varepsilon > 0$, *if we take* n_0 *sufficiently large, then* $[\varepsilon, 1-\varepsilon] \subset I$.

By taking into account the sequences $c_n(x)$ and $d_n(x)$ obtained in (4.7) and (4.8), the operators of type (4.1) will have the following expressions:

$$S_{1,n}^{(\alpha,\beta)*}(f,x) = (n+\beta+1)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{n+\beta+1}{n}x-\frac{2\alpha+1}{2n}\right)^{k} \tag{4.9}$$

$$\times\left(\frac{-(n+\beta+1)}{n}x+\frac{2n+2\alpha+1}{2n}\right)^{n-k}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt,$$

for any $x \in I$.

Lemma 4.1.7. *(See Lemma 3 in [110]) The operators* $S_{1,n}^{(\alpha,\beta)*}$ *in* (4.9) *satisfy*

$$S_{1,n}^{(\alpha,\beta)*}(e_0,x) = 1;$$
$$S_{1,n}^{(\alpha,\beta)*}(e_1,x) = x; \tag{4.10}$$
$$S_{1,n}^{(\alpha,\beta)*}(e_2,x) = \frac{n-1}{n}x^2+\frac{n+2\alpha+1}{n(n+\beta+1)}x-\frac{n(12\alpha+5)+3(2\alpha+1)^2}{12n(n+\beta+1)^2}$$

for $x \in I$.

Regarding the moments of the operators, we have the following result.

Lemma 4.1.8. *(See Lemma 4 in [110]) The following relations hold*

$$M_{n,0}\left(S_{1,n}^{(\alpha,\beta)*}\right)(x) = 1, \tag{4.11}$$

$$M_{n,1}\left(S_{1,n}^{(\alpha,\beta)*}\right)(x) = 0, \tag{4.12}$$

$$M_{n,2}\left(S_{1,n}^{(\alpha,\beta)*}\right)(x) = -\frac{x^2}{n}+\frac{n+2\alpha+1}{n(n+\beta+1)}x+O\left(\frac{1}{n^2}\right). \tag{4.13}$$

Lemma 4.1.9. *(See Lemma 5 in [110]) We have*

$$\lim_{n\to\infty} nM_{n,2}\left(S_{1,n}^{(\alpha,\beta)*}\right)(x) = x(1-x) \tag{4.14}$$

uniformly for $x \in I$. *Moreover, for any* $\varepsilon > 0$ *there exists an integer* $n_\varepsilon \geq n_0$, *sufficiently large, such that*

$$nM_{n,2}\left(S_{1,n}^{(\alpha,\beta)*}\right)(x) \leq \frac{1+\varepsilon}{4}, \tag{4.15}$$

for any $x \in I$ *and* $n \in \{1,2,\dots\}$ *such that* $n \geq n_\varepsilon$.

Theorem 4.1.10. *(See Theorem 2 in [110]) Let* $f \in C([0,1])$. *The following limit holds*

$$\lim_{m\to\infty} S_{1,m}^{(\alpha,\beta)*}(f) = f$$

uniformly on I. *Moreover, for every* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \{1,2,\dots\}$ *such that*

$$\left|S_{1,n}^{(\alpha,\beta)*}(f,x) - f(x)\right| \leq \left(1+\frac{\sqrt{1+\varepsilon}}{2}\right)\omega_1\left(f,\frac{1}{\sqrt{n}}\right),$$

for any $x \in I$ *and* $n \in \{1,2,\dots\}$, $n \geq n_\varepsilon$.

Further, we will present a graphical example of approximation. Here, we have considered the function $f(x) = \sin(20x)$, $n = 25$ iterations, $\alpha = 0.1$ and $\beta = 0.2$.

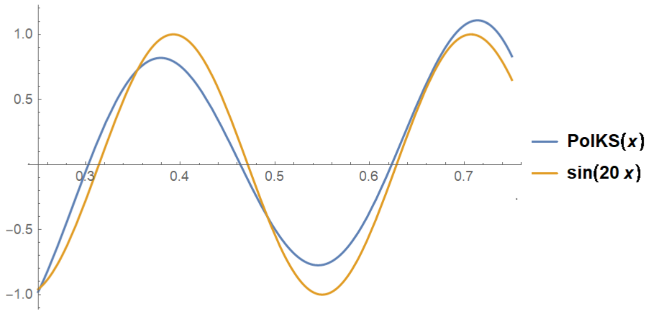


Figure 4.1: $\alpha = 0.1$, $\beta = 0.2$, $m = 25$ iterations

In Figure 4.1 it can be seen that the sequence of operators (blue) can be an approximation process for the function f (orange).

4.1.2 Stancu-Kantorovich type operators which preserve functions e_0 and e_2

Further, we shall treat operators of Stancu-Kantorovich type as in (4.1) for which we impose the preservation of test functions e_0 and e_2. In this sense, the operators are constructed such that:

$$\begin{gathered} \bullet_1 S_n^{(\alpha,\beta)*}(e_0, x) = 1 \\ \bullet_2 S_n^{(\alpha,\beta)*}(e_2, x) = x^2 \\ \bullet_3 \lim_{n\to\infty} S_n^{(\alpha,\beta)*}(e_1, x) = x \text{ uniformly on some interval.} \end{gathered} \tag{4.16}$$

Imposing the conditions above (4.16) and using the identities in (4.2) and (4.4) we obtain the following conditions on $c_n(x)$ and $d_n(x)$:

$$c_n(x) + d_n(x) = 1, \ \forall x \in I, \ n \in \{1, 2, \dots\}, \tag{4.17}$$

and the quadratic equation, in $c_n(x)$:

$$n(n-1)c_n^2(x) + 2n(1+\alpha)c_n(x) + \alpha(\alpha+1) + \frac{1}{3} = x^2(n+\beta+1)^2, \tag{4.18}$$

$x \in I$, and $n \in \{1, 2, \dots\}$.

Note that for $\alpha \geq 0$, $\beta \geq 0$, the discriminant

$$\delta_n(x) = 4n\left[n\left(\frac{2}{3}+\alpha\right) + \alpha^2 + \alpha + \frac{1}{3} + x^2(n-1)(n+\beta+1)^2\right] \tag{4.19}$$

of the quadratic equation (4.18) is positive.

We make the following notation:

$$\Delta_n(x) = \frac{\delta_n(x)}{4}.$$

By solving the equation (4.18) we get, for $n \geq 2$:

$$c_n(x) = \frac{-n(1+\alpha) + \sqrt{\Delta_n(x)}}{n(n-1)} \tag{4.20}$$

and, from relation (4.17), we obtain:

$$d_n(x) = \frac{n(n+\alpha) - \sqrt{\Delta_n(x)}}{n(n-1)}. \tag{4.21}$$

In order to apply Korovkin's Theorem, we need our operators to be positive. For that, we shall impose that $c_n(x)$ and $d_n(x)$, from (4.20) and (4.21), are positive. Thus, we obtain:

$$\frac{\sqrt{\alpha^2+\alpha+\frac{1}{3}}}{n+\beta+1} \leq x \leq \frac{\sqrt{n(n+2\alpha+1)+\alpha^2+\alpha+\frac{1}{3}}}{n+\beta+1}.$$

Lemma 4.1.11. *(See Lemma 6 in [110]) Let $0 < \varepsilon' < \frac{1}{2}$ be fixed. Then, there exists an integer $n_0 \in \{1, 2, \dots\}$, such that*

$$[\varepsilon', 1-\varepsilon'] \subset \left[\frac{\sqrt{\alpha^2+\alpha+\frac{1}{3}}}{n+\beta+1}; \frac{\sqrt{n(n+2\alpha+1)+\alpha^2+\alpha+\frac{1}{3}}}{n+\beta+1}\right], \tag{4.22}$$

for every $n \in \{1, 2, \dots\}$ such that $n \geq n_\varepsilon$ and α, β satisfying $\alpha \leq \beta$.

Remark 4.1.12. *Because the sequences $c_n(x)$ and $d_n(x)$ are positive on the interval* (4.22)*, from now on, we will consider $I = [\varepsilon', 1-\varepsilon']$, for all $\varepsilon' > 0$ and $n \geq n_0$.*

Going back to (4.1) with the expressions of $c_n(x)$ and $d_n(x)$ in (4.20) and (4.21), the operators become:

$$S_{2,n}^{(\alpha,\beta)*}(f,x) = \frac{n+\beta+1}{(n(n-1))^n} \sum_{k=0}^{n} \binom{n}{k} \left(-n(1+\alpha)+\sqrt{\Delta_n(x)}\right)^k \tag{4.23}$$

$$\times \left(n(n+\alpha)-\sqrt{\Delta_n(x)}\right)^{n-k} \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt,$$

for any $x \in I$ and $n \geq n_0$.

Lemma 4.1.13. *(See Lemma 7 in [110]) The operators $S_{2,n}^{(\alpha,\beta)*}$ from* (4.23) *satisfy:*

$$\begin{gathered} S_{2,n}^{(\alpha,\beta)*}(e_0,x) = 1; \\ S_{2,n}^{(\alpha,\beta)*}(e_1,x) = \frac{-(n+2\alpha+1)+2\sqrt{\Delta_n(x)}}{2(n+\beta+1)(n-1)}; \\ S_{2,n}^{(\alpha,\beta)*}(e_2,x) = x^2. \end{gathered} \tag{4.24}$$

for $x \in I$ and $n \geq n_0$.

Further, we will present the first three moments of the operator $S_{2,n}^{(\alpha,\beta)*}$.

Lemma 4.1.14. *(See Lemma 8 in [110]) The following relations hold*

$$M_{n,0}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) = 1, \tag{4.25}$$

$$M_{n,1}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) = -x + \frac{-(n+2\alpha+1)+2\sqrt{\Delta_n(x)}}{2(n+\beta+1)(n-1)}, \tag{4.26}$$

$$M_{n,2}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) = 2x\left(x - \frac{-(n+2\alpha+1)+2\sqrt{\Delta_n(x)}}{2(n+\beta+1)(n-1)}\right), \tag{4.27}$$

for any $x \in I$ and $n \in \{1,2,\dots\}$.

Lemma 4.1.15. *(See Lemma 9 in [110]) The following limits hold*

$$\lim_{n\to\infty} nM_{n,1}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) = \frac{1}{2}(x-1), \tag{4.28}$$

$$\lim_{n\to\infty} nM_{n,2}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) = x(1-x), \tag{4.29}$$

uniformly with respect to $x \in I$. Moreover, for any $\varepsilon > 0$ there exists $n_\varepsilon > n_0$ such that

$$M_{n,2}\left(S_{2,n}^{(\alpha,\beta)*}\right)(x) \leq \frac{1}{n}\left(\frac{1}{4}+\varepsilon\right), \tag{4.30}$$

for any $x \in I$ and $n \in \{1,2,\dots\}$ such that $n \geq n_\varepsilon$.

Theorem 4.1.16. *(See Theorem 3 in [110]) Let $f \in C([0,1])$. Then, we have*

$$\lim_{n\to\infty} S_{2,n}^{(\alpha,\beta)*}(f) = f$$

uniformly on I. Moreover, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \{1,2,\dots\}$ such that:

$$\left|\left(S_{2,n}^{(\alpha,\beta)*}f\right)(x) - f(x)\right| \leq \left(1+\sqrt{\frac{1}{4}+\varepsilon}\right)\omega_1\left(f, \frac{1}{\sqrt{n}}\right),$$

for any $x \in I$ and $n \in \{1,2,\dots\}$ such that $n \geq n_\varepsilon$.

For this operators we have obtained the following graphics for the functions $f(x) = 2x^3 - \frac{20}{7}x^2 + \frac{8}{7}x - \frac{1}{7}$ and $f(x) = \left|x - \frac{1}{2}\right|$, with $\alpha = 0.1$, $\beta = 0.65$ and $n = 50$.

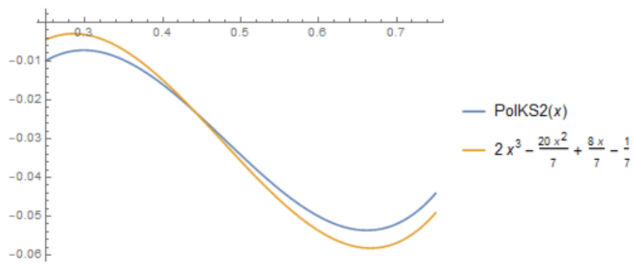


Figure 4.2: $\alpha = 0.1$, $\beta = 0.65$, $n = 50$ iterations

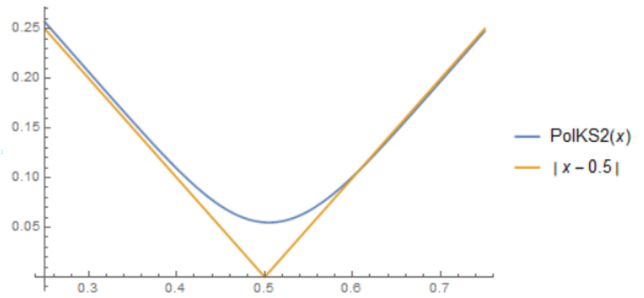


Figure 4.3: $\alpha = 0.1$, $\beta = 0.65$, $n = 50$ iterations

4.1.3 Stancu-Kantorovich type operators which preserve functions e_1 and e_2

In this section we will impose to the operators of Stancu-Kantorovich type in (4.1) to preserve the test functions e_1 and e_2. In this sense, operators should satisfy

$$\begin{aligned} &\bullet_1 \lim_{n\to\infty} S_{3,n}^{(\alpha,\beta)*}(e_0,x) = 1, \text{ uniformly on some interval,} \\ &\bullet_2 S_{3,n}^{(\alpha,\beta)*}(e_1,x) = x, \\ &\bullet_3 S_{3,n}^{(\alpha,\beta)*}(e_2,x) = x^2. \end{aligned} \tag{4.31}$$

In order to obtain the main results of this section, we will make the following notation

$$S_{3,n}^{(\alpha,\beta)*}(e_0,x) = 1 + w_n(x), \tag{4.32}$$

where $x \in I,\ n \in \{1, 2, \dots\}$ and $w_n : I \to \mathbb{R}$.

With the previous notation we can state the following remark.

Remark 4.1.17. *In order to obtain positive operators $S_{3,n}^{(\alpha,\beta)*}$, $n \in \{1, 2, \dots\}$, $0 \le \alpha \le \beta$, we shall impose that $S_{3,n}^{(\alpha,\beta)*}(e_0,x) \ge 0$, which implies*

$$1 + w_n(x) \ge 0,\ \forall x \in I,\ n \in \{1, 2, \dots\} \tag{4.33}$$

From the relation (4.32), we get

$$(c_n(x) + d_n(x))^n = 1 + w_n(x), \forall x \in I,\ n \in \{1, 2, \dots\}, \tag{4.34}$$

which implies

$$c_n(x) + d_n(x) = (1 + w_n(x))^{\frac{1}{n}}, \forall x \in I,\ n \in \{1, 2, \dots\}. \tag{4.35}$$

With the above considerations and imposing the conditions $\bullet_2$ and $\bullet_3$ from (4.31), we can state the following lemma.

Lemma 4.1.18. *(See Lemma 10 in [110]) In order to have $S_{3,n}^{(\alpha_\varepsilon,\beta)*}(e_1,x) = x$, $c_n(x)$ and $d_n(x)$ are the following:*

$$c_n(x) = \frac{n+\beta+1}{n}\left[x - \frac{2\alpha+1}{2(n+\beta+1)}(1 + w_n(x))\right](1 + w_n(x))^{\frac{1-n}{n}} \tag{4.36}$$

and

$$d_n(x) = (1 + w_n(x))^{\frac{1}{n}} \times \tag{4.37}$$
$$\left[1 - \frac{n+\beta+1}{m} \cdot \frac{1}{1+w_n(x)} \left(x - \frac{2\alpha+1}{2(n+\beta+1)}(1+w_n(x))\right)\right].$$

From condition $\bullet_2$ in (4.31), we get the following quadratic equation in $w_n(x)$:

$$\begin{gathered} w_n^2(x)\left[-5n-3-\alpha(1+n+\alpha)\right]+ \\ w_n(x)\{-12n\left[(n+1)^2+\beta(\beta+2n+2)\right]x^2+ \\ 12\left[(n+1)^2+2\alpha(1+n)+\beta(1+n+2\alpha)\right]x-2\left[5n+3+12\alpha(1+n+\alpha)\right]\}+ \\ +\{-12\left[(n+1)^2+\beta(\beta+2n+2)\right]x^2+ \\ +12\left[(n+1)^2+2\alpha(1+n)+\beta(1+n+2\alpha)\right]x-\left[5n+3+12\alpha(1+n+\alpha)\right]\}=0, \end{gathered}$$

with solutions denoted $w_{n,1}$ and $w_{n,2}$, $w_{n,2} < w_{n,1}$.

Remark 4.1.19. *It can be verified that the solutions of the quadratic equation above satisfy* $\lim_{n\to\infty} w_{n,2}(x) = -\infty$ *and* $\lim_{n\to\infty} w_{n,1}(x) = 0$, *uniformly for* $x \in (0,1)$.

From now on, in this section we will consider $w_n(x) = w_{n,1}(x)$.

In order to have a positive operator, the quantities $c_n(x)$ and $d_n(x)$ from relations (4.36) and (4.37) shall be positive. With these conditions, we obtain

$$x - \frac{2\alpha+1}{2(n+\beta+1)}(1+w_n(x)) \geq 0,$$

and

$$1 - \frac{n+\beta+1}{n} \cdot \frac{1}{1+w_n(x)} \left(x - \frac{2\alpha+1}{2(n+\beta+1)}(1+w_n(x))\right) \geq 0,$$

for all $x \in I,\ n \in \{1,2,\dots\}$ and $0 \leq \alpha \leq \beta$ which leads to:

$$\frac{2(n+\beta+1)}{2n+2\alpha+1}x - 1 \leq w_n(x) \leq \frac{2(n+\beta+1)}{2\alpha+1}x - 1, \tag{4.38}$$

for all $x \in I,\ n \in \{1,2,\dots\}$ and $0 \leq \alpha \leq \beta$.

Lemma 4.1.20. *(See Lemma 11 in [110]) Let* $0 < \varepsilon' < \frac{1}{2}$. *Then there exists* $n_0 \in \{1,2,\dots\}$ *such that relation* (4.38) *holds for any* $x \in [\varepsilon', 1-\varepsilon']$ *and* $n \in \{1,2,\dots\}$, $n \geq n_0$.

From now on, we will consider $I = [\varepsilon', 1-\varepsilon']$, with fixed $0 < \varepsilon' < \frac{1}{2}$.

We can write the operators in (4.1) as

$$S_{3,n}^{(\alpha,\beta)*}(f,x) = (n+\beta+1)\sum_{k=0}^{n}\binom{n}{k}(1+w_n(x))^{1-k}$$
$$\times\left(\frac{n+\beta+1}{n}\left(x-\frac{2\alpha+1}{2(n+\beta+1)}(1+w_n(x))\right)\right)^k$$
$$\times\left(1-\frac{n+\beta+1}{n(1+w_n(x))}\left(x-\frac{2\alpha+1}{2(n+\beta+1)}(1+w_n(x))\right)\right)^{n-k}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt.$$

Lemma 4.1.21. *(See Lemma 12 in [110]) For $x \in I$, $I = [\varepsilon', 1-\varepsilon']$, $0 < \varepsilon' < \frac{1}{2}$, and $n \in \{1, 2, \dots\}$, we have*

$$M_{n,0}\left(S_{3,n}^{(\alpha,\beta)*}\right)(x) = 1 + w_n(x),$$
$$M_{n,1}\left(S_{3,n}^{(\alpha,\beta)*}\right)(x) = -xw_n(x),$$
$$M_{n,2}\left(S_{3,n}^{(\alpha,\beta)*}\right)(x) = x^2 w_n(x).$$

Theorem 4.1.22. *(See Theorem 4 in [110]) We have*

$$\lim_{n\to\infty} S_{3,n}^{(\alpha,\beta)*}(f) = f$$

uniformly on $[\varepsilon', 1-\varepsilon']$, $0 < \varepsilon' < \frac{1}{2}$, for every $f \in C([0,1])$.

Remark 4.1.23. *Having in mind the result in [48] which proves there is no sequence of positive linear analytic operators $L : C([0,1]) \to C([0,1])$ that preserve the test functions e_1 and e_2, we can mention that the restriction of the image to $[\varepsilon', 1-\varepsilon']$, $0 < \varepsilon' < \frac{1}{2}$, is the maximum interval on which our operators remain positive.*

Further, we will provide some graphical example as a comparison between our operators $S_{3,n}^{(\alpha,\beta)*}$ denoted by $PolKS(x)$, and the operators $P(x)$ obtained by Indrea et al. in [62], which are a particular case of our operators with $\alpha = \beta = 0$.

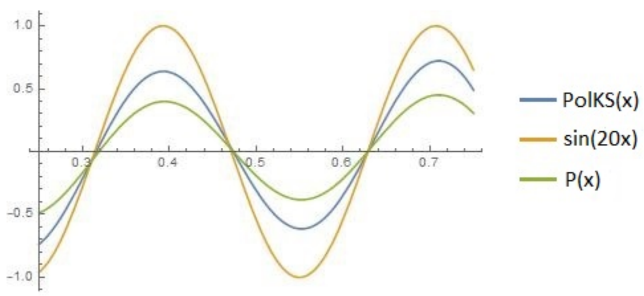


Figure 4.4: $f(x) = \sin(20x)$, $\alpha = 10$, $\beta = 20$, $n = 50$ iterations

Now, we consider the function $f(x) = |x - 0.5|$ and we obtain the following graphic:

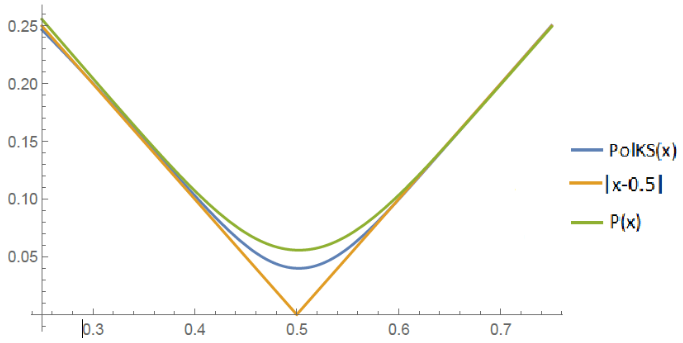


Figure 4.5: $f(x) = |x - 0.5|$, $\alpha = 10$, $\beta = 20$, $n = 50$ iterations

5 Non-positive Kantorovich type operators attached to some linear differential operators

In this chapter, we present some operators obtained as a generalization of Kantorovich operators using different kinds of linear differential operators. The results presented in this chapter were published in the following three papers: **Vasian, B. I.**, Approximation Properties of Some Non-positive Kantorovich Type Operators, 2022 Proceedings of International E-Conference on Mathematical and Statistical Sciences: A Selçuk Meeting (2022), 188-194; **Vasian, B. I.**, Voronovskaja type theorem for some non-positive Kantorovich type operators, Carpathian Journal of Mathematics Vol. 40, No. 1 (2024), 187-194 and **Vasian, B. I.**, Generalized Kantorovich operators, General Mathematics, Vol. 32, No.2, (2025), 67-83.

The first section of this chapter is dedicated to some general Bernstein-Kantorovich operators which are not positive on $[0,1]$, but can be used to uniform approximate all continuous functions on it.

In the second section, we will treat the most general Kantorovich type operators. The method presented there is useful for providing countless approximation operators, which are not positive on $[0,1]$.

5.1 Some general Bernstein-Kantorovich operators

In this section, we will provide a generalization of Bernstein-Kantorovich operators using a linear differential operator of order l with constant coefficients, D^l, and its corresponding anti-derivative operator I^l, having the property $D^l \circ I^l = Id$. We will prove that the convergence on all continuous functions on $[0,1]$ holds even though the operators constructed this way are not positive. The way of constructing our operators is actually Kantorovich's method presented in Section 2.8.3, meaning our operators will be obtained as a composition between the derivative operator D^l, Bernstein operator of order $n+l$ and the antiderivative operator I^l.

Let $l \in \{1,2,\dots\}$,

$$D^l f = \sum_{i=0}^{l} a_i f^{(i)}, \tag{5.1}$$

$a_0, a_1, a_2, \dots, a_l \in \mathbb{R}$. The corresponding antiderivative operator I^l is obtained from condition $D^l \circ I^l = Id$. This leads to:

$$\left(D^l \circ I^l\right)(f) = f,$$

that is equivalent with:

$$a_l \left(I^l f\right)^{(l)} + a_{l-1} \left(I^l f\right)^{(l-1)} + \cdots + a_1 \left(I^l f\right)' + a_0 \left(I^l f\right) = f, \tag{5.2}$$

which is a linear differential equation of order l with constant coefficients for which we know that $I^l f$ exists but it is not unique and it is of class $C^l([0,1])$. Since there is an infinity of such antiderivatives, a unique one can be obtained by imposing some initial conditions for differential equations such as, in a certain point the antiderivative $I^l f$ and its derivatives up to order $l-1$ should have particular values (see [15]). However, the exact form of the antiderivative operator does not play a role in our proofs, therefore, the initial conditions and their form are neglected.

The Kantorovich type operators that will be presented in this section are of the form

$$K^l = D^l \circ L \circ I^l, \tag{5.3}$$

where L is an operator.

Further we will consider $L = B_{n+l}$ in (5.3) and we will prove that these linear operators can approximate continuous functions on $[0,1]$ even though they are not positive operators.

The l-th derivative of B_{n+l} can be expressed in terms of finite difference of order l with the step $k = \frac{1}{n+l}$:

$$B_{n+l}^{(l)}(f,x) = \frac{(n+l)!}{n!} \sum_{k=0}^{n} \Delta^l_{\frac{1}{n+l}} f\left(\frac{k}{n+l}\right) p_{n,k}(x),\ x \in [0,1],\ f \in C([0,1]).$$

With all the considerations from above, let us define our Kantorovich type operators:

$$K_n^l(f,x) = \left(D^l \circ B_{n+l} \circ I^l\right)(f,x),\ x \in [0,1] \tag{5.4}$$

which can be written as

$$K_n^l(f,x) = D^l\left(B_{n+l} I^l f\right)(x). \tag{5.5}$$

For simplicity, let us denote $I^l f := F$

$$K_n^l(f,x) = \sum_{i=0}^{l} a_i \left[B_{n+l}(F,x)\right]^{(i)} \tag{5.6}$$

$$= \sum_{i=0}^{l} \frac{(n+l)!}{(n+l-i)!} a_i \sum_{j=0}^{n+l-i} \Delta^i_{\frac{1}{n+l}} F\left(\frac{j}{n+l}\right) p_{n+l-i,j}(x).$$

Remark 5.1.1. *The operators* (5.6) *are linear operators.*

Our aim is to prove that

$$\left\|K_n^l f - f\right\| \to 0,\ as\ n \to \infty$$

for all continuous functions on $[0,1]$.

5.1.1 Approximation result

In order to prove our main approximation the proof of the following lemma is essential:

Lemma 5.1.2. *(See Lemma 2.1 in [111]) Let I be a compact interval and $F \in C^k(I)$, then the following convergence holds:*

$$\lim_{n\to\infty} (n+l)^k \, \Delta^k_{\frac{1}{n+l}} F = F^{(k)}, \text{ uniformly on } I, \text{as } n \to \infty. \tag{5.7}$$

Now, we can state and prove the approximation result.

Theorem 5.1.3. *(See Theorem 2.2 in [111]) Let $f \in C([0,1])$. The following convergence holds:*

$$\lim_{n\to\infty} K_n^l f = f, \ uniformly \ on \ [0,1]. \tag{5.8}$$

Remark 5.1.4. *[111] The operators $K_n^l f(x)$ are not positive as the following example shows:*

Example 5.1.5. *[111] Let $n = 1$ and the differential operator be $D^1 f = f' - f$ which has a fixed corresponding antiderivative $I^1 f(x) = e^x \int_0^x e^{-t} f(t)\, dt$, which will be denoted by $I^1 f(x) := F(x)$ and chosen such that $F(0) = 0$. We consider the function*

$$f(t) = e^{-5t}, \ t \in [0,1].$$

Then:

$$K_1^1(f,1) = \frac{1}{3} e^{\frac{1}{2}} \left(e^{-3} - 1\right) - \frac{e}{6}\left(e^{-6} - 1\right) = -7.0288 \times 10^{-2},$$

which proves our remark.

5.1.2 Voronovskaja type result

In this section we will prove a Voronovskaja type theorem for the operators K_n^l. Let us introduce the following differential operator in order to simplify the notations:

$$D_y^l g(x) = \sum_{i=1}^{l} a_i y^i g^{(i-1)}(x). \tag{5.9}$$

Theorem 5.1.6. *(See Theorem 2.3 in [112]) Let $f \in C^2([0,1])$ and $F \in C^{l+2}([0,1])$, then*

$$\lim_{n\to\infty} n \left[K_n^l(f,x) - f(x)\right] = \frac{1}{2} D^l \left\{x(1-x)\left[F(x)\right]''\right\} \tag{5.10}$$

$$= \frac{1}{2} x(1-x) f''(x) + \left(\frac{1}{2} - x\right) \frac{\partial D_y^l F''(x)}{\partial y}\Big|_{y=1} - \frac{1}{2} \frac{\partial^2 D_y^l F'(x)}{\partial y^2}\Big|_{y=1}.$$

uniformly for $x \in [0,1]$.

5.1.3 Simultaneous approximation

In this section we will prove a simultaneous approximation result concerning operators K_n^l.

Theorem 5.1.7. *(See Theorem 3.4 in [112]) Let $f \in C^r([0,1])$ with $r \in \mathbb{N} \cup \{0\}$. Then:*

$$\lim_{n\to\infty} [K_n^l(f)]^{(r)} = f^{(r)}, \tag{5.11}$$

holds uniformly on $[0,1]$ and $F \in C^{l+r}([0,1])$.

5.1.4 Example

In this final section we will take a particular case of our operators and we will present some computations and graphics.

Let $D^*f = f'' - 3f' + 2f$ be a differential operator of order two and I^*, a fixed corresponding antiderivative operator, in the sense $D^* \circ I^* = Id$, with initial conditions $I^*f(0) = 0$ and $(I^*f)'(0) = 0$.

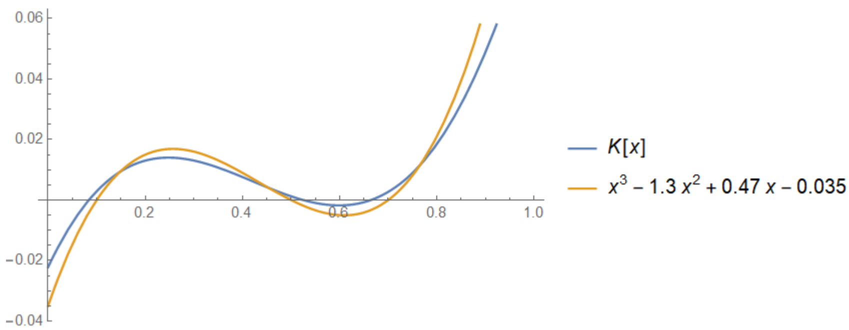
$$I^*f(x) = \int_0^x e^{x-t}\left(e^{x-t} - 1\right) f(t)dt, \;\; x \in [0,1]. \tag{5.12}$$

We denote $F(x) := I^*f(x)$.

Now, let us consider operators:

$$K_n^*(f,x) = \left(D^* \circ B_{n+2} \circ I^* f\right)(x) = D^*\left(B_{n+2}(F,x)\right), \;\; x \in [0,1]. \tag{5.13}$$

Now, we will consider function $f(x) = x^3 - 1.3x^2 + 0.47x - 0.035$. For this function, for $n = 30$ we have the following graphical process, using Wolfram Mathematica software:



5.2 Generalized Kantorovich operators

In this section we will introduce a new class of operators modified in Kantorovich's sense using some linear differential operators of order l with non constant coefficients. The operators constructed here can be used to approximate all continuous functions on $[0,1]$ even though they are not positive operators.

This section contains the most general results on this topic, generalizing the previous section by taking the differential operator D^l in a general way. The results are proved for any operator L satisfying some properties, not just for Bernstein operators.

To support our results, we will provide an approximation theorem and some Voronovskaja type theorems for the Kantorovich generalization of the Bernstein operators, Durrmeyer operators, Kantorovich operators, Stancu operators and U_n^ρ operators.

The results in this section can be found in paper **B. I. Vasian**, Generalized Kantorovich operators, General Mathematics, Vol. 32, No. 2 (2025), 67-83.

This topic was studied in particular cases in some papers such as [50, 83, 23, 109].

5.2.1 Construction of the operators

We will prove a uniform approximation result and a Voronovskaja type theorem for some Kantorovich type operators constructed using a more general differential operator: let $l \in \{1, 2, \dots\}$,

$$D^l g(x) = \sum_{i=0}^{l} a_i(x) g^{(i)}(x), \tag{5.14}$$

with $a_i(x) \in C(\mathbb{R}),\ i \in \{0, 1, \dots, l\}$. By an antiderivative operator corresponding to D^l we mean an operator I^l which satisfies $D^l \circ I^l = Id$. This condition leads to:

$$\left(D^l \circ I^l\right)(f) = f,$$

which is equivalent with:

$$a_l(x)\left(I^l f\right)^{(l)} + a_{l-1}(x)\left(I^l f\right)^{(l-1)} + \dots + a_1(x)\left(I^l f\right)' + a_0(x)\left(I^l f\right) = f. \tag{5.15}$$

Equation (5.15) is a linear differential equation of order l with non-constant coefficients. For this kind of differential equation we recall the following existence and unicity theorem:

Theorem 5.2.1. *[15] Suppose that $a_i \in C(I),\ i \in \{0, 1, \dots, l\}$, where $I \subset \mathbb{R}$ is an open interval with $a \in I$. Having $b_0, b_1, \dots, b_l \in \mathbb{R}$ such that the initial conditions $y^{(i)}(a) = b_i$ hold, $i \in \{0, 1, \dots, l\}$, the equation*

$$a_l(x) y^{(l)} + a_{l-1}(x) y^{(l-1)} + \dots + a_1(x) y' + a_0(x) y = f(x),\ f \in C(I), \tag{5.16}$$

has an unique solution $y \in C^l(I)$.

From the above theorem we have that for equation (5.15) there exists solutions, and by imposing some initial conditions we will find a unique one. We mention that for the construction of the operators studied in this paper, the exact expression of antiderivative operators does not play an important role since choosing a different antiderivative operator, our operator will be different, but the approximation processes we study do not depend on the choice of the antiderivative operator.

Remark 5.2.2. *(See Remark 2.2. in [113]) From Theorem 5.2.1 we have that the solution of* (5.15) *exists on an open interval I. In order to prove our results, let us take I such that* $[0,1] \subset I$.

Definition 5.2.3. *[113] Let $(L_n)_n$ be a sequence of linear and positive operators on $C([0,1])$. We introduce the following Kantorovich type operators*

$$K_n = D^l \circ L_n \circ I^l, \tag{5.17}$$

which have the expression

$$\begin{aligned} K_n(f,x) &= \left(D^l \circ L_n \circ I^l\right)(f,x) \\ &= D^l\left(L_n(F,x)\right) \\ &= \sum_{i=0}^{l} a_i(x) L_n^{(i)}(F,x), \ x \in [0,1], \ f \in C[0,1], \end{aligned} \tag{5.18}$$

where $F(x) = I^l f(x)$.

In order to prove our results we will need the following.
Let I be a compact interval and

$$M_{n,k}(x) = L_n\left(\frac{(t-x)^k}{k!}, x\right), \ x \in I.$$

In paper [51] it was proved the following result.

Theorem 5.2.4. *[51] Let $(L_n)_n$ be a sequence of positive and linear operators on $C(I)$, I compact, such that L_n is convex of order $r-1$, with $r \geq 0$ an integer. Suppose that $M_{n,4}(x) = o(M_{n,2}(x))$ uniformly for $x \in I$, and $M_{n,0}(x) = 1$, and that there exists two functions $a, b : I \to \mathbb{R}$ such that $\lim_{n\to\infty} nM_{n,1}(x) = a(x)$ and $\lim_{n\to\infty} nM_{n,2}(x) = b(x)$. Then, for $f \in C^{r+2}(I)$ the following limit holds uniformly:*

$$\lim_{n\to\infty} n\left[L_n^{(r)}(f,x) - f^{(r)}(x)\right] = \left[a(x) f'(x) + b(x) f''(x)\right]^{(r)}, \ x \in I. \tag{5.19}$$

5.2.2 Approximation properties

For the operators K_n introduced above we can state the following approximation results.

Theorem 5.2.5. *(See Theorem 4.1 in [113]) Let $(L_n)_n$ be a sequence of linear and positive operators on $C([0,1])$, which has the property of simultaneous approximation $\left\|L_n^{(k)} g - g^{(k)}\right\| \to 0$ uniformly on $[0,1]$, $g \in C^k([0,1])$, $k = \overline{0,l}$, $l \in \{0,1,2\ldots\}$. Then the following convergence holds*

$$\|K_n f - f\| \to 0, \ uniformly \ on \ [0,1], \tag{5.20}$$

with $f \in C([0,1])$.

Using Theorem 5.2.4 we can prove the following Voronovskaja type theorem for our operators K_n.

Theorem 5.2.6. *(See Theorem 4.2 in [113]) Let* $(L_n)_n$ *be a sequence of positive and linear operators on* $C([0,1])$ *such that* L_n *is convex of order* $r-1$*, with* $r \geq 0$ *an integer. Suppose that* $M_{n,4}(x) = o(M_{n,2}(x))$ *uniformly for* $x \in [0,1]$*, and* $M_{n,0}(x) = 1$*, and that there exists two functions* $a, b : [0,1] \to \mathbb{R}$ *such that* $\lim_{n\to\infty} nM_{n,1}(x) = a(x)$ *and* $\lim_{n\to\infty} nM_{n,2}(x) = b(x)$*. Let* $K_n = D^l \circ L_{n+l} \circ I^l$*. Then, for* $f \in C^{r+2}([0,1])$ *the following limit holds uniformly:*

$$\lim_{n\to\infty} n\left[K_n(f,x) - f(x)\right] = \sum_{i=0}^{l} a_i(x)\left[a(x)F'(x) + b(x)F''(x)\right]^{(i)} \qquad (5.21)$$
$$= D^l\left(a(x)F'(x) + b(x)F''(x)\right),\ x \in [0,1].$$

5.2.3 Simultaneous approximation result

In this case we consider $a_i(x) = a_i \in \mathbb{R},\ i \in \{0,1,\ldots,l\},\ l \geq 0$, real constants and the differential operator

$$D^{*l} f(x) = \sum_{i=0}^{l} a_i f^{(i)}(x), \qquad (5.22)$$

and I^{*l} a corresponding antiderivative operator with respect to $D^{*l} \circ I^{*l} g = g$, which leads to a linear differential equation of order l with constant coefficients $a_i \in \mathbb{R}$. For this kind of equation it is well known that solutions exist but they are not unique until we impose some initial conditions which are not needed for the following result. Denote $I^{*l} f := F^*$.

Let L_n be a sequence of operators and define K_n^* as:

$$K_n^* = D^{*l} \circ L_n \circ I^{*l}. \qquad (5.23)$$

For operators K_n^* we can state the following simultaneous approximation result:

Theorem 5.2.7. *(See Theorem 5.1 in [113]) Let* $(L_n)_n$ *be a sequence of operators having the property that* $(L_n g)^{(j)} \to g^{(j)}$ *uniformly for* $x \in [0,1]$ *and* $g \in C^j([0,1]),\ 0 \leq j \leq r+l,\ r \geq 0$*. Then*

$$\lim_{n\to\infty} (K_n^* f)^{(r)} = f^{(r)} \text{ uniformly on } [0,1], \qquad (5.24)$$

$f \in C^r([0,1])$ *and* $F^* \in C^{r+l}([0,1])$.

5.2.4 Particular cases

In this section we will study the generalized Kantorovich modification for some well known operators.

In order to simplify the notations we introduce the following differential operator

$$D_y^l g(x) = \sum_{i=1}^{l} y^i a_i(x) g^{(i-1)}(x). \qquad (5.25)$$

Bernstein operators

Let B_n be Bernstein operators defined in (1.1):

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \ x \in [0,1], \ f \in C([0,1]),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for $0 \le k \le n$.

Let $L_n = B_{n+l}$. Therefore,

$$\begin{aligned} K_n^B(f,x) &= \left(D^l \circ B_{n+l} \circ I^l\right)(f,x) \\ &= \sum_{i=0}^{l} a_i(x) B_{n+l}^{(i)}(F,x), \ x \in [0,1], \ f \in C([0,1]). \end{aligned} \tag{5.26}$$

Theorem 5.2.8. *(See Theorem 6.3 in [113]) The following convergence*

$$\lim_{n\to\infty} K_n^B(f) = f \tag{5.27}$$

holds uniformly on $[0,1]$ *and* $f \in C([0,1])$.

Theorem 5.2.9. *(See Theorem 6.4 in [113]) Let* $a_i \in C^2([0,1])$ *and* $F \in C^{l+2}([0,1])$. *The following limit holds:*

$$\begin{aligned} &\lim_{n\to\infty} n\left[K_n^B(f,x) - f(x)\right] \\ &= \frac{x(1-x)}{2} D^l F''(x) + \frac{1-2x}{2} \frac{\partial D_y^l F''(x)}{\partial y}\Big|_{y=1} - \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}\Big|_{y=1}, \end{aligned} \tag{5.28}$$

$f \in C^2([0,1])$.

Durrmeyer operators

Let D_n be Durrmeyer operators defined in (1.1):

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt, \ x \in [0,1], \ f \in C([0,1]),$$

with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for $0 \le k \le n$.

Durrmeyer operators are useful in approximation of functions $f \in L_1([0,1])$.

We now choose $L_n = D_{n+l}$ in our construction, therefore

$$\begin{aligned} K_n^D(f,x) &= \left(D^l \circ D_{n+l} \circ I^l\right)(f,x) \\ &= \sum_{i=0}^{l} a_i(x) D_{n+l}^{(i)}(F,x), \ x \in [0,1], \ f \in C([0,1]). \end{aligned} \tag{5.29}$$

Theorem 5.2.10. *(See Theorem 6.6 in [113]) The following convergence*

$$\lim_{n\to\infty} K_n^D(f) = f \tag{5.30}$$

holds uniformly on $[0,1]$ *and* $f \in C([0,1])$.

Theorem 5.2.11. *(See Theorem 6.7 in [113]) Let* $a_i \in C^2([0,1])$ *and* $F \in C^{l+2}([0,1])$. *The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^D(f,x) - f(x)\right] \tag{5.31}$$
$$= x(1-x)D^lF''(x) + (1-2x)D^lF'(x) + (1-2x)\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1}$$
$$-2\frac{\partial D_y^l F'(x)}{\partial y}|_{y=1} - \frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1},\ f \in C^2([0,1]).$$

Kantrovich operators

Let $\tilde{K}_n$ be Kantorovich operators defined in (1.1):

$$\tilde{K}_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x \in [0,1],\ f \in C([0,1]),$$

with $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, for $0 \le k \le n$.

For these operators we have that the simultaneous approximation result holds, see [50].

We choose $L_n = \tilde{K}_{n+l}$ in our construction, therefore

$$K_n^{\tilde{K}}(f,x) = \left(D^l \circ \tilde{K}_{n+l} \circ I^l\right)(f,x) \tag{5.32}$$
$$= \sum_{i=0}^{l} a_i(x)\,\tilde{K}_{n+l}^{(i)}(F,x),\ x \in [0,1],\ f \in C([0,1]).$$

Now, because the simultaneous approximation result holds for $K_n^{\tilde{K}}$, we can apply Theorem 5.2.6 and we get the following uniform approximation result:

Theorem 5.2.12. *(See Theorem 6.9 in [113]) The following convergence*

$$\lim_{n\to\infty} K_n^{\tilde{K}}(f) = f \tag{5.33}$$

holds uniformly for $x \in [0,1]$ *and* $f \in C([0,1])$.

Theorem 5.2.13. *(See Theorem 6.10 in [113]) Let* $a_i \in C^2([0,1])$ *and* $F \in C^{l+2}([0,1])$. *The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^{\tilde{K}}(f,x) - f(x)\right] \tag{5.34}$$
$$= \frac{x(1-x)}{2}D^lF''(x) + (1-2x)D^lF'(x) + \frac{1-2x}{2}\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1}$$
$$-\frac{\partial D_y^l F'(x)}{\partial y}|_{y=1} - \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1},\ f \in C^2([0,1]).$$

Stancu operators

Let $S_n^{\alpha,\beta}$ be Bernstein-Stancu operators defined in (**??**):

$$S_n^{\alpha,\beta}(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \ x \in [0,1], \ f \in C([0,1]),$$

with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for $0 \le k \le n$, and $\frac{k+\alpha}{n+\beta} \in [0,1], \ 0 \le k \le n$.

We choose $L_n = S_{n+l}^{\alpha,\beta}$ in our construction, therefore

$$K_n^S(f,x) = \left(D^l \circ S_{n+l}^{\alpha,\beta} \circ I^l\right)(f,x) \tag{5.35}$$
$$= \sum_{i=0}^{l} a_i(x) \left(S_{n+l}^{\alpha,\beta}\right)^{(i)}(F,x), \ x \in [0,1], \ f \in C([0,1]).$$

Because the simultaneous approximation result holds for K_n^S, we can state the following uniform approximation result using Theorem 5.2.5:

Theorem 5.2.14. *(See Theorem 6.12 in [113]) The following convergence*

$$\lim_{n\to\infty} K_n^S(f) \to f \tag{5.36}$$

holds uniformly on $[0,1]$ *and* $f \in C([0,1])$.

Theorem 5.2.15. *Let* $M_{n,m} = \frac{1}{m!} S_n^{\alpha,\beta}((e_1 - xe_0)^m, x)$. *Then*

$$(n+\beta)(m+1)M_{n,m+1}(x) \tag{5.37}$$
$$= x(1-x)\left[M'_{n,m}(x) + mM_{n,m-1}(x)\right] + (\alpha - \beta x)M_{n,m}(x), \ x \in [0,1].$$

Theorem 5.2.16. *(See Theorem 6.13 in [113]) Let* $a_i \in C^2([0,1])$ *and* $F \in C^{l+2}[0,1]$. *The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^S(f,x) - f(x)\right] \tag{5.38}$$
$$= \frac{x(1-x)}{2} D^l F''(x) + (\alpha - \beta x) D^l F'(x) + \frac{1-2x}{2} \frac{\partial D_y^l F''(x)}{\partial y}|_{y=1}$$
$$-\frac{1}{2} \frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1} - \beta \frac{\partial D_y^l F'(x)}{\partial y}|_{y=1}, \ x \in [0,1], \ f \in C^2([0,1]).$$

U_n^ρ operators

U_n^ρ operators are defined as:

$$U_n^\rho(f,x) = \sum_{k=0}^{n} F_{n,k}^\rho(f) p_{n,k}(x), \ f \in C([0,1]), \ x \in [0,1],$$

with

$$F_{n,k}^\rho(f) = \int_0^1 f(t) \mu_{n,k}^\rho(t)\, dt \ \textit{and}$$
$$\mu_{n,k}^\rho(t)\, dt = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

where $B(\cdot,\cdot)$ is Euler's Beta function and $\rho > 0$.

Now, we take $L_n = U_{n+l}^{\rho}$ and we get

$$UK_n^{\rho}(f,x) = \left(D^l \circ U_{n+l}^{\rho} \circ I^l\right)(f,x), \; f \in C([0,1]), \; x \in [0,1]. \tag{5.39}$$

Theorem 5.2.17. *(See Theorem 6.16 in [113]) The following convergence*

$$\lim_{n\to\infty} UK_n^{\rho}(f) \to f \tag{5.40}$$

holds uniformly on $[0,1]$, $f \in C([0,1])$ *and* $\rho \in (0,\infty]$.

Theorem 5.2.18. *(See Theorem 6.17 in [113]) Let* $a_i \in C^2([0,1])$ *and* $F \in C^{l+2}([0,1])$. *The following limit holds*

$$\lim_{n\to\infty} n\left[UK_n^{\rho}(f,x) - f(x)\right] \tag{5.41}$$

$$= \frac{\rho+1}{\rho}\left[\frac{x(1-x)}{2}D^l F''(x) + \frac{1-2x}{2}\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1} - \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1}\right]. \tag{5.42}$$

where $x \in [0,1]$, $f \in C^2([0,1])$.

5.2.5 Nonpositivity of operators

Proposition 5.2.19. *(See Proposition 7.1 in [113]) Operators* K_n^B, K_n^D, K_n^S, $K_n^{\tilde{K}}$ *and* UK_n^{ρ} *are linear but not positive operators.*

6 Double weighted second order modulus of continuity

In this chapter we introduce a second order modulus of smoothness with two weight functions in order to obtain estimates of the degree of approximation of functions with fast growth to infinity, by general positive linear operators which preserve polynomials of degree one. We will also give an example for Szász-Mirakjan operators. The results in this chapter are based on the paper published in R. Păltănea, **B. I. Vasian**, Double weighted modulus, submitted.

The aim of the present work is to give general quantitative convergence results for the pointwise approximation by positive linear operators of functions on interval $[0, \infty)$ with a fast growth to infinity, using a new special second order modulus of smoothness. Also, we will mention some consequences for uniform approximation on compact sets and for weighted approximation.

The new modulus we introduce uses two weight functions. The first one is $\varphi(x) = \sqrt{x}$, $x \in [0, \infty)$, which was already used in the construction of Ditzian-Totik modulus on interval $[0, \infty)$. The second one, denoted by Ψ, has the role of weighting the growth of functions to infinity.

The method described is a direct one and uses a "canonical" sequence which can be attached to a point in interval $(0, \infty)$ which we will define later. This method was used for the first time in estimating the rate of approximation by general positive linear operators in terms of Ditzian-Totik modulus in [47] and developed in [80], [24].

Some examples are given for the case of the Szász-Mirakjan operators, in the last section of this chapter.

6.1 Definitions and basic results

We denote $I = [0, \infty)$ and we use the definitions mentioned in Chapter 2, Section 2.1 for the spaces $\mathcal{F}(I)$, $C(I)$, $C^2(I)$. If $f \in C(I)$ and $b > 0$, denote $\|f\|_{[0,b]} = \max_{x\in[0,b]} |f(x)|$. Let $e_i(t) = t^i$, $(t \in I)$, for $i = 0, 1, 2, \ldots$ and Π_k, the space of polynomials with degree at most k.

For a function $f \in \mathcal{F}(I)$ and three points $0 \le a < y < b$, let us denote

$$\Delta(f, a, y, b) = \frac{b-y}{b-a} f(a) + \frac{y-a}{b-a} f(b) - f(y). \tag{6.1}$$

We consider the function $\varphi(t) = \sqrt{t}$, $(t \in I)$. Also, let $\Psi \in \mathcal{F}(I)$ be an increasing function such that $\Psi(0) > 0$.

Definition 6.1.1. *[87] For $h > 0$ and $f \in \mathcal{F}(I)$, let*

$$\omega_2^{\Psi,\varphi}(f, h) = \sup\left\{ \frac{|\Delta(f, a, y, b)|}{\Psi(y)},\ 0 \le a < y < b,\ b - a \le 2h\varphi\left(\frac{a+b}{2}\right) \right\} \tag{6.2}$$

and

$$B^h_{\Psi,\varphi}(I) = \{ f \in \mathcal{F}(I),\ \omega_2^{\Psi,\varphi}(f, h) < \infty \}. \tag{6.3}$$

It is easy to show that $\omega_2^{\Psi,\varphi}$ is a second order modulus on the space $B^h_{\Psi,\varphi}(I)$. Note that, if $0 < h_1 < h_2$, then $B^{h_1}_{\Psi,\varphi}(I) \subset B^{h_2}_{\Psi,\varphi}(I)$.

Lemma 6.1.2. *[87] Let $h > 0$ and function $\Theta_h(u) = u + h^2 - h\sqrt{4u + h^2}$, $u \in I$.*

i) For any numbers $0 \le t < u$ and $h > 0$, the condition $u - t \le 2h\sqrt{\frac{u+t}{2}}$ is equivalent to inequality $t \ge \Theta_h(u)$.

ii) Let $a > 0$ and $\eta \in (0, 1)$. If $h \le (1 - \eta)\sqrt{\frac{a}{2(1+\eta)}}$, then $\Theta_h(u) \ge \eta u$, $(u \ge a)$.

Lemma 6.1.3. *[87] Let $\Psi \in \mathcal{F}(I)$ be an increasing function such that $\Psi(0) > 0$. If for $f \in C(I)$, there exists $\eta \in (0, 1)$ such that*

$$\lim_{x\to\infty} \frac{f(x)}{\Psi(\eta x)} = 0, \tag{6.4}$$

then

i) for any $h > 0$ we have $\omega_2^{\Psi,\varphi}(f, h) < \infty$;

ii) we have

$$\lim_{h\to 0+} \omega_2^{\Psi,\varphi}(f, h) = 0. \tag{6.5}$$

Lemma 6.1.4. *[87] Let $\Psi \in \mathcal{F}(I)$ be an increasing function, with $\Psi(0) > 0$. If $f \in C^2(I)$ satisfies the following condition: there exist the constants $M > 0$ and $\eta \in (0, 1)$ such that*

$$\|f''\|_{[0,x]} \frac{x}{\Psi(\eta x)} \le M,\ (x > 0), \tag{6.6}$$

then

$$\omega_2^{\Psi,\varphi}(f, h) < \infty,\ (h > 0),\ \textit{and} \lim_{h\to 0+} \omega_2^{\Psi,\varphi}(f, h) = 0. \tag{6.7}$$

Example 6.1.5. *[87]* If $f(x) = x^\gamma$, $(x \in I)$, with $\gamma \ge 2$, and $\Psi(x) = x^\alpha + 1$, $(x \in I)$ with $\alpha > \gamma - 1$, then $\lim_{h\to 0+} \omega_2^{\Psi,\varphi}(f, h) = 0$. One can apply Lemma 6.1.4.

Example 6.1.6. *[87]* If $f \in C(I)$ and there exist $M > 0$, $\gamma > 0$ such that $|f(x)| \le Me^{\gamma x}$, $(x \in I)$, and $\Psi(x) = e^{\alpha x}$, $(x \in I)$, $\alpha > \gamma$, then $\lim_{h\to 0+} \omega_2^{\Psi,\varphi}(f, h) = 0$. One can apply Lemma 6.1.3.

6.2 Canonical sequence attached to a point

In order to obtain estimates with modulus $\omega_2^{\Psi,\varphi}$ we define the canonical sequence attached to a point $y \in (0, \infty)$ and to a number $h > 0$, such that $y \ge h^2$, as follows.

Definition 6.2.1. *[87] Let $h > 0$ and $y \ge h^2$. Let $q \ge 1$ such that $y = q^2h^2$. The canonical sequence attached to y and h is the sequence $(x_j)_{j\ge -2r}$:*

$$0 \le x_{-2r} \le x_{1-2r} < \ldots < x_{-1} < x_0 = y < x_1 < \ldots, \tag{6.8}$$

where $r = [q]$ (the lest integer less than or equal to q) and

$$x_j = \begin{cases} (q + k)^2h^2, & \text{for } j = 2k,\ k \in \mathbb{Z},\ k \ge -r \\ (q + k)(q + k + 1)h^2, & \text{for } j = 2k + 1,\ k \in \mathbb{Z},\ k \ge -r. \end{cases} \tag{6.9}$$

Remark 6.2.2. *[87]*

i) The term $x_{-2r-1} \geq 0$ is not defined because $(q-r-1)(q-r) < 0$.

ii) For $q \in \mathbb{N}$ we have $0 = x_{-2r} = x_{-2r+1} < x_{-2r+2} < \ldots$. For $q \notin \mathbb{N}$ we have $0 < x_{-2r} < x_{-2r+1} < \ldots$

iii) We have

$$x_{2k+1} - x_{2k} = h\varphi(x_{2k}), \text{ for } k \geq -r; \tag{6.10}$$

$$x_{2k} - x_{2k-1} = h\varphi(x_{2k}), \text{ for } k \geq -r+1. \tag{6.11}$$

Lemma 6.2.3. *[87] We have*

$$x_{j+1} - x_{j-1} \leq 2h\varphi\left(\frac{x_{j+1}+x_{j-1}}{2}\right), \ \forall j \geq 1-2r, \tag{6.12}$$

and consequently, for any $f \in B^h_{\Psi,\varphi}(I)$

$$|\Delta(f, x_{j-1}, x_j, x_{j+1})| \leq \Psi(x_j)\omega_2^{\Psi,\varphi}(f,h), \ j \geq 1-2r. \tag{6.13}$$

Lemma 6.2.4. *[87] Let $h > 0$ and $y = q^2h^2$, with $q \geq 1$. Let $f \in B^h_{\Psi,\varphi}(I)$ be such that $f(y - h\varphi(y)) = 0$ and $f(y + h\varphi(y)) = 0$. Then*

$$|f(u) - f(y)| \leq \Psi(u)\left(1 + \frac{(u-y)^2}{yh^2}\right)\omega_2^{\Psi,\varphi}(f,h), \ \forall u > y + h\varphi(y). \tag{6.14}$$

Lemma 6.2.5. *[87] Under conditions of Lemma 6.2.4, we have:*

$$|f(t) - f(y)| \leq \Psi(y)\left(1 + 4\frac{(t-y)^2}{yh^2}\right)\omega_2^{\Psi,\varphi}(f,h), \ \forall\, 0 \leq t < y - h\varphi(y), \tag{6.15}$$

(if there exists such t).

Let us fix $y > 0$, $h > 0$ and $\Psi \in \mathcal{F}(I)$, an increasing function, such that $\Psi(0) > 0$. For $f \in B^h_{\Psi,\varphi}(I)$, let us consider the function:

$$\Lambda_f(s) = \left[\left(1 + \frac{(s-y)^2}{yh^2}\right)\Psi(s) + \left(1 + 4\frac{(s-y)^2}{yh^2}\right)\Psi(y)\right]\omega_2^{\Psi,\varphi}(f,h), \ (s \in I). \tag{6.16}$$

Lemma 6.2.6. *[87] Under conditions of Lemma 6.2.4 we have*

$$|f(s) - f(y)| \leq (\Psi(s) + \Psi(y))\omega_2^{\Psi,\varphi}(f,h), \ (s \in [y - h\varphi(y), y + h\varphi(y)]). \tag{6.17}$$

Corollary 6.2.7. *[87] In conditions of Lemma 6.2.4, we have:*

$$|(u-y)(f(t)-f(y)+(y-t)(f(u)-f(y))| \leq (u-y)\Lambda_f(t) + (y-t)\Lambda_f(u), \tag{6.18}$$

for all points $0 \leq t < y < u$.

Lemma 6.2.8. *[87] Let $h > 0$ and $0 < y \leq h^2$. Let $f \in B^h_{\Psi,\varphi}(I)$ such that $f(0) = 0$ and $f(2h^2) = 0$. Then relation (6.18) is satisfied for all points $0 \leq t < y < u$.*

6.3 The main results

In order to obtain the main result we shall use of the following general result.

Lemma 6.3.1. *[87] Let μ be a positive measure on interval J and let F be the functional defined by μ, i. e. $F(f) = \int_J f(t)d\mu(t)$, $f \in \mathcal{L}_\mu(J)$. Let y be an interior point of J. Suppose that $\Pi_1 \subset \mathcal{L}_\mu(J)$, $F(e_0) = 1$, $F(e_1) = y$ and $F(|e_1 - ye_0|) > 0$. Let $f \in \mathcal{L}_\mu(J)$ and $\Phi \in \mathcal{L}_\mu(J)$ be such that $\Phi \geq 0$ and*

$$|(u-y)(f(t)-f(y))+(y-t)(f(u)-f(y))| \leq (u-y)\Phi(t)+(y-t)\Phi(u), \quad (6.19)$$

for all $t, u \in J$, $t < y < u$. Then:

$$|F(f) - f(y)| \leq F(\Phi). \quad (6.20)$$

Remark 6.3.2. In [80]-Theorem 2.1.1 a more general result is given, with a different proof.

Recall that $I = [0, \infty)$. Let us fix an increasing function $\Psi : I \to (0, \infty)$ and consider a sequence of positive linear operators $L_n : V \to \mathcal{F}(I)$, of the form

$$L_n(f, y) = \int_{[0,\infty)} f(t)d\mu_{n,y}(t), \ (f \in V, \ y \in I, \ n \in \mathbb{N}), \quad (6.21)$$

where $\{\mu_{n,y}, \ (n, y) \in \mathbb{N} \times I\}$ is a family of positive measures and by V we mean $V = \bigcap_{(n,y)\in\mathbb{N}\times I} \mathcal{L}_{\mu_{n,y}}(I)$. Also suppose that

$$\Pi_2 \subset V, \text{ and } \Psi\Pi_2 \subset V \quad (6.22)$$

$$L_n(e_i) = e_i, \ i = 0, 1, \text{ and } L_n(|e_1 - ye_0|)(y) \neq 0, \text{ for } n \in \mathbb{N}, \ y \in I. \quad (6.23)$$

Also, using the notations given in Section 6.1 we state our main result.

Theorem 6.3.3. *[87] If $(L_n)_n$ is a sequence of positive linear operators of type (6.21) satisfying conditions (6.22) and (6.23). Then for any $f \in B^h_{\Psi,\varphi}(I) \cap V$, $n \in \mathbb{N}$, $y \in I$ and $h > 0$ we have:*

$$|L_n(f,y) - f(y)| \leq \left[\Psi(y)\left(1 + 4L_n\left(\left(\frac{e_1 - ye_0}{h\varphi(y)}\right)^2, y\right)\right) + L_n\left(\Psi\cdot\left(e_0 + \left(\frac{e_1 - ye_0}{h\varphi(y)}\right)^2\right), y\right)\right]\omega_2^{\Psi,\varphi}(f,h). \quad (6.24)$$

In the particular case where $\Psi(s) = 1$, $(s \in I)$, denote $\omega_2^{1,\varphi}(f,h)$ simply by $\omega_2^{\varphi}(f,h)$ and $B^h_{\Psi,\varphi}(I)$ by $B^h_\varphi(I)$. In this case we obtain a simpler result.

Corollary 6.3.4. *[87] If the conditions of Theorem 6.3.3 are satisfied, and $\Psi = e_0$, then inequality*

$$|L_n(f,y) - f(y)| \leq \left[2 + 5L_n\left(\left(\frac{e_1 - ye_0}{h\varphi(y)}\right)^2, y\right)\right]\omega_2^{\varphi}(f,h) \quad (6.25)$$

holds for any $f \in B^h_\varphi(I) \cap V$, $n \in \mathbb{N}$, $y \in I$ and $h > 0$.

Corollary 6.3.5. *[87] Let $(L_n)_n$ be a sequence of positive linear operators as in Theorem 6.3.3. Let $b > 0$. Suppose that Ψ is increasing, $\Psi(0) = 1$ and $\|L_n(\Psi)\|_{[0,b]} < \infty$. Denote*

$$h_n^b = \sup_{y\in(0,b]} y^{-1} L_n(\Psi(e_1 - y)^2, y),\ (n \in \mathbb{N}), \tag{6.26}$$

and suppose that $h_n^b < \infty$. If $f \in V$ satisfies the condition $\omega_2^{\Psi,\varphi}(f, h_n^b) < \infty$, $n \in \mathbb{N}$, then

$$\|L_n(f) - f\|_{[0,b]} \le \Big(5\Psi(b) + \|L_n(\Psi)\|_{[0,b]} + 1\big)\omega_2^{\Psi,\varphi}\Big(f, \sqrt{h_n^b}\Big). \tag{6.27}$$

Consequently, if $\lim_{n\to\infty} h_n^b = 0$, for each $b > 0$ and $\lim_{h\to 0} \omega_2^{\Psi,\varphi}(f, h) = 0$, then $(L_n(f))_n$ is uniform convergent on compact sets to f.

6.4 Applications to Szász-Mirakjan operators

Szász-Mirakjan operators are defined by:

$$S_n(f, y) = e^{-ny} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} f\left(\frac{k}{n}\right),\ (y \in I), \tag{6.28}$$

where $f \in \mathcal{F}(I)$ is a function for which the series is convergent. These operators can be represented using a family of measures $\mu_{n,y} = e^{-ny} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} \delta_{k/n}$, where δ_z is the Dirac measure of the point z.

Remark 6.4.1. *[87]* If $f \in \mathcal{F}(I)$ satisfies condition $|f(y)| \le M e^{\gamma y}$, $\forall\ y \in I$, where $M > 0$ and $\gamma > 0$ are constants, then $S_n(f)(y)$ is well-defined for all $y \in I$ and $n \in \mathbb{N}$. Indeed, using the inequality $n\left(e^{\frac{\gamma}{n}} - 1\right) \le e^{\gamma} - 1$, $n \in \mathbb{N}$, $\alpha > 0$, we have

$$e^{-ny} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} \left| f\left(\frac{k}{n}\right)\right| \le M e^{-ny} \sum_{k=0}^{\infty} \frac{(nye^{\frac{\gamma}{n}})^k}{k!} = M e^{ny(e^{\frac{\gamma}{n}} - 1)} \le M e^{(e^{\gamma} - 1)y}.$$

Then the series in (6.28) is absolutely convergent.

Denote $E(I) = \{f \in \mathcal{F}(I),\ \exists M > 0,\ \exists \gamma > 0,\ |f(t)| \le M e^{\gamma t},\ (t \in I)\}$.

Theorem 6.4.2. *Let $\Psi(t) = e^{\alpha t}$, $(t \in I)$, $\alpha > 0$. Consider the function*

$$H_\alpha(y) = 5e^{\alpha y} + e^{(e^{\alpha} - 1)y}(1 + e^{\alpha} + y(e^{\alpha} - 1)^2),\ (y \in I). \tag{6.29}$$

Let $f \in E(I)$ and $n \in \mathbb{N}$ be such that $\omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right) < \infty$. We have

$$|S_n(f, y) - f(y)| \le H_\alpha(y)\omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right),\ (y \in I), \tag{6.30}$$

and consequently, for each $b > 0$:

$$\|S_n(f) - f\|_{[0,b]} \le H_\alpha(b)\omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right). \tag{6.31}$$

Corollary 6.4.3. *[87] If $f \in E(I)$, then sequence $(S_n(f))_n$ is uniformly convergent on compact sets to f.*

Moreover, if $f \in C(I)$ satisfies condition $|f(x)| \leq Me^{\gamma x}$, $(x \in I)$, with $M > 0$, $\gamma > 0$, then for each $b > 0$ relation (6.31) holds for $n \in \mathbb{N}$, where $\alpha > \gamma$ and $\Psi(x) = e^{\alpha x}$, $(x \in I)$. Also $\lim_{n\to\infty} \omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right) = 0$.

Finally we consider the weighted approximation. For $\beta > 0$, let the weight function $\exp_{(-\beta)}(x) = e^{-\beta x}$, $(x \in I)$. For $f \in \mathcal{F}(I)$, denote

$$\|f\|_{(-\beta)} = \sup_{x \in I} |f(x)|e^{-\beta x}. \tag{6.32}$$

Corollary 6.4.4. *[87] Let $f \in C(I)$, be such that $|f(x)| \leq Me^{\gamma x}$, $(x \in I)$, where $M > 0$, $\gamma > 0$. If $\beta > \max\{\gamma, e^{\gamma} - 1\}$, then:*

$$\lim_{n\to\infty} \|S_n(f) - f\|_{(-\beta)} = 0. \tag{6.33}$$

Moreover, if there is α, β, such that $\beta > \max\{\alpha, e^{\alpha} - 1\}$ and $\alpha > \gamma$, then:

$$\|S_n(f) - f\|_{(-\beta)} \leq \sup_{y \in I}(H_\alpha(y)e^{-\beta y})\omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right), \tag{6.34}$$

where $\Psi(x) = e^{\alpha x}$, $(x \in I)$, $\sup_{y\in I}(H_\alpha(y)e^{-\beta y}) < \infty$ and $\lim_{n\to\infty} \omega_2^{\Psi,\varphi}\left(f, \frac{1}{\sqrt{n}}\right) = 0$.

7 Conclusions

The results presented in this thesis bring new contributions to approximation theory.

We have proved the usefulness of considering operators which are not positive by providing estimation results that are improved on the interval on which operators considered are not positive.

We also have introduced a method of generating approximation operators of Kantorovich type defined using an arbitrary linear differential operator with constant or non-constant coefficients.

Moreover, we have introduced a new second order modulus which is helpful in obtaining estimates of the degree of approximation of functions with fast growth to infinity, by general sequences of positive linear operators.

In what regards further development, this thesis opens new directions of research. One of these is the approximation of functions using operators that are not mandatory positive operators and research the improvements they can bring.

Also, in what concerns the modulus introduced, the problem remains open to considering the second weight function convex instead of increasing.

In conclusion, this thesis brings results that significantly advances the approximation theory by linear operators by introducing novel methods and concepts.

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