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GENERALIZATIONS OF THERMOELASTIC MEDIA

SUMMARY

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INTRODUCTION

Micropolar media

In the study of continuum mechanics, micropolar media are represented by materials that take into account the effects of microrotations and micromoments. In other words, in addition to the classical displacements and stresses considered in traditional elastic or viscoelastic media, micropolar media also take into account small rotations of material particles and the moments acting on them. The concept of micropolar media was developed by the Cosserat brothers and later expanded by other researchers, in particular by the Turkish-born professor A. Cemal Eringen. In his works from the 1960s, Eringen proposed the micropolar theory to describe the behavior of materials that cannot be adequately explained by classical continuum mechanics. Eringen showed how the micropolar theory can be applied in various fields, such as: composite materials (carbon fiber, laminated materials), granular materials (sand, soils, dust and metal powders), porous structures (metallic and polymer foams, rocks and geological materials), biomechanics (bone, cartilage), liquid crystals and polymers and nanotechnology (carbon nanotubes, graphene). Each of these materials presents characteristics that cannot be adequately captured by classical continuum models, thus justifying the need for the micropolar approach.

Dipolar media

Dipolar media in the context of elasticity are materials in which the mechanical properties are influenced by internal dipole moments. This extends the classical theory of elasticity to take into account the effects caused by the non-uniform distribution of electric charges and dipole moments at the microscopic level. The first to carry out studies on these media are Mindlin, Rivlin and Green, as in [46]. The degrees of freedom of each moment corresponding to this theory are twelve in number and include nine microstrains and three translations.

Thesis structure: In the present work, the original results are contained in 4 chapters as follows:

Chapter I - General aspects of isotropic micropolar media

The first chapter of this paper comprises results corresponding to isotropic micropolar media and is divided into three sections.

In the first section, the variational principle and the continuous dependence for an isotropic and homogeneous micropolar medium are obtained. For this purpose, we start from an alternative presentation of the problem with initial and boundary data, characterized by equations (1.7), (1.8), (1.11), with the conditions (1.13), (1.14), relations analogous to those in [7]-[8], as well as in [10]-[11]. Next, a theorem is obtained that provides an alternative characterization of the mixed problem and, moreover, we obtain the variational principle of the universal poro-thermoelastic theory with a single relaxation parameter for an isotropic and homogeneous micropolar medium. In the last part of the section, we aim to obtain the continuous dependence of the solutions of the problem with respect to initial and boundary data and with respect to the loads in the present context.

The purpose of Section II is the study of the plane deformation of isotropic

micropolar materials in the equilibrium theory, where in addition to displacement and absolute temperature, the particles of the mentioned materials exhibit pores and microrotations. The determined solution helps to study the effects of heat sources and pores on the deformation of the body. So, using the equilibrium equations (1.56)-(1.59) from [1], we obtain the solutions of the field equations. Later we obtain a result that helps to study the mentioned effects, and in the last part of this section, we consider a cylindrical hole in an elastic space, which also contains the domain $B = \{x : x_1^2 + x_2^2 > r_1^2, x_3 \in \mathbb{R}, (r_1 > 0)\}$. Obtaining the deformation involves determining the functions $\theta, \varphi, u_\alpha$ and ϕ_α , more precisely the solutions of the formed system, which were denoted by V, W, Q and U .

More considerations can be found in [22-24] and a more extensive presentation in [30].

In section III, the physical characteristics of thermomechanical bodies are investigated in short time intervals. The emphasis is placed on the transmission of thermal energy, the evolution from a single relaxation time to the dual phase-shifted (DPL) model being clearly necessary. Therefore, τ_q and τ_θ are introduced as two lag times of the analyzed model, more precisely they represent characteristics that assume the achievement of thermal equilibrium, as well as the existence of collisions between electrons and photons. In the first part of this section, two shear waves are obtained, undamped in time and unaffected by porosity and/or temperature. In the next step, starting from the dispersion equation, five longitudinal waves are obtained, and in the last stage, more precisely the numerical part, the effects of thermoelastic coupling with microrotations on longitudinal waves are highlighted.

It is necessary to take into account that, in our study, the relation (1.110) was obtained from the constitutive relation for a micropolar, isotropic and homogeneous thermoelastic medium, using the Taylor series expansion with respect to time, up to the 2nd degree. The description of the effect of porosity on elasticity is made by Cowin-Nunziano in [13]-[14], and in [8], [16], [18] the thermal response is described, related to the time differential formulation of the Tzou model. The form of the solution (1.114) is in accordance with [13].

The study of wave propagation with two lag times in a porous and isotropic micropolar thermoelastic material represents both an evolution and a synthesis of previous studies. Some results in this regard can be found in [35]-[44] and [69].

Chapter II- Advanced solutions in isotropic micropolar media

This chapter is devoted to the study of solutions in terms of complex potentials for Cosserat media with pores. Thus, the first step involves the determination of the plane deformation within the equilibrium theory of micropolar, homogeneous, isotropic bodies with pores. Using the constitutive equations (2.1)-(2.3), the geometric equations (2.4) and the mass-free equilibrium equations (2.5)-(2.7) from [1], we focus on addressing the fundamental boundary-value problems of the plane deformation theory, see also [45]-[47]. Subsequently, we obtain a description of the displacement, microrotations and pores using complex analytic functions and two real functions, based on the homogeneous Helmholtz equations, described in [4]. Next, the structure of potential functions is studied for several domains

of interest and we apply the method of complex variables, without introducing stress functions, to solve the Kirsch problem [6]. The last part of this section is dedicated to the numerical study, where the graphs corresponding to the complex potentials and the distributions of stress and displacement in an isotropic micropolar medium are obtained.

Chapter III - Anisotropic micropolar media

In the first section of this chapter, uniqueness and instability in two-temperature thermoelasticity are studied. Many studies dedicated to classical thermoelasticity have used a heat propagation equation that is based on the classical Fourier law. Consequently, the heat flux vector depends on the temperature gradient and, therefore, the thermal signals will propagate with infinite speed. This contradicts the principle of causality, and in order to avoid this contradiction, a series of new theories of thermoelasticity have emerged that propose different alternatives to the classical heat conduction equation. Thus, various models have emerged, of which the best known in the literature are Green and Lindsay[31], Lord and Shulman[48], Green and Naghdi[49]-[51], More-Gibson-Thompson[52] or [53]. In all these models, thermal waves propagate with finite velocities and all the results of these generalized theories are more general and more realistic from the point of view of physics than in the classical theory. In our study we propose a new variation of temperature, which depends on two temperatures, by modifying the relationship between the two temperatures, namely the thermodynamic temperature and the conductive temperature. There are many studies that take into account the two temperatures, among which we mention [54-57]. Other generalizations of the heat conduction equation can be found in many articles, among which we mention [58-60] and [64-68].

Our uniqueness result is obtained assuming that the initial energy is not strictly positive (strictly positive energy contradicts the assumption that the solutions coincide). Other uniqueness results are based on the assumption that the elastic tensor is positive definite. However, there are concrete thermoelastic situations in which the positive definition of the elastic tensor cannot be guaranteed.

Our result on exponential instability is also obtained from the assumption that the initial energy is not strictly positive. It should be emphasized that our mixed problem is considered both in the theory in which it is assumed to depend on the variation of both temperatures, and in the theory in which there is no dependence on the variation of the conductive temperature, but on the variation of the thermodynamic temperature. The calculations are, however, quite similar in both situations, which is why the demonstrations are made in detail only in the case of the dependence on the variation of the thermodynamic temperature.

In the second section, the purpose of the study is represented by the formulation of the mixed problem with initial and boundary data, within the Cosserat theory of thermodynamics of media and obtaining qualitative results for the solutions of the formulated problem. One of the reasons why the Cosserat theory of thermoelastic media captured the interest of many specialists was that this theory predicts the finite speed of heat signals, as most non-classical theories of

thermoelasticity did. This theory, initiated by the Cosserat brothers, introduced a mechanics of continuous solids based on the principle that each point of the medium has the six degrees of freedom, just like a rigid body. Since the appearance of this theory, but especially in the recent period, a lot of works have been published that highlight its advantage over the classical theory of thermoelasticity, but also its practical importance, as in [19-20] or in [101-113]. Specialists appreciate that a natural fibrous composite, such as human bone or animal bone, has a tension and bending behavior that is more faithfully described by the Cosserat elasticity than by the classical elasticity. Similar results to those in this section have been obtained for classical thermoelastic media, among which we mention [28], [29] and [114]. In some cases they were based on the Laplace transform. In other works, these results were possible due to the reformulation of the initial mixed problem, namely the energy equation. Neither of the two procedures is used in our study. The plan of this section is as follows: We synthesize the main equations and conditions that characterize the mixed problem from the Cosserat thermodynamics theory, namely the equations of motion, the energy equation, the initial data and the boundary conditions. We also specify which are the regularity conditions imposed on the functions we work with, which allow us to obtain the proposed results. Subsequently, we formulate and prove the main results of our study. So we present two reciprocity results, a uniqueness result, and the variational principle, which extends similar principles from classical thermoelasticity.

Many studies have been dedicated to micropolar environments, among which we mention [78-85] and [91].

Chapter IV - Dipolar Media

In the first section of Chapter III, results of uniqueness and instability are obtained, following the same course as in the case of micropolar media. In addition, the motivation for considering the effect due to the dipolar structure should be mentioned. In the opinion of many researchers, it is known that this effect makes an important contribution to the deformation of media. It is sufficient to refer to media that have a granular structure, for example polymers, human bones or graphite.

Section II is dedicated to the effect of voids and internal state variables. The reason why we have taken into account voids in the material is due to the significant increase in the number of studies dedicated to the theory of porous media in recent years. The initiating work of this theory is considered to be [5]. As we know, in this theory a new degree of freedom appears, associated with the independent rotation of material particles around their center of mass, for a more complex description of the behavior of materials in terms of mechanical properties (see, for example, [70]-[74]). The importance of porous media is seen in the geophysical context (geological materials, for example, rocks or soils) and in the case of granular materials (artificially obtained). Of course, the first studies were developed in the linear case, without taking into account the thermal effect (see, for example, [73]). Then the studies addressed the thermoelasticity of bodies with voids [75]. Other works on this topic are [76-77]. Also in our work we consider a

dipolar structure, with a particular case of microrotations which was introduced by Eringen. Later many works related to microstructured media appeared which increased the importance of this theory.

The dipolar structure, as a particular case of microstructure, aims to eliminate, at least partially, some contradictions in the classical elasticity theory (one, very well known, heat waves propagate with infinite speed). Some remarkable researchers have addressed this type of structure and obtained conclusive results, among which we highlight [61-63]. Different features of the generalized theory are addressed in [53], [59-60] and [84-90].

The reason for studying the effect of internal state variables is represented by the fact that recently the number of studies dedicated to the theory of media with internal state variables has significantly increased. Internal state variables can be considered a means of evaluating the mechanical properties of a body, but also many other effects, such as electrical, magnetic, chemical. For the first time, internal state variables were taken into account to describe the evolution of viscoelastic bodies in the theory of thermoelasticity (Chiriță [23]). Then it was found that these variables are useful for studying the behavior of other types of materials as well. So, in the study of Nachlinger and Nunziato [93], internal state variables are used for the evolution of finite deformations without heat conduction, for a one-dimensional body. Sherburn, Horstemeyer, Bammann and Baumgardner [94] describe the implications of the so-called geological material, such as silicate rocks and others. In [95] it is shown that hysteresis of bodies, the phenomenon by which a material or a physical system does not instantly return to its initial state after being subjected to an external action (e.g. forces, magnetic or electric fields), can be described using a set of internal state variables. In [96] we find a model for an amorphous thermoplastic using a set of internal state variables that motivates this thermodynamic approach. In [97], the authors develop an elastic-viscoplastic theory for the deformation of polymeric and metallic glasses. A gradient theory based on state variables is presented in [98], and this theory provides a consistent framework and strong coupling to prescribe dissipation, the process by which energy is lost or degraded, and energy storage. For some results on antiplane states in an anisotropic elastic body and for orthotropic or isotropic plates, see [99-100].

Notations

- The time derivative is introduced by a dot above a letter;
- By convention, indices are understood to have the integer values 1, 2, or 3;
- The summation rule regarding repeating indices is given by Einstein's summation convention;
- For the partial derivative of a function f with respect to the spatial variables x_j , we will use the notation $f_{,j}$;
- For a bounded domain Ω in the three-dimensional Euclidean space $\mathbb{R}^3$, we denote by $\partial\Omega$ its boundary and $\bar{\Omega}$ its closure;
- $Ox_i, i = 1, 2, 3$, represents the fixed orthogonal axes system to which the body's motion is related;
- Functions belonging to the regularity class $C^{\alpha,\beta}$ are differentiable functions of

order at most α with respect to the time variables and of order at most β with respect to the spatial variable.

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Capitol 1

General aspects of isotropic micropolar media

1.1 Variational principle and continuous dependence

1.1.1 Basic equations

We consider an isotropic and homogeneous micropolar material occupying a regular region B in three-dimensional Euclidean space. The closure and boundary of B are denoted by $\overline{B}$ and ∂B , and $\partial B_i, (i = 1, 2, 3, 4)$ are subsets of the boundary. According to [2]-[3] and [7], the basic equations describing the evolution of a porous isotropic micropolar thermoelastic medium defined on the domain $\overline{B} \times [0, \infty)$ are:

- the constitutive equations:

$$t_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) + k(u_{i,j} + \epsilon_{ijk} \phi_k) + \xi \varphi \delta_{ij} - (2\lambda + 2\mu + k) \nu \theta \delta_{ij}, \quad (1.1)$$

$$m_{ij} = \alpha \phi_{k,k} \delta_{ij} + \gamma \phi_{j,i} + \epsilon \phi_{i,j} + \xi \varphi \delta_{ij}, \quad (1.2)$$

$$g = -\xi u_{k,k} - \psi \phi_{k,k} - a\varphi - b\theta, \quad (1.3)$$

$$h_i = d\varphi_{,i}, \quad (1.4)$$

$$\rho\eta = (3\lambda + 2\mu + k) \nu u_{k,k} - b\varphi + c\theta; \quad (1.5)$$

- equations of motion:

$$t_{ji,j} + \rho F_i = \rho \ddot{u}_i, \quad (1.6)$$

$$m_{ji,j} - \epsilon_{ijk} t_{jk} + \rho G_i = \rho I_{ij} \ddot{\phi}_i; \quad (1.7)$$

- the equation for the evolution of voids.

$$h_{i,i} + g + \rho L = \rho \chi \ddot{\varphi}; \quad (1.8)$$

- heat equations:

$$r\dot{\varrho}_{,i} = k\theta_i - \varrho_i, \quad (1.9)$$

$$k\varphi_{,kk} + a_0\left(1 + r\frac{\partial}{\partial t}\right)\left[-(3\lambda + 2\mu + k)\nu\dot{u}_{k,k} + b\dot{\varphi} - c\dot{\theta} + \frac{\rho r}{\theta_0}\right] = 0; \quad (1.10)$$

- energy equation:

$$\rho\theta_0\dot{\eta} = \varrho_{i,i} + \rho r; \quad (1.11)$$

and the geometric equations:

$$e_{ij} = u_{i,j} + \epsilon_{ijk}\phi_k, \quad \chi_{ij} = \phi_{i,j}. \quad (1.12)$$

In the previous equations we used the notations present in the table below.

Notations	Physical Interpretation
λ, μ	constants Lamé
ν	linear thermal expansion
η	entropy per unit of mass
k	thermal conductivity
$\alpha, \gamma, \epsilon, \xi, \zeta, a, b, c, d, I$	constitutive moduli
f_i	mass force
G_i	mass torque
L	extrinsic force of body
r	internal heat source
χ	inertia coefficient
I_{ij}	components of microinertia tensor
t_{ij}	components of tension tensor
m_{ij}	components of tensor micromomentum
ϱ_i	components of the thermal conduction vector
u_i	components of the displacement vector
ϕ_i	components of the microrotation vector
φ	volume fraction change
ρ	mass density
ϵ_{ijk}	Ricci tensor
δ_{ij}	Kronecker symbol

Tabel 1.1: Notations

To define the mixed problem, in our context, we need initial conditions in the following form:

$$\begin{aligned} u_i(0, x) = u_i^0(x), \quad \dot{u}_i(0, x) = u_i^1(x); \quad \phi_i(0, x) = \phi_i^0(x), \quad \dot{\phi}_i(0, x) = \phi_i^1(x); \\ \varphi(0, x) = \varphi^0(x), \quad \theta(0, x) = \theta^0(x), \quad \varrho_i(0, x) = \varrho_i^0(x); \end{aligned} \quad (1.13)$$

and the boundary conditions in the general form:

$$\begin{aligned} u_i = \tilde{u}_i, \text{ on } \partial B_1 \times [0, \infty); \quad t_{ij}n_j = \tilde{t}_i \text{ on } \partial B_1^c \times [0, \infty); \\ \phi_i = \tilde{\phi}_i \text{ on } \partial B_2 \times [0, \infty); \quad m_{ij}n_j = \tilde{m}_i \text{ on } \partial B_2^c \times [0, \infty); \\ \varphi = \tilde{\varphi} \text{ on } \partial B_3 \times [0, \infty); \quad h_in_i = \tilde{h} \text{ on } \partial B_3^c \times [0, \infty); \end{aligned} \quad (1.14)$$

$$\theta = \tilde{\theta} \text{ on } \partial B_4 \times [0, \infty), \quad \varrho_i n_i = \tilde{\varrho} \text{ on } \partial B_4^c \times [0, \infty),$$

where $\partial B_1, \partial B_2, \partial B_3$ and ∂B_4 with their complements $\partial B_1^c, \partial B_2^c, \partial B_3^c$ and ∂B_4^c , sunt submulțimi ale lui ∂B which satisfy the conditions:

$$\begin{cases} \overline{\partial B_1} \cup \partial B_1^c = \overline{\partial B_2} \cup \partial B_2^c = \overline{\partial B_3} \cup \partial B_3^c = \overline{\partial B_4} \cup \partial B_4^c = \partial B, \\ \partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \partial B_4 \cap \partial B_4^c = \emptyset, \end{cases}$$

and n_i are components of the external normal to ∂B .

We call admissible state, an ordered collection of functions $S = \{u_i, \phi_i, \varphi, e_{ij}, \chi_{ij}, t_{ij}, m_{ij}, h_i, g, \theta\}$, functions that satisfy the following conditions:

- a) $u_i, \phi_i, \varphi \in C^{1,2}; \quad t_{ij}, m_{ij}, h_{i,i}, g \in C^{1,0}; \quad \theta \in C^{1,0};$
- b) $t_{ij} = t_{ji}, m_{ij} = m_{ji}, e_{ij} = e_{ji}; \chi_{ij} = \chi_{ji} \text{ on } \bar{B} \times [0, \infty).$

We consider the problem P , with initial and boundary data, consisting of the system of equations (1.1)- (1.12), with the conditions (1.13), (1.14).

1.1.2 Alternative formulation. Variational theorem

In this section we present an alternative to the P problem with initial and boundary data. We start by defining the corresponding convolution of two space and time functions, as follows:

$$(u * v)(x, t) = \int_0^t u(x, t - \tau) v(x, \tau) d\tau; \quad (x, t) \in \bar{B} \times [0, \infty), \quad (1.15)$$

which takes place for any $x \in B$ and where u and v are defined on $\bar{B} \times [0, \infty)$ and continue in relation to $t \in [0, \infty)$. Let us recall the properties of convolution:

$$u_1 * u_2 = u_2 * u_1, \quad (1.16)$$

$$u_1 * (u_2 + u_3) = (u_1 * u_2) + (u_1 * u_3), \quad (1.17)$$

$$u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3, \quad (1.18)$$

$$u_1 * u_2 = 0 \Rightarrow u_1 = 0 \text{ sau } u_2 = 0. \quad (1.19)$$

We will consider, next, the functions $m(t)$ and $n(t)$ so that

$$m(t) = t \text{ and } p(t) = 1, \quad t \in [0, \infty). \quad (1.20)$$

Furthermore, we set the following functions: f_i, g_i, l and ω defined on $\bar{B} \times [0, \infty)$ as follows:

$$f_i = \rho m * F_i + \rho [t u_i^1(x) + u_i^0(x)], \quad (1.21)$$

$$g_i = \rho m * G_i + I_{ij} [t \phi_i^1(x) + \phi_i^0(x)], \quad (1.22)$$

$$l = \rho m * L + \rho \chi [t \varphi^1(x) + \varphi^0(x)], \quad (1.23)$$

$$\omega = \rho p * r + \rho \theta_0 \eta. \quad (1.24)$$

To achieve the desired results, we will use the following properties:

$$m * \ddot{w}(x, t) = w(x, t) - [t\dot{w}(x, 0) + w(x, 0)], \quad (1.25)$$

$$p * \dot{w}(x, t) = w(x, t) - w(x, 0), \quad (1.26)$$

$$\begin{aligned} m * \dot{w}(x, t) &= p * (p * \dot{w}(x, t)) = p * [w(x, t) - w(x, 0)] \\ &= p * w(x, t) - tw(x, 0), \end{aligned} \quad (1.27)$$

where $w(x, t)$ and $\dot{w}(x, t)$ are defined on $\bar{B} \times [0, \infty)$ and are continuously differentiable on $[0, \infty)$, as in [11].

Taking all this into account, we can formulate the following theorem, which provides the necessary and sufficient conditions for the existence and uniqueness of the solution to the problem.

Teoremă 1. *The functions $u_i, \phi_i, \varphi, t_{ij}, m_{ij}, \eta, \varrho_i$ verify the relations (1.6) - (1.9), (1.11) and the initial conditions (1.13) $\Leftrightarrow$ satisfy the following system of equations,*

$$\begin{cases} m * t_{ji,j} + f_i = \rho u_i, \\ m * (m_{ji,j} - e_{ijk} t_{jk}) + g_i = I_{ij} \phi_i, \\ m * (h_{i,i} + g) + l = \rho \chi \varphi, \\ p * \varrho_{i,i} + \omega = \rho \theta_0 \eta. \end{cases} \quad (1.28)$$

Based on Theorem 1, we can reformulate the mixed problem as follows:

Let $S = \{u_i, \phi_i, \varphi, \theta, e_{ij}, \psi_{ij}, t_{ij}, m_{ij}, h_i, g, \}$ be an admissible state. We say that S is a solution of the mixed problem if and only if it satisfies the system (1.28), the constitutive equations, the geometric equations, and the initial and boundary conditions.

1.1.3 The variational principle

In this section, we formulate the variational principle in the context of the generalized poro-thermoelastic theory with a single relaxation parameter for an isotropic and homogeneous micropolar medium. To obtain the variational formulation, we will consider Theorem 1 and the new formulation of the mixed problem, presented in the previous section.

This theorem introduces the energy functional $\Omega_t(s)$, the minimum of which represents the condition of equilibrium and stability for the physical system considered.

Teoremă 2. *Let Σ be the set of admissible states. If for $\forall S \in \Sigma$ and for*

$\forall t \in [0, \infty)$, we define the functional $\Omega_t\{S\}$ on S by

$$\begin{aligned}
\Omega_t\{S\} = & \int_B m * [\lambda u_{k,k} \delta_{ij} * e_{ij} + \mu(u_{i,j} + u_{j,i}) * e_{ij} + k(u_{i,j} + \epsilon_{ijk} \phi_k) * e_{ij} \\
& + \frac{\xi \delta_{ij}}{b} [\rho \eta - (3\lambda + 2u + k) \nu u_{k,k} - c\theta] * (e_{ij} + \psi_{ij} + \rho \eta \frac{b}{\xi \delta_{ij}})] dB \\
& - \int_B m * [(3\lambda + 2\mu + k) \nu \theta] * (\delta_{ij} e_{ij} - \frac{\theta_{,i}}{\theta} u_i) dB - \int_B m * t_{ij} * e_{ij} dB \\
& - \int_B m * m_{ij} * \psi_{ij} dB - \int_B m * b \varphi * \rho \eta dB + \int_B m * (\alpha \phi_{k,k} \delta_{ij} + \gamma \phi_{j,i} + \epsilon \phi_{i,j}) * \psi_{ij} dB \\
& - \int_B \{m * [(\lambda + \mu) u_{k,ki} + (\mu + v) u_{ik,k}] - \rho u_i\} * u_i dB \\
& - \int_B \{m * [(\alpha + v) \theta_{k,ki} + \epsilon \phi_{i,kk} - 2k \phi_i] - \rho I_{ij} \phi_i\} * \phi_j dB \\
& - \int_B m * (k \epsilon_{ijk} \phi_{k,j} * u_i - k \epsilon_{ij} u_{j,k} * \phi_j) dB - \int_B m * \xi \varphi_{,i} (u_i + \phi_j) dB \\
& - \int_B m * \rho (\varphi_i * u_i + G_i * \phi_j) dB + \int_B (e_{ij} - u_{i,j} - \epsilon_{ijk} \phi_k) * t_{ij} dB \\
& + \int_B m * (\psi_{ij} - \phi_{i,j}) * m_{ij} dB + \int_B m * (\omega - \rho \theta_0 \eta) * \theta dB \\
& + \int_B m * r * (\varrho_{i,j} * h_{i,i} + g * \varrho_i) dB + \int_B r * (L - \rho \chi \varphi) * \varrho_i dB \\
& - \int_{\partial B_1} m * t_i * \tilde{u}_i dA + \int_{\partial B_1^c} m * (t_i - \tilde{t}_i) * u_i dA - \int_{\partial B_2} m * m_i * \tilde{\phi}_i dA \\
& + \int_{\partial B_2^c} m * (m_i - \tilde{m}_i) * \phi_i dA - \int_{\partial B_3} m * h_i * \tilde{\varphi} dA + \int_{\partial B_3^c} m * (h_i - \tilde{h}_i) * \varphi dA \\
& - \int_{\partial B_4} m * \varrho_i * \tilde{\theta} dA + \int_{\partial B_4^c} m * (\varrho - \tilde{\varrho}) * \theta dA, \tag{1.29}
\end{aligned}$$

then

$$\delta \Omega_t\{S\} = 0, t \in [0, \infty), \tag{1.30}$$

if and only if S satisfies the considered mixed problem.

1.1.4 Continuous dependence

To obtain a simpler form of the equations, dimensionless parameters were used and subsequently some notations for convenience. We will further determine the continuous dependence of the solutions with respect to the initial data and external loads.

Let $Z^\alpha = \{u_i^\alpha; \phi_i^\alpha; \varphi^\alpha \theta^\alpha\}$, $\alpha = 1, 2$, two solutions corresponding to the following external data, with the same boundary conditions

$$\Xi^\alpha = \{F_i, G_i, L, r, \tilde{u}_i = 0, \tilde{\phi}_i = 0, \tilde{\varphi} = 0, \tilde{\varphi} = 0, \tilde{t}_i = 0, \tilde{m}_i = 0, \tilde{h}_i = 0,$$

$$\tilde{\varrho}_i = 0, u_i^1, u_i^0, \phi_i^1, \phi_i^0, \varphi^0, \theta^0, \varrho_i^0\}. \quad (1.31)$$

If we note $u_i = u_i^2 - u_i^1, \phi_i = \phi_i^2 - \phi_i^1, \varphi = \varphi^2 - \varphi^1, \theta = \theta^2 - \theta^1$, then $Z = \{u_i, \phi_i, \varphi, \theta\}$ is a solution to the mixed problem with external data:

$$\Xi = \{F_i, G_i, L, r, \tilde{u}_i = 0, \tilde{\phi}_i = 0, \tilde{\varphi} = 0, \tilde{\theta} = 0, \tilde{t}_i = 0, \tilde{m}_i = 0, \tilde{h}_i = 0, \\ \tilde{\varrho}_i = 0, u_i^1, u_i^0, \phi_i^1, \phi_i^0, \varphi^0, \theta^0, \varrho_i^0\},$$

where

$$f_i = f_i^2 - f_i^1, G_i = G_i^2 - G_i^1, r = r^2 - r^1, u_i^0 = u_i^{0(2)} - u_i^{0(1)}, \phi_i^0 = \phi_i^{0(2)} - \phi_i^{0(1)}, \\ \theta^0 = \theta^{0(2)} - \theta^{0(1)}, \varrho_i^0 = \varrho_i^{0(2)} - \varrho_i^{0(1)}, u_i^1 = u_i^{1(2)} - u_i^{1(1)}, \phi_i^1 = \phi_i^{1(2)} - \phi_i^{1(1)}. \quad (1.32)$$

We note this problem with Γ and we introduce the function τ on $[0, t_1]$, so that

$$\tau = \frac{1}{2} \int_B [a_{14} \dot{u}_i \dot{u}_i + a_{14} \dot{\phi}_i \dot{\phi}_i + 2Y] dB, \quad (1.33)$$

where

$$2Y = a_{14} \lambda_1 \mu_{k,k} e_{ij} \delta_{ij} + 2a_{14} \mu_1 e_{ij} (u_{i,j} + u_{j,i}) + a_{14} a_{11} \chi_{ij} \chi_{ij} \\ + a_{14} a_9 \varphi^2 + a_{10} a_{14} \varphi_{,j} \varphi_{,j} + a_{12} a_{14} \phi_{ij} \phi_{ij} + \frac{a_4}{a_{13}} c \theta^2 + \frac{a_4^2}{a_{13}} \theta_{,j} \theta_{,j}. \quad (1.34)$$

Let Y be a quadratic form in the variables $e_{ij}, \phi_{i,j}, \varphi$ and θ , such that

$$y_1(e_{ij} e_{ij} + \phi_{i,j} \phi_{i,j} + \varphi^2 + \theta^2) \leq Y \leq y_2(e_{ij} e_{ij} + \phi_{i,j} \phi_{i,j} + \varphi^2 + \theta^2), \quad (1.35)$$

for any t and any variables, where y_1 and y_2 are considered two positive constants.

We deduce

$$\dot{Y} = a_{14} t_{ij} \dot{e}_{ij} + a_{14} m_{ij} \dot{\chi}_{ij} + a_{14} \varphi \dot{\varphi} + a_5 \theta \dot{\eta}. \quad (1.36)$$

Therefore, from equations (1.33) and (1.36), we deduce:

$$\dot{\tau} = \int_B a_{14} \ddot{u}_i \dot{u}_i + a_{14} \ddot{\phi}_i \dot{\phi}_i + a_{14} t_{ij} \dot{e}_{ij} + a_{14} m_{ij} \dot{\chi}_{ij} + a_{14} \varphi \dot{\varphi} + a_5 \theta \dot{\eta} dB. \quad (1.37)$$

Furthermore, the present lemma is a basis for the demonstration of the following theorem, ensuring that the energy variation corresponding to the solution of the problem does not grow uncontrollably, being restrained by external forces.

Lemă 1. *Let $\{u_i, \phi_i, \varphi, \theta\}$ be a solution of the problem Γ . Then*

$$\dot{\tau} \leq \int_B (a_1 a_{14} F_i \dot{u}_i + a_2 a_{14} G_i \dot{\phi}_i + a_{14} a_3 L \varphi + a_5 r \theta) dB. \quad (1.38)$$

We now define the functions P and Q on $[0, t_1]$ as follows:

$$P = \left(\int_B [\dot{u}_i \dot{u}_i + \dot{u}_i \dot{\phi}_i + \dot{\phi}_i \dot{\phi}_i + \varphi^2 + \theta^2 + e_{ij} e_{ij} + \chi_{ij} \chi_{ij}] dB \right)^{\frac{1}{2}}; \quad (1.39)$$

$$Q = \left(\int_B [(a_1 a_{14} F_i)^2 + (a_2 a_{14} G_i)^2 + (a_5 L)^2 + (a_5 r)^2] dB \right)^{\frac{1}{2}}. \quad (1.40)$$

The continuous dependence of the solutions on the initial data and on the external loads is implied by the following theorem.

Teoremă 3. *Let there be two strictly positive constants, H_1, H_2 and consider $\{u_i, \phi_i, \theta\}$ as a solution to the problem Γ . Then, there exists constants σ_1, σ_2 , such that,*

$$P(t) \leq \sigma_1 P(0) + \sigma_2 \int_0^t Q(s) ds, \quad t \in [0, t_1]. \quad (1.41)$$

1.2 Plane deformation

We consider a body, which at a given moment occupies the region B of the three-dimensional Euclidean space and is bounded by the smooth surface ∂B . Let O_{x_k} , ($k = 1, 2, 3$) be a system of orthogonal axes to which the motion is reported. In addition to the equations describing the isotropic micropolar medium in the theory of thermoelasticity, presented previously, we introduce the equilibrium equations in local forms:

$$t_{ji,j} + \rho F_i = 0, \quad (1.42)$$

$$m_{ji,j} - \epsilon_{ijk} t_{jk} + \rho G_i = 0, \quad (1.43)$$

$$h_{i,i} + g + \rho L = 0, \quad (1.44)$$

$$q_{i,i} + \rho r = 0. \quad (1.45)$$

The surface force vector t_i , the surface force torque m_i , the heat flux q and the normal component of the hole evolution vector N at a regular point on ∂B , are defined by:

$$t_i = t_{ji} n_j, \quad m_i = m_{ji} n_j, \quad q = q_j n_j, \quad N = h_j n_j. \quad (1.46)$$

Knowing that the elastic potential is a positive quadratic form, we will consider that the elastic moduli satisfy the relations imposed by it:

$$3\lambda + 2\mu > 0, \mu > 0, \varepsilon > 0, 3\lambda + 2\mu > 3k. \quad (1.47)$$

Next, we consider the region B as the interior of a right cylinder, where Σ is the open cross-section, Π is the lateral area, and L is the boundary of Σ . We choose the Cartesian system so that the generators of B are parallel to the x_3 axis.

Let us introduce the variables that describe the plane deformation of B , in a plane parallel to the Ox_1x_2 plane.

$$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0, \quad \phi_\alpha = \phi_\alpha(x_1, x_2), \quad \varphi = \varphi(x_1, x_2),$$

$$\theta = \theta(x_1, x_2), \forall (x_1, x_2) \in \Sigma. \quad (1.48)$$

From the local forms of the equilibrium laws, from (1.12) and from (1.2) it follows that $e_{ij}, \chi_{ij}, t_{ij}, m_{ij}, g, h_i, \rho\eta$ and $r\dot{q}_{,i}$ are independent of x_3 .

The deformation tensors e_{ij} and χ_{ij} , defined by the geometric equations, take the form:

$$e_{\alpha,\beta} = u_{\alpha,\beta} + \epsilon_{\alpha\beta\rho} \phi_\rho, \quad \chi_{\alpha\beta} = \phi_{\alpha,\beta}. \quad (1.49)$$

The non-zero constitutive variables are $t_{\alpha\beta}, t_{33}, m_{\alpha\beta}, h_{\alpha}, r\dot{q}_{,\alpha}$. More:

$$t_{\alpha\beta} = \lambda u_{\rho,\rho} \delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}) + k u_{\alpha,\beta} \phi_{\rho} + \xi \varphi \delta_{\alpha\beta} - (2\lambda + 2\mu + k) \nu \theta \delta_{\alpha\beta} \quad (1.50)$$

$$m_{\alpha\beta} = -\alpha \phi_{\rho,\rho} \delta_{\alpha\beta} - \gamma \phi_{\beta,\alpha} + \epsilon_{\alpha,\beta} + \xi \varphi \delta_{\alpha\beta}, \quad (1.51)$$

$$g = -\xi u_{\alpha,\alpha} - \psi \phi_{\alpha,\alpha} - a\varphi - b\theta, \quad (1.52)$$

$$h_{\alpha} = d\varphi_{,\alpha}, \quad (1.53)$$

$$\rho\eta = (3\lambda + 2\mu + k) \sigma u_{\alpha,\alpha} - b\varphi + c\theta, \quad (1.54)$$

$$\dot{q}_{,\alpha} = k\theta_{\alpha} - q_{\alpha}. \quad (1.55)$$

Therefore, we will consider that the body loads are independent of x_3 , and $f_3 = 0$. Therefore, the equilibrium equations reduce to:

$$t_{\alpha\beta,\beta} + \rho f_{\alpha} = 0, \quad (1.56)$$

$$m_{\alpha\beta,\beta} + \rho G_{\alpha} = 0, \quad (1.57)$$

$$h_{\alpha,\alpha} + g + \rho L = 0, \quad (1.58)$$

$$q_{\alpha,\alpha} + \rho S = 0. \quad (1.59)$$

The relations (1.46), at a regular point of Π , become:

$$t_{\alpha} = t_{\beta\alpha} n_{\beta}, \quad m_{\alpha} = m_{\beta\alpha} n_{\beta}, \quad t_3 = 0, \quad q = q_{\alpha} n_{\alpha}, \quad N_i = h_{\beta i} n_{\beta}, \quad \text{on } L, \quad (1.60)$$

where, $n_{\alpha} = \cos(n_x, x_{\alpha})$ and where we denote by n_x the unit vector of the external normal to L .

According to the geometric equations, constitutive equations and equilibrium equations, it is necessary to check the boundary conditions. In the case of the first problem, the boundary conditions are:

$$u_{\alpha} = \tilde{u}_{\alpha}, \quad \phi_{\alpha} = \tilde{\phi}_{\alpha}, \quad \theta = \tilde{\theta}, \quad \varphi = \tilde{\varphi} \quad \text{on } L, \quad (1.61)$$

where $\tilde{u}_{\alpha}$, $\tilde{\phi}$, $\tilde{\theta}$ and $\tilde{\varphi}$ are prescribed functions. In the case of the second problem, the boundary conditions are:

$$t_{\beta\alpha} n_{\beta} = \tilde{t}_{\alpha}, \quad m_{\beta\alpha} n_{\beta} = \tilde{m}_{\alpha}, \quad \tilde{q} = q_{\beta} n_{\beta}, \quad h_{\alpha i} n_{\alpha} = \tilde{N}_i, \quad \text{pe } L, \quad (1.62)$$

where the given functions $\tilde{t}_{\alpha}$, $\tilde{m}_{\alpha}$, $\tilde{q}$ and $\tilde{N}_j$ are independent of x_3 .

From (1.49)-(1.59), it can be deduced that $u_{\alpha}, \phi_{\alpha}, \theta$, and φ satisfy the equations:

$$(\lambda + \mu) u_{\rho,\rho\alpha} + (\mu + k) u_{\alpha,\rho\rho} + \xi \varphi_{,\alpha} - (3\lambda + 2\mu + k) \sigma \theta_{,\alpha} = -\rho f_{\alpha}, \quad (1.63)$$

$$(\alpha + \gamma) \phi_{\rho,\rho\alpha} + \epsilon \phi_{\alpha,\rho\rho} + \zeta \varphi_{,\alpha} - 2k \phi_{\alpha} = -\rho G_{\alpha}, \quad (1.64)$$

$$k \Delta \theta = -\rho S, \quad (1.65)$$

$$d\varphi_{,\rho\rho} - \xi u_{\rho,\rho} - \zeta \phi_{\rho,\rho} - a\varphi - b\theta = -\rho L, \quad \text{pe } \Sigma. \quad (1.66)$$

The first boundary value problem involves finding the functions $u_\alpha, \phi_\alpha, \theta$, and φ that satisfy the above equations, on Σ and the boundary conditions (1.61). Obviously, from the constitutive equations and from (1.47), we can express the boundary conditions in terms of $u_\alpha, \phi_\alpha, \theta, \varphi$.

In the case of equilibrium, we divide this problem into two, the first involving finding the functions θ and ϕ_α and the second the functions u_α and φ . We assume, in this section, that:

$$k > 0, d_2 > 0, d_6 > 0, kd_2 - d_1d_3 > 0, d_4 + d_5 + d_6 > 0. \quad (1.67)$$

It is important to note that the restrictions imposed by the Clausius-Duhem inequality on the constitutive coefficients, (Grof, 1969), have been taken into account in obtaining the above conditions.

1.2.1 Solution for the field equations

We start by introducing some notations, such as:

$$c_1 = \lambda + 2\mu + k, \quad (1.68)$$

$$c_2 = d + \frac{-a + d_2}{\Delta}, \quad (1.69)$$

$$m_1 = \left(\frac{2k}{h}\right)^{\frac{1}{2}}, \text{ where } h = \alpha + \gamma + \varepsilon, \quad (1.70)$$

$$m_2 = \left(d_2/c_2\right)^{\frac{1}{2}}, \quad (1.71)$$

$$m_3 = \left(\frac{a}{d_6} + \Delta\right)^{\frac{1}{2}}, \quad (1.72)$$

$$\kappa_1 = -c_2(3\lambda + 2\mu + k)\sigma - c_1b\zeta, \quad (1.73)$$

$$\kappa_2 = -c_2(3\lambda + 2\mu + k), \quad (1.74)$$

$$\kappa_3 = 0. \quad (1.75)$$

Taking these notations into account, it follows from (1.47) and from (1.67) that m_1^2, m_2^2 , and $m_3^2 > 0$. We introduce the operators:

$$C_1 = c_1h\Delta(\Delta - m_1^2), \quad (1.76)$$

$$C_2 = kc_2\Delta(\Delta - m_2^2), \quad (1.77)$$

$$C_3 = d_6(\Delta - m_3^2), \quad (1.78)$$

$$B_1 = hc_1(\lambda + \mu)(\Delta - m_1^2), \quad (1.79)$$

$$B_2 = h\sigma c_1(3\lambda + 2\mu + k)(\Delta - m_1^2)(c_2\Delta - d_2), \quad (1.80)$$

$$B_3 = 0. \quad (1.81)$$

The following theorem gives us a solution to the system of equations corresponding to the environment.

Teoremă 4. *We consider the functions*

$$u_\alpha = -c_1 C_1 \Gamma_\alpha + B_1 \Gamma_{\rho, \rho\alpha} - B_2 f_{,\alpha} - C_3 B_3 g_{\rho, \rho\alpha}, \quad (1.82)$$

$$\phi_\alpha = c_1^2 \mu \Delta \psi_\alpha + c_1 (\kappa_1 \Delta - \kappa_2) \Delta l_\alpha + k \zeta c_1 C_1 \Delta \Delta C_3 g_{\rho, \rho\alpha}, \quad (1.83)$$

$$\theta = -c_1 (c_2 \Delta - d_2) C_1 l, \quad (1.84)$$

$$\varphi = c_1 C_1 C_2 g - c_1 [k(d_4 - d_5) \Delta C_1 g_{\rho, \rho} - c_1 d_3 C_1 l]. \quad (1.85)$$

If $\Gamma_\alpha, \psi_\alpha \in C^6(\Sigma), l \in C^8(\Sigma)$, and $g \in C^{10}(\Sigma)$, satisfy the equations:

$$(\mu + k) \Delta c_1 C_1 \Gamma_\alpha = \rho F_\alpha; \quad (1.86)$$

$$\mu c_1 C_1 \psi_\alpha = \rho G_\alpha; \quad (1.87)$$

$$c_1 C_1 C_2 l = \rho S; \quad (1.88)$$

$$c_1 C_1 C_2 C_3 g = -\rho L, \quad (1.89)$$

then $u_\alpha, \phi_\alpha, \theta$ and φ satisfy (1.63)-(1.66).

1.2.2 The effect of heat sources and pores

In order to study the influences of heat sources and pores on deformation, we will use the solution obtained in the theorem presented in the previous section.

We assume that

$$\rho F_\alpha = 0, \rho G_\alpha = 0, \rho S = \delta(x - y), \rho L = 0.$$

where $y(y_\alpha)$ is a fixed point, and δ is the Dirac distribution.

In this case, the relations (1.86)-(1.89) are satisfied, if we consider

$$\Gamma_\alpha = 0, \psi_\alpha = 0, l = \omega \text{ and } g = 0.$$

The function ω is a solution to the equation:

$$\Delta \Delta (\Delta - m_1^2) (\Delta - m_2^2) \omega = \gamma \delta(x - y), \quad (1.90)$$

where we used the notation γ for $(\varepsilon k c_1^2 c_2)^{-1}$.

Substituting $\Gamma_\alpha = 0, \psi_\alpha = 0, l = \omega$ and $g = 0$ in the relations (1.86)-(1.89) we obtain the functions $u_\alpha^{(1)}(x, y), \phi_\alpha^{(1)}(x, y), \theta^{(1)}(x, y)$ and $\varphi^{(1)}(x, y)$. Therefore,

$$u_\alpha^{(1)}(x, y) = -B_2 \omega_{,\alpha}; \quad (1.91)$$

$$\phi_\alpha^{(1)}(x, y) = c_1 (\kappa_1 \Delta - \kappa_2) \Delta \omega_\alpha; \quad (1.92)$$

$$\theta^{(1)}(x, y) = -c_1 (c_2 \Delta - d_2) C_1 \omega; \quad (1.93)$$

$$\varphi^{(1)}(x, y) = -c_1 b C_1 \omega. \quad (1.94)$$

Next, we will have the following hypotheses:

* m_1, m_2, m_3 sunt different,

* $\omega_s, (s = 1, 2, 3, 4)$, functions that satisfy the following equations :

$\Delta\omega_1 = M$, $\Delta\Delta\omega_2 = M$, $(\Delta - m_1^2)\omega_3 = M$, $(\Delta - m_2^2)\omega_4 = M$, where M is a given function.

Therefore, we can formulate the solution of the equation

$$\Delta\Delta(\Delta - m_1^2)(\Delta - m_2^2)\omega = M,$$

as follows:

$$\omega = \sum_{s=1}^4 z_s \omega_s,$$

where, the constants z_s , ($s = 1, 2, 3, 4$), are given by:

$$z_1 = \frac{m_1^2 + m_2^2}{m_1^4 m_2^4}, \quad z_2 = \frac{1}{m_1^2 m_2^2}, \quad z_3 = \frac{1}{m_1^4 (m_1^2 - m_2^2)}, \quad z_4 = -\frac{1}{m_2^4 (m_1^2 - m_2^2)}. \quad (1.95)$$

For $M = \delta(x - y)$, the functions ω_s , ($s = 1, 2, 3, 4$) are given by:

$$\omega_1 = \frac{1}{2\pi} \ln r, \quad \omega_2 = \frac{1}{8\pi} r^2 \ln r, \quad \omega_3 = -\frac{1}{2\pi} K_0(m_1 r), \quad \omega_4 = -\frac{1}{2\pi} K_0(m_2 r),$$

$$r = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{\frac{1}{2}}, \quad (1.96)$$

where we used the notation K_0 for the modified Bessel function of order 0. Thus, for equation (1.90), we have the following solution:

$$\omega = \frac{\gamma}{2\pi} [z_1 \ln r + \frac{1}{4} z_2 r^2 \ln r - z_3 K_0(m_1 r) - z_4 K_0(m_2 r)]. \quad (1.97)$$

The functions $u_\alpha^{(1)}$ and $\phi_\alpha^{(1)}$ represent the displacement and microrotation. In what follows, we will study the effect of pores. Thus, we assume that

$$\rho F_\alpha = 0, \quad \rho G_\alpha = 0, \quad \rho S = 0, \quad \rho L = \delta_{\alpha\beta} \delta(x - y),$$

where β is fixed. So, we will have $\Gamma_\alpha = 0$, $\psi_\alpha = 0$, $l = 0$ and $g = \delta_{\alpha\beta} \Omega$. In this case, from (1.86)-(1.89), it follows that Ω is a solution of the following equation:

$$\Delta\Delta(\Delta - m_1^2)(\Delta - m_2^2)(\Delta - m_3^2)\Omega = \gamma_1 \delta(x - y), \quad (1.98)$$

where $\gamma_1 = (k\epsilon d_6 c_1^2 c_2)^{-1}$. Therefore, we obtain from the relations (1.86)-(1.89), the functions:

$$u_\alpha^{(1+\beta)}(x, y), \quad \phi_\alpha^{(1+\beta)}(x, y), \quad \theta^{(1+\beta)}(x, y), \quad \varphi^{(1+\beta)}(x, y).$$

1.2.3 Plane deformation of an elastic space with a cylindrical hole

We consider an elastic space with a cylindrical hole. We assume that the domain $B = \{x : x_1^2 + x_2^2 > r_1^2, x_3 \in \mathbb{R}\}$, ($r_1 > 0$), is occupied by an elastic material with microstructure. This material will undergo a plane deformation parallel to the plane Ox_1x_2 . Knowing these, the domain Σ is defined by $\Sigma =$

$\{x : x_1^2 + x_2^2 > r_1^2, x_3 = 0\}$. Furthermore, we will assume that body loads are absent and the surface of the circular inclusion is free of surface forces.

The problem we will study involves determining the functions $\theta, \varphi, u_\alpha$ and ϕ_α , which must satisfy the following equations:

$$(\lambda + \mu)u_{\rho,\rho\alpha} + (\mu + k)u_{\alpha,\rho\rho} + \xi\varphi_{,\alpha} - (3\lambda + 2\mu + k)\sigma\theta_{,\alpha} = 0, \quad (1.99)$$

$$(\alpha + \gamma)\phi_{\rho,\rho\alpha} + \varepsilon\phi_{\alpha,\rho\rho} + \zeta\varphi_{,\alpha} - 2k\phi_\alpha = 0, \quad (1.100)$$

$$k\Delta\theta = 0, \quad (1.101)$$

$$d\varphi_{,\rho\rho} - \xi u_{\rho,\rho} - \zeta\phi_{\rho,\rho} - a\varphi - b\theta = 0, \quad \text{pe } \Sigma. \quad (1.102)$$

We consider the solution to have the form

$$\theta = V(r), \varphi = W(r), u_\alpha = x_\alpha U(r) \text{ and } \phi_\alpha = x_\alpha Q(r),$$

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$. Therefore, they must satisfy the equations:

$$\begin{aligned} &(\lambda + \mu)x_\alpha r^2 U + (\mu + k)x_\alpha r^2 U + \xi r W \\ &-(3\lambda + 2\mu + k)\sigma r V = 0, \end{aligned} \quad (1.103)$$

$$(\alpha + \gamma - 2\kappa)x_\alpha r^2 Q + \varepsilon x_\alpha r^2 Q + \zeta r W = 0, \quad (1.104)$$

$$k\Delta V = 0, \quad (1.105)$$

$$dr^2 W - \xi x_\alpha r U - \zeta x_\alpha r Q - aW - bV = 0, \quad \text{pe } \Sigma. \quad (1.106)$$

By solving the previous system we obtain the following forms for V, W and Q :

$$V = C_1 + B_1 \ln r, \quad (1.107)$$

$$\tau_3 x_\alpha r Q = \frac{\tau_4}{m_1^2} W - \tau_4 C_2 - \tau_4 B_2 \ln r, \quad (1.108)$$

$$W = \tau_1 \left(\frac{C_1 + B_1 \ln r}{m_1^2} - B_1 \Delta \ln r \right) + \tau_4 \left(\frac{C_2 + B_2 \ln r}{m_1^2} - B_2 \Delta \ln r \right) \quad (1.109)$$

$$-\frac{\tau_2}{m_1^2} N_1 + N_3 k_0(m, r).$$

The function U , is determined immediately by substituting the functions V, W and Q into the corresponding relation.

1.3 Wave Propagation with two delay times

1.3.1 Preliminaries

A linear, isotropic, and homogeneous micropolar thermoelastic continuum with pores occupies a fairly regular region B of three-dimensional Euclidean space, and its thermodynamic evolution is governed by the existence of two delay times (τ_q and τ_θ).

In the following equation we used the Taylor series expansion up to the second order for the double phase-lag thermal model.

$$q_r + \tau_q \dot{q}_r + (\tau_q^2) \ddot{q}_r = k\theta_{,r} + \tau_\theta k \dot{\theta}_{,r} \text{ in } \bar{B} \times [0, \infty). \quad (1.110)$$

Regarding the delay times, using (1.110) and adding the second law of thermodynamics we obtain (as in [12]):

$$0 < \tau_q \leq 2\tau_\theta. \quad (1.111)$$

We will assume that in the equations of motion the mass torque and the mass force are zero. It is important to note that the constitutive moduli:

$$\lambda, \mu, a, b, d, \xi, \alpha, \epsilon, \gamma,$$

verify the following conditions:

$$\rho > 0, \mu > 0, (\kappa + 2\mu)a > \xi^2, d > 0, a > 0, b > 0. \quad (1.112)$$

By substituting the constitutive equations and the geometric equations in (1.7)-(1.9) we obtain the equations present in the following system, in the variables: $u(x, t), \phi(x, t), \varphi(x, t), \theta(x, t)$,

$$\begin{cases} (\mu + \kappa)u_{j,kj} + \lambda u_{k,kj} + \mu u_{k,jj} + \xi \varphi_{,k} - (3\lambda + 2\mu + \kappa)\sigma\theta_{,k} + gF_k = \rho \ddot{u}_k, \\ \alpha \phi_{j,jk} + \gamma \phi_{k,jj} + c\phi_{j,kj} + \psi \varphi_{,k} + \rho G_k = \rho I_{kj} \ddot{\phi}_k, \\ d\varphi_{,kk} - \xi u_{k,k} - \psi \phi_{k,k} - a\varphi - b\theta + \rho L = \rho \tau \ddot{\varphi}, \\ \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2}\right) \left[(3\lambda + 2\mu + \kappa)\sigma \dot{u}_{k,k} \theta_0 - b\dot{\varphi} \theta_0 + c\theta_0 \dot{\theta} - \rho r\right] = k\theta_{,kk} + \tau_\theta \dot{\theta}_{,kk}. \end{cases} \quad (1.113)$$

From the analysis of equations (1.113) it is seen that the mechanical deformation is influenced by the thermal effect and the porous structure.

1.3.2 A plane wave analysis

We mention that in order to have irreversible thermoelastic processes, the temperature dissipation must also be taken into account.

First, we will assume that the loads: F_k, G_k, L and r are zero. We seek solutions for the system (1.113) in the following form:

$$\begin{cases} u_r(x, t) = Re\{\frac{i}{\chi} A_r e^{i\chi \sin(x_s n_s - \nu t)}\}, \\ \Phi_r(x, t) = Re\{\frac{i}{\chi} B_r e^{i\chi \sin(x_s n_s - \nu t)}\}, \\ \varphi(x, t) = Re\{C e^{i\chi \sin(x_s n_s - \nu t)}\}, \\ \theta(x, t) = Re\{D \theta_0 e^{i\chi \sin(x_s n_s - \nu t)}\}, \end{cases} \quad (1.114)$$

where $|\mathbf{A}| + |\mathbf{B}| + |C| + |D| \neq 0$, and A_r, B_r represent the components of two constant complex vectors A and B , respectively, while C and D are complex constants, and $i^2 = -1$ is the imaginary unit.

The wave propagates in the direction of the normal vector n , and the wave number is χ . Assuming $\nu \in \mathbb{C}$, so:

$$\nu = Re(\nu) + iIm(\nu), \quad (1.115)$$

where the real part indicates the wave propagation speed, while the imaginary part indicates the damping in time.

Later we will use the notations:

$$\psi(\nu) = 1 - i r \nu \chi, \quad (1.116)$$

$$\nu = i c_2 w,$$

$$c_1 = \sqrt{\frac{\kappa + 2\mu}{\rho}}, c_2 = \sqrt{\frac{\mu}{\rho}}, c_3 = \sqrt{\frac{\lambda}{\rho}}.$$

Given (1.115) and the notation above, we impose the following restrictions $Re(w) \leq 0, Im(w) \leq 0$, and we obtain:

$$(1 + w^2)A_k + \left[\left(\frac{c_1^2}{c_2^2} - 1 \right) A_j n_j + \frac{c_3^2}{c_2^2} A_k n_j - \frac{\xi}{\rho c_2^2} C + \frac{3c_3^2 + c_1^2}{c_2^2} \sigma D \theta_0 \right] n_k = 0, \quad (1.117)$$

and

$$\begin{cases} \left(w^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} \right) A_k n_k - \frac{\xi}{\rho c_2^2} C + \theta_0 \frac{3c_3^2 + c_1^2}{c_2^2} \sigma D = 0, \\ (\gamma + \alpha + \epsilon + \rho I_{kj} c_2^2 w^2) B_k n_k - \xi C = 0, \\ -\frac{\xi}{\chi^2} A_k n_k - \frac{\psi}{\chi^2} B_k n_k + \left(d + \frac{a}{\chi^2} + \rho \tau c_2^2 w^2 \right) C + \frac{b}{\chi^2} \theta_0 D = 0, \\ -\frac{c_2 w \sigma \rho (3c_3^2 + c_1^2)}{\chi} \psi(w) A_k n_k - \frac{b c_2 w}{\chi} \psi(w) C + \left[k(1 + \tau \theta \chi c_2 w) + \frac{c c_2 w \theta_0}{\chi} \psi(w) \right] D = 0, \end{cases} \quad (1.118)$$

where

$$\psi(w) = 1 + \tau_q \chi c_2 w + \frac{\tau_q^2}{2} \chi^2 c_2^2 w^2. \quad (1.119)$$

1.3.3 Study of shear waves

To study the solutions of shear waves (transverse waves, which appear as a result of periodic shearing of an elastic medium), it is necessary to consider that $A_k n_k = 0, B_k n_k = 0$, with $(A_1, A_2, A_3) \neq 0$ and $(B_1, B_2, B_3) \neq 0$, from which we deduce that (1.117) and (1.118) become:

$$(1 + w^2)A_k + \left[-\frac{\xi}{\rho c_2^2} C + \frac{3c_3^2 + c_1^2}{c_2^2} \sigma D \theta_0 \right] n_k = 0, \quad (1.120)$$

$$\begin{cases} -\frac{\xi}{\rho c_2^2} C + \theta_0 \frac{3c_3^2 + c_1^2}{c_2^2} \sigma D = 0, \\ -\xi C = 0, \\ \left(d + \frac{a}{\chi^2} + \rho \tau c_2^2 w^2 \right) C + \frac{b}{\chi^2} \theta_0 D = 0, \\ -\frac{bc_2 w}{\chi} \psi(w) C + \left[k(1 + \tau_\theta \chi c_2 w) + \frac{cc_2 w \theta_0}{\chi} \psi(w) \right] D = 0. \end{cases} \quad (1.121)$$

We further observe that the following equality follows from this.

$$(1 + w^2)A_k = 0. \quad (1.122)$$

From the relation (1.122), using $Re(w) \leq 0, Im(w) \leq 0$ we deduce that

$$w = w_5 \equiv -i, \text{ that is } \nu = \nu_5 \equiv c_2 = \sqrt{\frac{\mu}{\rho}}. \quad (1.123)$$

Therefore c_2 , the wave propagation speed, is as in the classical theory. Moreover, the above system admits only the trivial solution $C = 0$ and $D = 0$, because

$$\text{rang} \begin{pmatrix} \frac{-\xi}{\rho c_2^2} & \frac{3c_3^2 + c_1^2}{c_2^2} \sigma \theta_0 \\ -\xi & 0 \\ d + \frac{a}{\chi^2} + \rho \tau c_2^2 & \frac{b}{\chi^2} \theta_0 \\ \frac{ibc_2}{\chi} \psi(-i) & k(1 + \tau_\theta \chi c_2(-i)) + \frac{cc_2 \theta_0}{\chi} \psi(-i) \end{pmatrix} = 2. \quad (1.124)$$

Next we will consider a second-order minor corresponding to the above matrix and we will assume that it is zero. So we have:

$$D_1 \equiv \begin{vmatrix} \frac{-\xi}{\rho c_2^2} & \frac{3c_3^2 + c_1^2}{c_2^2} \sigma \theta_0 \\ -\xi & 0 \end{vmatrix} = 0, \text{ from which we obtain : } \xi \sigma \theta_0 \frac{3c_3^2 + c_1^2}{c_2^2} = 0, \text{ which implies } \xi = 0 \text{ or } 3c_3^2 + c_1^2 = 0.$$

In both cases, one of the relations of the system (1.113) will be decoupled from the others, which contradicts the assumption made previously.

Without affecting generality, we will consider the axis x_1 , which coincides with the direction of propagation, since $n_1 = 1, n_2 = n_3 = 0$. Obviously, $A_i n_i = 0$ and $B_i n_i = 0$ imply $A_1 = 0$ and $B_1 = 0$, from which we deduce that the shear waves are not damped in time and there is no effect on them due to pores or temperature. This type of wave can be recognized by: $U^{(1)} = \{u_r^{(1)}, \phi_r^{(1)}, \varphi^{(1)}, \theta^{(1)}\}$ and $U^{(2)} = \{u_r^{(2)}, \phi_r^{(2)}, \varphi^{(2)}, \theta^{(2)}\}$, where

$$\begin{aligned}
u_1^{(1)}(x_1, t) &= 0, u_2^{(1)}(x_1, t) = -\frac{1}{\chi} \sin \chi(x_1 - c_2 t), u_3^{(1)}(x_1, t) = 0, \\
\phi_1^{(1)}(x_1, t) &= 0, \phi_2^{(1)}(x_1, t) = -\frac{1}{\chi} \sin \chi(x_1 - c_2 t), \phi_3^{(1)}(x_1, t) = 0, \\
\varphi^{(1)}(x_1, t) &= 0, \theta^{(1)}(x_1, t) = 0,
\end{aligned} \tag{1.125}$$

and

$$\begin{aligned}
u_1^{(2)}(x_1, t) &= 0, u_2^{(2)}(x_1, t) = -\frac{1}{\chi} \sin \chi(x_1 - c_2 t), u_3^{(2)}(x_1, t) = 0, \\
\phi_1^{(2)}(x_1, t) &= 0, \phi_2^{(2)}(x_1, t) = -\frac{1}{\chi} \sin \chi(x_1 - c_2 t), \phi_3^{(2)}(x_1, t) = 0, \\
\varphi^{(2)}(x_1, t) &= 0, \theta^{(2)}(x_1, t) = 0.
\end{aligned} \tag{1.126}$$

Considering the above, we conclude that transverse waves are not dispersive, are not damped in time, and are not influenced by thermal and/or pore effects.

1.3.4 Study of longitudinal waves

Returning to the system (1.118), and using $A_k n_k \neq 0$, $B_k n_k \neq 0$, we deduce that the algebraic system admits nontrivial solutions, consequently the discriminant of the system must be 0.

$$D(w) = \begin{vmatrix} w^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} & 0 & -\frac{\xi}{\rho c_2^2} & \frac{3c_3^2 + c_1^2}{c_2^2} \\ 0 & M(w) & -\xi & 0 \\ -\frac{\xi}{\chi^2} & -\frac{\psi}{\chi^2} & d + \frac{a}{\chi^2} + \rho \tau c_2^2 w^2 & \frac{b}{\chi^2} \theta_0 \\ -\frac{\sigma \rho c_2 (3c_3^2 + c_1^2)}{k \chi} w \psi(w) & 0 & -\frac{b c_2}{k \chi} w \psi(w) & N(w) \end{vmatrix} = 0,$$

where

$$M(w) = \gamma + \alpha + \epsilon + \rho I_{kj} c_2^2 w^2,$$

$$N(w) = 1 + \tau \theta \chi c_2 w + \frac{\theta_0 c_2 c}{k \chi} w \psi(w).$$

We now propose to determine the solutions for the waves. To do this, we need to solve the dispersion equation considering $Re(w) \leq 0$, $Im(w) \leq 0$, a situation discussed earlier. First, we study the case where the coupling effects are equal to 0. Accordingly, we assume

$$C = \{\xi, 3c_3^2 + c_1^2, b, \psi\} = \{0, 0, 0, 0\}.$$

In this context, the above equation $D(w) = 0$, reduces to:

$$\left(w^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2}\right) M(w) \left(d + \frac{a}{\chi^2} + \rho \tau c_2^2 w^2\right) N(w) = 0, \tag{1.127}$$

whose roots, verify the condition $Re(w) \leq 0, Im(w) \leq 0$:

$$w_1^{(0)} = -\frac{\sqrt{c_1^2 + c_3^2}}{c_2}i, \text{ so that } \nu_1^{(0)} = \sqrt{c_1^2 + c_3^2}, \quad (1.128)$$

$$w_2^{(0)} = -\frac{\sqrt{\gamma + \alpha + \epsilon}}{\sqrt{\rho I_{kj} c_2}}i \text{ so that } \nu_2^{(0)} = \sqrt{\frac{\gamma + \alpha + \epsilon}{\rho I_{kj}}}, \quad (1.129)$$

$$w_3^{(0)} = -\sqrt{\frac{d\chi^2 + a}{\rho\tau}} \frac{1}{c_2\chi}i \text{ so that } \nu_3^{(0)} = \frac{\sqrt{d\chi^2 + a}}{\chi\sqrt{\rho\tau}}, \quad (1.130)$$

and the rest of the roots are determined from the polynomial

$$L(w) = \Gamma_3 w^3 + \Gamma_2 w^2 + \Gamma_1 w + \Gamma_0, \quad (1.131)$$

where

$$\Gamma_3 = \frac{\theta_0 c_2^3 \tau_q^2 \chi c}{2k}, \Gamma_2 = \frac{\theta_0 c_2^2 c \tau_q}{k}, \Gamma_1 = \tau_\theta \chi c_2 + \frac{\theta_0 c_2 c}{k\chi}, \Gamma_0 = 1. \quad (1.132)$$

Since the physical constants involved are strictly positive, we deduce that $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 > 0$, which implies $L(w) > 0$ for any $w > 0$, so there are no positive real solutions. Furthermore, the discriminant of the cubic equation is positive, which implies the existence of one real root and two complex conjugate roots.

$$L(0) = \Gamma_0 > 0, L\left(-\frac{1}{\tau_\theta \chi c_2}\right) = -\frac{\theta_0 c}{k\chi^2 \tau_\theta} \psi\left(-\frac{1}{\tau_\theta \chi c_2}\right) < 0.$$

Neglecting the terms τ_q and τ_q^2 , equation (1.131) becomes $\Gamma_1 w + \Gamma_0 = 0$. Thus, the negative real root is given by:

$$w_4^{(0)} \in \left(-\frac{1}{\tau_\theta \chi c_2}, 0\right). \quad (1.133)$$

Regarding the complex conjugate roots, we will use Cardano's formula to be able to determine their real part. Furthermore, the complex roots are:

$$w_5^{(0)} = -\gamma - i\delta = -\left(\frac{2}{3c^2 \tau_q \chi} + \frac{u^* + v^*}{2}\right) - i\frac{\sqrt{3}}{2}(u^* - v^*), \quad (1.134)$$

$$w_6^{(0)} = -\gamma + i\delta = -\left(\frac{2}{3c^2 \tau_q \chi} + \frac{u^* + v^*}{2}\right) + i\frac{\sqrt{3}}{2}(u^* - v^*), \quad (1.135)$$

cu $\nu_5^{(0)} = ic_2 w_5^{(0)}, \nu_6^{(0)} = ic_2 w_6^{(0)}$.

Without affecting the generality of the analysis, we choose the axis x_1 , corresponding to the propagation direction, with $n_1 = 1, n_2 = 0$ and $n_3 = 0$.

Regarding the root $\nu = \nu_1$, there is the following wave for the displacement:

$$\begin{cases} u_1^{(1)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{A_1^{(1)} e^{i\chi(x_1 - c_2 b_1 t)}\} e^{-\chi c_2 a_1 t}, \\ \phi_1^{(1)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{B_1^{(1)} e^{i\chi(x_1 - c_2 b_1 t)}\} e^{-\chi c_2 a_1 t}, \\ \varphi^{(1)}(x_1, t) = \text{Re}\{C^{(1)} e^{i\chi(x_1 - c_2 b_1 t)}\} e^{-\chi c_2 a_1 t}, \\ \theta^{(1)}(x_1, t) = \theta_0 \text{Re}\{D^{(1)} e^{i\chi(x_1 - c_2 b_1 t)}\} e^{-\chi c_2 a_1 t}, \end{cases} \quad (1.136)$$

where

$$\begin{aligned} A_1^{(1)} &= \begin{vmatrix} M(w_1) & -\xi & 0 \\ -\frac{\psi}{\chi^2} & d + \frac{a}{\chi^2} + \rho\tau c_2^2 w_1^2 & \frac{b}{\chi^2} \theta_0 \\ 0 & -\frac{bc_2 w_1}{\chi} \psi(w_1) & N(w_1) \end{vmatrix} = \\ &= M(w_1) \left\{ \left(d + \frac{a}{\chi^2} + \rho\tau c_2^2 w_1^2 \right) N(w_1) + \frac{b^2 c_2 \theta_0}{\chi^3} \psi(w_1) \right\}, \end{aligned}$$

$$\begin{aligned} B_1^{(1)} &= \begin{vmatrix} 0 & -\xi & 0 \\ -\frac{\xi}{\chi^2} & d + \frac{a}{\chi^2} + \rho\tau c_2^2 w_1^2 & \frac{b}{\chi^2} \theta_0 \\ -\frac{\sigma\rho c_2(3c_3^2 + c_1^2)}{k\chi} w_1 \psi(w_1) & -\frac{bc_2}{k\chi} w_1 \psi(w_1) & N(w_1) \end{vmatrix} = \\ &= \frac{\sigma\xi b \theta_0}{k\chi^3} \rho c_2 (3c_3^2 + c_1^2) w_1 \psi(w_1) + \frac{\xi^2}{\chi^2} N(w_1), \end{aligned}$$

$$\begin{aligned} C^{(1)} &= \begin{vmatrix} 0 & M(w_1) & 0 \\ -\frac{\xi}{\chi^2} & -\frac{\psi}{\chi^2} & \frac{b}{\chi^2} \theta_0 \\ -\frac{\sigma\rho c_2(3c_3^2 + c_1^2)}{k\chi} w_1 \psi(w_1) & 0 & N(w_1) \end{vmatrix} = \\ &= M(w_1) \left[\frac{\xi}{\chi^2} N(w_1) - \frac{\sigma b \theta_0}{k\chi^3} \rho c_2 (3c_3^2 + c_1^2) w_1 \psi(w_1) \right], \end{aligned}$$

$$\begin{aligned} D^{(1)} &= \begin{vmatrix} 0 & M(w_1) & -\xi \\ -\frac{\xi}{\chi^2} & -\frac{\psi}{\chi^2} & d + \frac{a}{\chi^2} + \rho\tau c_2^2 w_1^2 \\ -\frac{\sigma\rho c_2(3c_3^2 + c_1^2)}{k\chi} w_1 \psi(w_1) & 0 & -\frac{bc_2}{k\chi} w_1 \psi(w_1) \end{vmatrix} = \\ &= \frac{c_2}{k\chi} w_1 \psi(w_1) \left\{ \frac{\sigma\xi\psi}{\chi^2} \rho(3c_3^2 + c_1^2) - M(w_1) \left[\left(d + \frac{a}{\chi^2} + \rho\tau c_2^2 w_1^2 \right) \sigma\rho(3c_3^2 + c_1^2) + \frac{\xi b}{\chi^2} \right] \right\}. \end{aligned}$$

Regarding the root $\nu = \nu_2$, there is the following microrotational wave:

$$\begin{cases} u_1^{(2)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{A_1^{(2)} e^{i\chi(x_1 - c_2 b_2 t)}\} e^{-\chi c_2 a_2 t}, \\ \phi_1^{(2)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{B_1^{(2)} e^{i\chi(x_1 - c_2 b_2 t)}\} e^{-\chi c_2 a_2 t}, \\ \varphi^{(2)}(x_1, t) = \text{Re}\{C^{(2)} e^{i\chi(x_1 - c_2 b_2 t)}\} e^{-\chi c_2 a_2 t}, \\ \theta^{(2)}(x_1, t) = \theta_0 \text{Re}\{D^{(2)} e^{i\chi(x_1 - c_2 b_2 t)}\} e^{-\chi c_2 a_2 t}, \end{cases} \quad (1.137)$$

For the root $\nu = \nu_3$, there is the following porous longitudinal wave:

$$\begin{cases} u_1^{(3)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{A_1^{(3)} e^{i\chi(x_1 - c_2 b_3 t)}\} e^{-\chi c_2 a_3 t}, \\ \phi_1^{(3)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{B_1^{(3)} e^{i\chi(x_1 - c_2 b_3 t)}\} e^{-\chi c_2 a_3 t}, \\ \varphi^{(3)}(x_1, t) = \text{Re}\{C^{(3)} e^{i\chi(x_1 - c_2 b_3 t)}\} e^{-\chi c_2 a_3 t}, \\ \theta^{(3)}(x_1, t) = \theta_0 \text{Re}\{D^{(3)} e^{i\chi(x_1 - c_2 b_3 t)}\} e^{-\chi c_2 a_3 t}, \end{cases} \quad (1.138)$$

For the root $\nu = \nu_4$, there is the following longitudinal thermal wave:

$$\begin{cases} u_1^{(4)} = \left\{ -\frac{1}{\chi} M(w_4) \left[\left(d + \frac{a}{\chi^2} + \rho \tau c_2^2 a_4^2 \right) \frac{3c_3^2 + c_1^2}{c_2^2} + \frac{\xi b}{\rho c_2^2 \chi^2} \theta_0 \right] \right. \\ \quad \left. + \frac{\xi \psi(3c_3^2 + c_1^2)}{c_2^2 \chi^3} \right\} \sin(\chi x_1) e^{-\chi c_2 a_4 t}, \\ \phi_1^{(4)} = -\frac{\xi^2}{\chi^2} \left[\left(w_4^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} \right) b \theta_0 + \frac{\xi(3c_3^2 + c_1^2)}{c_2^2} \right] \sin(\chi x_1) e^{-\chi c_2 a_4 t}, \\ \varphi^{(4)} = \frac{1}{\chi^2} M(w_4) \left[\left(w_4^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} \right) b \theta_0 + \frac{\xi(3c_3^2 + c_1^2)}{c_2^2} \right] \cos(\chi x_1) e^{-\chi c_2 a_4 t}, \\ \theta^{(4)} = \left\{ M(w_4) \left[\left(w_4^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} \right) \left(d + \frac{a}{\chi^2} + \rho \tau c_2^2 a_4^2 \right) - \frac{\xi^2}{\chi^2 \rho c_2^2} \right] \right. \\ \quad \left. - \frac{\xi \psi}{\chi^2} \left(w_4^2 + \frac{c_1^2}{c_2^2} + \frac{c_3^2}{c_2^2} \right) \right\} \cos(\chi x_1) e^{-\chi c_2 a_4 t}. \end{cases} \quad (1.139)$$

And, regarding the root $\nu = \nu_5$, there is the following longitudinal wave corresponding to the two delay times:

$$\begin{cases} u_1^{(5)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{A_1^{(5)} e^{i\chi(x_1 - c_2 b_5 t)}\} e^{-\chi c_2 a_5 t}, \\ \phi_1^{(5)}(x_1, t) = -\frac{1}{\chi} \text{Im}\{B_1^{(5)} e^{i\chi(x_1 - c_2 b_5 t)}\} e^{-\chi c_2 a_5 t}, \\ \varphi^{(5)}(x_1, t) = \text{Re}\{C^{(5)} e^{i\chi(x_1 - c_2 b_5 t)}\} e^{-\chi c_2 a_5 t}, \\ \theta^{(5)}(x_1, t) = \theta_0 \text{Re}\{D^{(5)} e^{i\chi(x_1 - c_2 b_5 t)}\} e^{-\chi c_2 a_5 t}, \end{cases} \quad (1.140)$$

Above we used the notations

$$M(w_s) = \gamma + \alpha + \epsilon + \rho I_{kj} c_2^2 w_s^2,$$

$$N(w_s) = 1 + \tau_\theta \chi c_2 w_s + \frac{c c_2 \theta_0}{k \chi} w_s \psi(w_s), \text{ cu } s \in \{1, 2, 3, 4, 5\}$$

In the determinant corresponding to the dissipation equation $D(w) = 0$, one can observe the dependence of the propagation velocities of the four longitudinal waves. This dependence is achieved using τ_θ and $\psi(w)$ (obviously in the phase shift parameters).

We can therefore conclude that the propagation of longitudinal waves is affected by the coupling of elastic deformations and microrotations with porous and thermal effects. At the same time, the coupling with porous effects gives rise to a porous longitudinal wave, and taking into account thermal effects with DPL times, implies the appearance of a longitudinal wave corresponding to the two delay times. It is also noted that the coupling with thermal effects has consequences on all longitudinal waves in time (amplitude damping).

For a better understanding and highlighting of the effects of the mentioned coupling on the propagation velocities, a numerical study on a chosen model is necessary.

1.3.5 Numerical simulation

In this section we aim to gain a better understanding of the behavior of phase velocities and the decomposition of the longitudinal wave solutions derived in the previous section. We note that we used the Wolfram Mathematica software for this purpose.

Symbol	Material parameter	Value	Unit of measurement
σ	Coefficient of linear thermal expansion	2.33×10^{-4}	K^{-1}
λ	First Lamé modulus	9.4×10^{10}	Nm^{-2}
μ	Second Lamé modulus	4×10^{10}	Nm^{-2}
κ	Thermal conductivity	1×10^{10}	$\text{W m}^{-1}\text{K}^{-1}$
ξ	Pore parameter	1.13849×10^{10}	Nm^{-2}
α	Micropolar constant	0.020×10^{10}	Nm^{-2}
γ	Micropolar constant	0.779×10^{-9}	N
ϵ	Micropolar constant	0.002×10^{10}	Nm^{-2}
a	Pore parameter	1.475	
b	Porothermic coupling coefficient	-2×10^6	$\text{Nm}^{-2}\text{K}^{-1}$
c	Pore diffusion coefficient	2.9×10^4	$\text{m}^2\text{sec}^{-2}\text{K}^{-1}$
I	Balanced inertia	0.2×10^{-19}	m^2
τ	balanced inertia	1.753×10^{-15}	m^{-2}
ρ	Density	1.74×10^3	Kg m^{-3}

Table 1.2: Material parameter values

Regarding the numerical simulations, the coefficients in Table 1.2 were chosen for magnesium in [17] and [18]. Also, $\chi = 1 \times (-b)^{-1}$, $\tau_q = 5 \times 10^{-4}\text{s}$ and $\tau_\theta = 1 \times 10^{-3}\text{s}$ are considered. These values were chosen by referring to works such as [15] and [16] and taking into account the condition (1.115). Thus, the following cases are considered to compare the velocities.

C_1 : Uncoupled case: $\xi = \psi = \sigma(3c_3^2 + c_1^2) = b = 0$;

C_2 : Poroelastic case with microrotations: $\sigma(3c_3^2 + c_1^2) = b = 0$;

C_3 : Thermoelastic case with microrotations: $\xi = \psi = b = 0$;

C_4 : Coupled case with microrotations.

Reference model	Uncoupled	Thermoelastic with microrotations
ν_1	10196.9	$-4.815.82i$
ν_2	-3.61747×10^{19}	2.51433×10^{12}
ν_3	-7.951×10^{12}	1.39079×10^{12}
ν_4	$-0.0000189777i$	$-0.0000280988i$
ν_5	$5.65685 \times 10^9 + 9.48887 \times 10^{-6}i$	$1.58605 \times 10^8 + 1.57416 \times 10^6i$

Table 1.3: Numerical comparison of $\nu_1, \nu_2, \nu_3, \nu_4$ and ν_5

Following the simulations, the values presented in Table 1.3 can be distinguished: the uncoupled case and the thermoelastic case with microrotations.

Figures 1-8 show the graphics for the cases where we do not have standing waves:

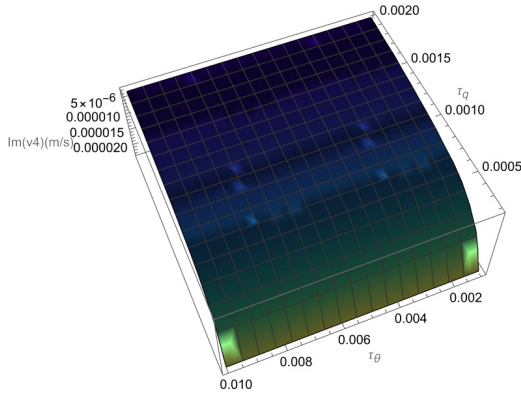


Figure 1.1: The uncoupled case for $Im(\nu_4)$. Behavior for $Im(\nu_4)$ varying as a function of

$$\tau_q \in [2 \times 10^{-4}\text{s}, 2 \times 10^{-3}\text{s}]$$

$$\tau_\theta \in [1 \times 10^{-3}\text{s}, 1 \times 10^{-2}\text{s}].$$

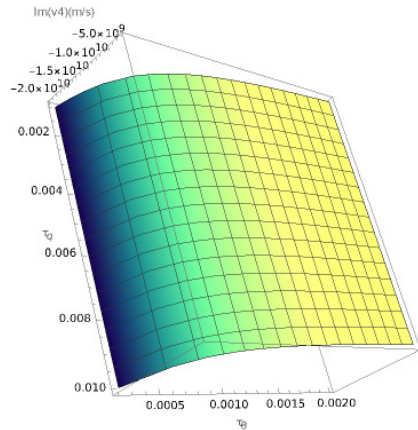


Figure 1.2: Thermoelastic case for $Im(\nu_4)$. Behavior for $Im(\nu_4)$ varying as a function of

$$\tau_q \in [2 \times 10^{-4} \text{ s}, 2 \times 10^{-3}\text{s}]$$

$$\tau_\theta \in [1 \times 10^{-3}\text{s}, 1 \times 10^{-2}\text{s}].$$

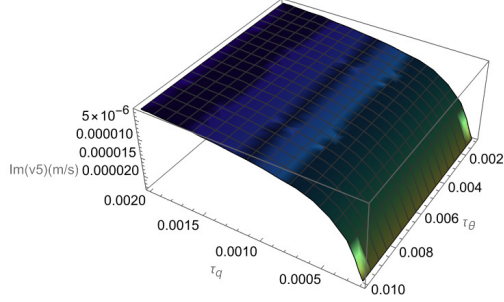


Figure 1.3: The uncoupled case for $Im(\nu_5)$. Behavior for $Im(\nu_5)$ varying with

$$\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$$

$$\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s].$$

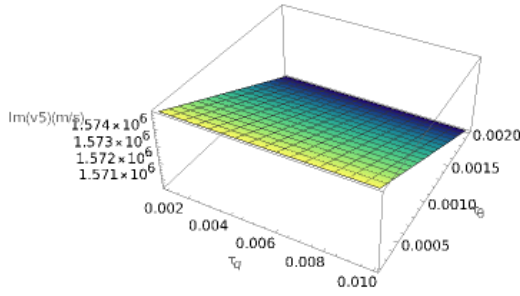


Figure 1.4: Thermoelastic case for $Im(\nu_5)$. Behavior for $Im(\nu_5)$ varying as a function of

$$\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$$

$$\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s].$$

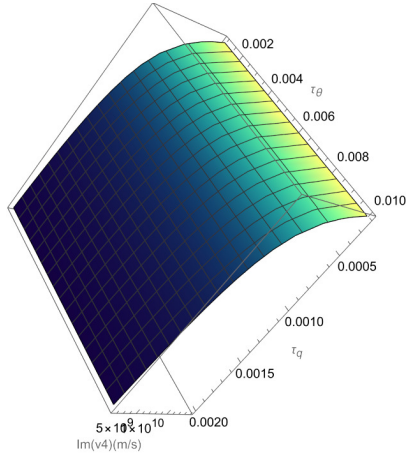


Figure 1.5: The uncoupled case for $Re(\nu_4)$. Behavior for $Re(\nu_4)$ varying with

$$\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$$

$$\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s].$$

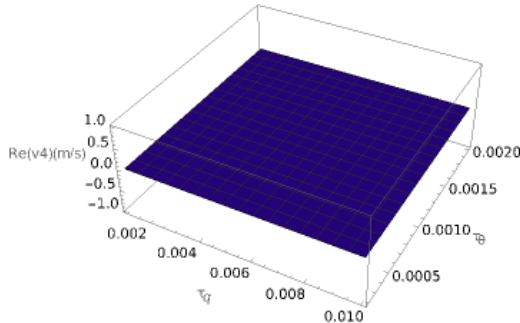


Figure 1.6: Thermoelastic case for $Re(\nu_4)$. Behavior for $Re(\nu_4)$ varying as a function of

$$\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$$

$$\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s].$$

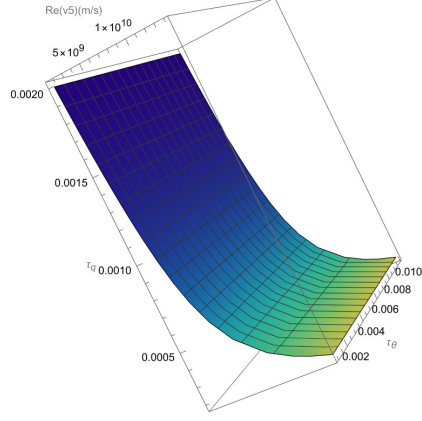


Figure 1.7: The uncoupled case for $Re(\nu_5)$. Behavior for $Re(\nu_5)$ varying as a function of
 $\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$
 $\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s]$.

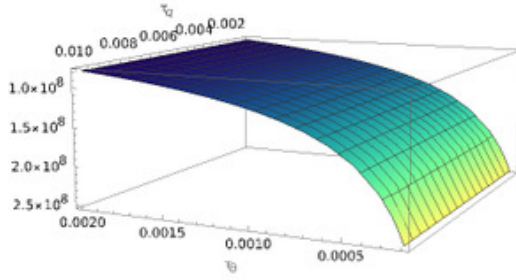


Figure 1.8: Thermoelastic case $Re(\nu_5)$. Behavior for $Re(\nu_5)$ varying as a function of
 $\tau_q \in [2 \times 10^{-4}s, 2 \times 10^{-3}s]$
 $\tau_\theta \in [1 \times 10^{-3}s, 1 \times 10^{-2}s]$.

The 8 analyzed graphs (Figures 1.1-1.8) provide a detailed insight into the behavior of longitudinal waves in the case of a DPL model, comparing the uncoupled and coupled scenarios. The analysis of the graphs shows that thermoelastic coupling plays a crucial role in stabilizing and controlling longitudinal waves. While the uncoupled scenario highlights the high sensitivity of the system to parameter variations, thermoelastic coupling minimizes these effects, favoring a more efficient and stable propagation.

Capitol 2

Advanced Solutions in Isotropic Micropolar Media

2.1 Solutions in Terms of Complex Potentials

In this part of the paper we will investigate the plane deformation, within the framework of equilibrium theory, for homogeneous and isotropic micropolar porous bodies, using the constitutive equations, geometric equations and equilibrium equations without mass forces. Our emphasis is on treating fundamental boundary value problems of the plane deformation theory. Subsequently, we obtain a description of the displacement, microrotations and pores using complex analytic functions and two real functions. In this sense, we use the homogeneous Helmholtz equations [4].

2.1.1 Field equations

We consider B a bounded domain in three-dimensional Euclidean space, with ∂B its boundary and n the external normal to the surface ∂B . Assuming that we have a porous micropolar elastic medium occupying the domain B , we relate the motion of the body to a system of orthogonal axes $Ox_i (i = 1, 2, 3)$.

The basic equations describing the evolution of an isotropic Cosserat medium with voids are the following:

- the constitutive equations:

$$t_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) + k(u_{i,j} + \varepsilon_{ijk} \phi_k) + \xi \varphi \delta_{ij}, \quad (2.1)$$

$$m_{ij} = \alpha \phi_{k,k} \delta_{ij} + \gamma \phi_{j,i} + \psi \phi_{i,j} + \zeta \varepsilon_{sji} \varphi_{,s}, \quad (2.2)$$

$$h_i = d\varphi_{,i}, \quad (2.3)$$

-the geometric equations

$$e_{ij} = u_{i,j} + \varepsilon_{ijk} \phi_k, \quad \chi_{ij} = \phi_{i,j}, \quad (2.4)$$

-equilibrium equations (body charges are absent)

$$t_{ji,j} = 0, \quad (2.5)$$

$$m_{ji,j} + \varepsilon_{irs} t_{rs} = 0, \quad (2.6)$$

$$h_{i,i} + g = 0. \quad (2.7)$$

Consider

$$t_i = t_{ji} n_j, \quad m_i = m_{ji} n_j, \quad N_i = h_{ji} n_j, \quad (2.8)$$

where N_i is the generalized surface force at a regular point on ∂B , t_i is the surface force vector and m_i is the surface force couple.

We will further assume that the internal energy density is a positive quadratic form, from which we have:

$$\begin{aligned} \gamma + \beta > 0, \quad \gamma - \beta > 0, \quad \gamma + \beta + 3\alpha > 0, \quad d > 0, \\ \kappa > 0, \quad \kappa + 2\mu > 0, \quad \kappa + 2\mu + 3\lambda > 0. \end{aligned} \quad (2.9)$$

The notations used are in accordance with Table 1.1.

2.1.2 Plane deformation problem

In this part of the paper, we will consider that B is the interior of a right cylinder whose cross-section is Σ and whose lateral boundary is Π . This configuration is represented in an orthogonal coordinate system, such that its generators are parallel to the x_3 axis. We denote by L the contour corresponding to the cross-section. The plane deformation is considered to be parallel to the (Ox_1x_2) plane. As in the previous section, dedicated to plane deformation, we obtain the following system, corresponding to the displacement, microrotation and pores. (Δ is the Laplacian)

$$\begin{aligned} (\lambda + \mu)u_{\rho,\rho\alpha} + (\mu + \kappa)\Delta u_\alpha + \xi\varphi_{,\alpha} + \kappa\varepsilon_{3\alpha\beta}\phi_{,\beta} &= 0, \\ \psi\Delta\varphi + \kappa\varepsilon_{3\alpha\beta}u_{\beta,\alpha} - 2k\phi &= 0, \\ d\Delta\varphi - \xi u_{\rho,\rho} - a\varphi &= 0. \end{aligned} \quad (2.10)$$

The steps to obtain the system are found in section {1.1.5}

2.1.3 Complex potentials

In this section we will work in the system (2.10), whose relations will be rewritten in complex coordinates and integrated directly. In other words, we will determine the displacement using a pair of complex analytic functions, and the microrotation and the change in volume fraction using real functions that verify the homogeneous Helmholtz equations [4]. First we introduce the complex coordinates:

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad (2.11)$$

and complex displacement

$$U = u_1 + iu_2. \quad (2.12)$$

So, we get

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}, \quad u_{\alpha, \alpha} = \frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}}, \quad \varepsilon_{3\alpha\beta} u_{\beta, \alpha} = i \left(\frac{\partial \bar{U}}{\partial \bar{z}} - \frac{\partial U}{\partial z} \right). \quad (2.13)$$

Considering (2.13), we will rewrite the relations of the system (2.10) in the following form

$$\begin{aligned} 2(\kappa + \mu) \frac{\partial^2 U}{\partial z \partial \bar{z}} + (\mu + \lambda) \frac{\partial}{\partial \bar{z}} \left(\frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}} \right) + \xi \frac{\partial \varphi}{\partial \bar{z}} - i\kappa \frac{\partial \phi}{\partial \bar{z}} &= 0, \\ 4\psi \frac{\partial^2 \phi}{\partial z \partial \bar{z}} - i\kappa \left(\frac{\partial U}{\partial z} - \frac{\partial \bar{U}}{\partial \bar{z}} \right) - 2\kappa \phi &= 0, \\ 4d \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} - \xi \left(\frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}} \right) - a\varphi &= 0, \end{aligned} \quad (2.14)$$

and using notations for convenience and a set of operations, we obtain the corresponding form of the function φ :

$$\varphi = M_1 - \frac{\xi}{2dm^2(\lambda + \kappa + 2\mu)} [\Gamma(z) + \bar{\Gamma}'(\bar{z})], \quad (2.15)$$

where M_1 is a real function that satisfies

$$4 \frac{\partial^2 M_1}{\partial z \partial \bar{z}} - m^2 M_1 = 0, \quad (2.16)$$

the form corresponding to the function ϕ :

$$\phi = P - \frac{i\kappa}{2\gamma p^2(\mu + \kappa)} [\Gamma'(z) - \bar{\Gamma}'(\bar{z})], \quad (2.17)$$

where the real function P satisfies

$$\left(4 \frac{\partial^2}{\partial z \partial \bar{z}} - p^2 \right) P = 0 \quad (2.18)$$

and

$$U = \eta_1 \Gamma(z) - \eta_2 \bar{\Gamma}(\bar{z}) z - \bar{\omega}(\bar{z}) + 4iq_1 \frac{\partial P}{\partial z} - 4q_2 \frac{\partial M_1}{\partial \bar{z}}, \quad (2.19)$$

where ω is an analytic function on z

Previously, we were able to obtain in the relations (2.15), (2.17) and (2.19) a representation of the functions φ, ϕ and U in terms of complex analytic functions Γ, ω and real functions M and P .

Using simple calculations, the following form of the stresses results:

$$\begin{aligned}
t_{11} + t_{22} &= \frac{a(2\mu + 2\lambda + \kappa) - 2\xi^2}{2dm^2(2\mu + \kappa + \lambda)}[\Gamma'(x) + \bar{\Gamma}'(\bar{z})] + \frac{\xi(2\mu + \kappa)}{2\mu + \kappa + \lambda}M_1, \\
t_{11} + it_{12} - t_{22} + it_{21} &= -2(2\mu + k)\left[\eta_2\bar{\Gamma}''(\bar{z})z + \bar{\omega}(\bar{z}) - 4iq_1\frac{\partial^2 P}{\partial \bar{z}^2} + 4q_2\frac{\partial^2 M_1}{\partial \bar{z}^2}\right], \\
t_{21} - t_{12} &= \gamma p^2 P, \\
m_{13} - im_{23} &= 2\gamma\frac{\partial P}{\partial z} + 2i\zeta\frac{\partial M_1}{\partial z} - i\left[\frac{\kappa}{p^2(\mu + \kappa)} + \frac{\zeta\xi}{dm^2(2\mu + \lambda + \kappa)}\right]\Gamma''(z), \\
h_1 - ih_2 &= 2d\frac{\partial M_1}{\partial z} - \frac{\xi}{m^2(2\mu + \lambda + \kappa)}\Gamma''(z).
\end{aligned} \tag{2.20}$$

We express the boundary conditions in the following form:

$$\begin{aligned}
\eta_1\Gamma(z) - \eta_2x\bar{\Gamma}'(\bar{z}) - \bar{\omega}(\bar{z}) + 4iq_1\frac{\partial P}{\partial \bar{z}} - 4q_2\frac{\partial M_1}{\partial \bar{z}} &= \tilde{u}(\tau), \\
P(z, \bar{z}) - \frac{i\kappa}{2\gamma p^2(\kappa + \mu)}[\Gamma'(z) - \bar{\Gamma}'(\bar{z})] &= \phi(\tau), \\
M_1(z, \bar{z}) - \frac{\xi}{2dm^2(\lambda + \kappa + 2\mu)}[\Gamma(z) + \bar{\Gamma}'(\bar{z})] &= \varphi(\tau), \quad z \in L,
\end{aligned} \tag{2.21}$$

where $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$. Furthermore, the boundary conditions can take the following form:

$$\begin{aligned}
(2\mu + \kappa)\frac{d}{ds}\{\eta_2[\Gamma(z) + z\bar{\Gamma}'(\bar{z})] + \bar{\omega}(\bar{z}) - 4iq_1\frac{\partial P}{\partial \bar{z}} + 4q_2\frac{\partial M_1}{\partial \bar{z}}\} &= \tilde{t}(\tau), \\
Im\{[2\gamma\frac{\partial P}{\partial z} + 2i\zeta\frac{\partial M_1}{\partial z} - iw_1\Gamma''(z)]\frac{dz}{\tau}\} &= \tilde{m}(\tau), \\
Im\{[2d\frac{\partial M_1}{\partial z} - w_2\Gamma''(z)]\frac{dz}{d\tau}\} &= \tilde{N}(\tau), \quad z \in L,
\end{aligned} \tag{2.22}$$

where $\tilde{t}(\tau) = -\tilde{t}_2 + i\tilde{t}_1$.

2.1.4 Construction of potentials

In this section, we aim to derive the structure of the potentials Γ, ω, P , and M_1 and explore their arbitrariness in different domains of interest. We analyze the differences between the configurations of the following sets of potentials (Γ, ω, P, M_1) and $(\Gamma^*, \omega^*, P^*, M_1^*)$, corresponding to the same functions $t_{\alpha\beta}, m_{\alpha 3}$ and h_α .

According to (2.20), it is necessary that

$$\begin{aligned}
Re[\Gamma'(z)] &= Re[\Gamma^{*'}(z)], \quad M_1 = M_1^*, \quad P = P^*, \\
\eta_2 z \bar{\Gamma}''(\bar{z}) + \bar{\omega}'(\bar{z}) &= \eta_2 z \bar{\Gamma}^{*''}(\bar{z}) + \bar{\omega}^{*'}(\bar{z}),
\end{aligned}$$

where $Re[\]$ represents the real part for $\]$. Therefore, we deduce that

$$\begin{aligned}
\Gamma(z) &= \Gamma^*(z) + iX_z + \rho_1, \\
\omega(z) &= \omega^*(z) + \rho_2, \\
M_1 &= M_1^*, \\
P &= P^*,
\end{aligned} \tag{2.23}$$

where X is a real constant, and ρ_1, ρ_2 are complex constants

We can fix the origin of the coordinates at Σ so that X, ρ_1, ρ_2 satisfy the conditions

$$\Gamma(0) = 0, \operatorname{Im}\{\Gamma'(0)\} = 0, \omega(0) = 0, \quad (2.24)$$

which ensures uniformity for Γ and ω .

We obtain the following form of the complex potentials,

$$\begin{aligned} \Gamma(z) &= \Gamma_1(z) + \sum_{k=1}^m (zX_k + Y_k) \log(z - z_k), \\ \omega(z) &= \omega_1(z) + \sum_{k=1}^m Z_k \log(z - z_k), \end{aligned} \quad (2.25)$$

where z_k represents a point in the simply connected region Σ_k , bounded by L_k . Γ_1 and ω_1 are uniform analytic functions on Σ , Y_k and Z_k are complex constants, and A_k are real constants.

From (2.25) we deduce:

$$\begin{aligned} \Gamma(z) &= -\frac{1}{2\pi} \sum_{k=1}^n (S_1^{(k)} + iS_2^{(k)}) \log(z - z_k) + \Gamma_1(z), \\ \omega(z) &= \frac{1}{2\pi} \eta_1 \sum_{k=1}^m (S_1^{(k)} - iS_2^{(k)}) \log(z - z_k) + \omega_1(z), \end{aligned} \quad (2.26)$$

where we have written the resultants of the stress vector applied to the contour as $S_1^{(k)}, S_2^{(k)}$.

Teoremă 5. *Let Σ be an unbounded domain with the contours $L_1, L_2, \dots, L_m$ as internal bounded regions. If we assume that the origin $z = 0$ is outside the section Σ and that $h_\alpha, t_{\alpha\beta}$, and $m_{\alpha\beta}$ are bounded in the vicinity of the boundary point and for $|z| = \chi$ sufficiently large, then we have the following representations:*

$$\begin{aligned} \Gamma(z) &= -\frac{1}{2\pi} (R_1 + iR_2) \log z + (a_1 + ia_2)z + \Gamma_0(z), \\ \omega(z) &= \frac{1}{2\pi} \eta_1 (R_1 - iR_2) \log z + (b_1 + ib_2)z + \omega_0(z), \\ P(z, \bar{z}) &= \sum_{n=0}^{\infty} (P_n e^{in\theta} + \bar{P}_n e^{-in\theta}) K_n(\tau\chi), \\ M(z, \bar{z}) &= \sum_{n=0}^{\infty} (M_n e^{in\theta} + \bar{M}_n e^{-in\theta}) K_n(w\chi). \end{aligned} \quad (2.27)$$

2.1.5 Stresses around a circular hole

In this part of the paper, we will use the results obtained in the previous sections. By using boundary conditions and complex potentials, the following theorem allows the analysis of stresses and strains in materials with circular hole under external loading.

Teoremă 6. *Let $\Sigma_1 = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 > \chi^2\}$ be an unbounded domain with a circular hole centered at the origin and with radius χ . Assuming that a uniform axial stress acts on the body in the x_1 direction, at infinity we have:*

$$t_{11}^* = Q, \quad t_{12}^* = 0, \quad t_{21}^* = 0, \quad t_{22}^* = 0, \quad m_{\alpha 3}^* = 0, \quad h_{\alpha}^* = 0, \quad (2.28)$$

where Q is a given constant. The boundary conditions on the boundary of the circular hole become:

$$\begin{aligned} \eta_2[\Gamma(z) + z\bar{\Gamma}'(\bar{z})] + \bar{\omega}(\bar{z}) - 4iq_1 \frac{\partial P}{\partial \bar{z}} + 4q_2 \frac{\partial M_1}{\partial \bar{z}} &= 0, \\ \text{Im}\{[2\gamma \frac{\partial P}{\partial z} - iw_1 \Gamma''(z)] \frac{dz}{ds}\} &= 0, \\ \text{Im}\{[2d \frac{\partial M_1}{\partial z} - w_2 \Gamma''(z)] \frac{dz}{ds}\} &= 0, \quad \text{for } |z| = \chi. \end{aligned} \quad (2.29)$$

Using these and (2.29) we obtain the components of the complex potentials as follows:

$$\begin{aligned} \Gamma(z) &= \frac{1}{4\eta_2(2\mu + \kappa)} Qz + \frac{1}{z} D_1, \\ \omega(z) &= \frac{1}{z} E_1 + \frac{1}{z^3} E_3 - \frac{1}{2(2\mu + \kappa)} Qz, \\ P(z, \bar{z}) &= iH_1 \left(\frac{z}{\bar{z}} - \frac{\bar{z}}{z} \right) K_2(\tau\chi), \\ M_1(z, \bar{z}) &= H_2 \left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right) K_2(w\chi), \quad \chi = (z\bar{z})^{1/2}, \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} D_1 &= \frac{1}{2(2\mu + \kappa)F} Q\xi^2, \quad E_1 = -\frac{1}{2(2\mu + \kappa)} Q\chi^2, \\ E_3 &= \frac{1}{2(2\mu + \kappa)F} [\eta_2 + 1q_1\chi\tau TK_3(\tau\chi) + 2q_2\chi mHK_3(m\chi)], \\ H_1 &= TD_1, \quad H_2 = HD_1, \quad F = \eta_2 + 2q_1\tau a_2 TK_1(\tau\chi) + 2q_2m\chi K_3(m\chi) \\ T &= \frac{4}{2\chi^4\omega} \{8dK_2(m\chi) + 2m\chi\xi[K_1(m\chi) + K_3(\tau\chi)]\}, \\ H &= \frac{4}{3\chi^4\omega} \{8\gamma K_2(\tau\chi) - 2\tau\chi\xi[K_1(\tau\chi) + K_3(\tau\chi)]\}, \\ \Omega &= \frac{16d\gamma}{\chi^2} K_2(\tau\chi)K_2(m\chi) + 4m\tau\xi^2[K_1(m\chi) + K_3(m\chi)][K_1(\tau\chi) + K_3(\tau\chi)]. \end{aligned} \quad (2.31)$$

Let

$$e^{i\theta}U = u_\chi + iu_\theta,$$

where, the components u_χ and u_θ are in polar coordinates. From (2.15), (2.17), (2.19) and (2.30) it follows that

$$\begin{aligned} u_\chi + iu_\theta &= \frac{\eta_1 - \eta_2}{4\eta_2(2\mu + \kappa)}Q\chi - \frac{1}{\chi}(E_1 - \eta_1 D_1) + u \cos 2\theta + iv \sin 2\theta, \\ \phi &= -2[H_1 K_2(\tau\chi) - \frac{\kappa}{2\gamma\tau^2(\mu + \kappa)\chi^2}D_1] \sin 2\theta, \\ \varphi &= -\frac{\xi Q}{4dm^2\eta_2(\lambda + \kappa + 2\mu)(\kappa + 2\mu)} + 2[H_2 K_2(m\chi) \\ &\quad + \frac{\xi}{2dm^2(\lambda + \kappa + 2\mu)\chi^2}] \cos 2\theta, \end{aligned}$$

where

$$\begin{aligned} u &= \frac{1}{\chi}\eta_2 D_1 + \frac{Q\chi}{2(2\mu + \kappa)} - \frac{1}{\chi^3}E_3 + \frac{2}{\chi}q_1 H_1 K_2(\tau\chi) \\ &\quad + 2mq_2 H_2[K_1(m\chi) + K_3(m\chi)], \\ v &= \frac{1}{\chi}\eta_2 D_1 - \frac{Q\chi}{2(\kappa + 2\mu)} - \frac{1}{r^3}E_3 + 2q_1 H_1[K_1(\tau\chi) + K_3(\tau\chi)] \\ &\quad + \frac{8}{rq_2}H_2 K_2(m\chi). \end{aligned}$$

Similarly, using (2.20), (2.29) we obtain the stresses.

2.1.6 Numerical simulation

The graphs below correspond to an isotropic magnesium crystal with pores and are obtained using the "Wolfram Mathematica" computing system. The values used can be found in [9]. The obtained graphs represent the real and imaginary parts of the complex potentials $\Gamma(z)$ and $\omega(z)$, obtained previously. They allow us to visualize the variation of the potentials in the complex. More precisely, these graphs help to understand the stress and displacement fields in the material, to identify critical regions and to predict the behavior of the material under different conditions.

Figure 2.1: Behavior of $Re(\Gamma(z))$ and $Im(\Gamma(z))$

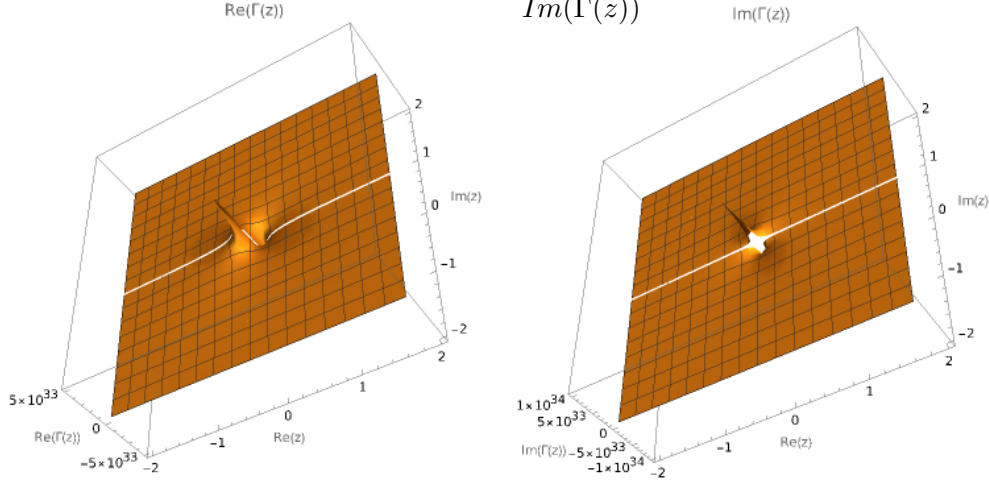
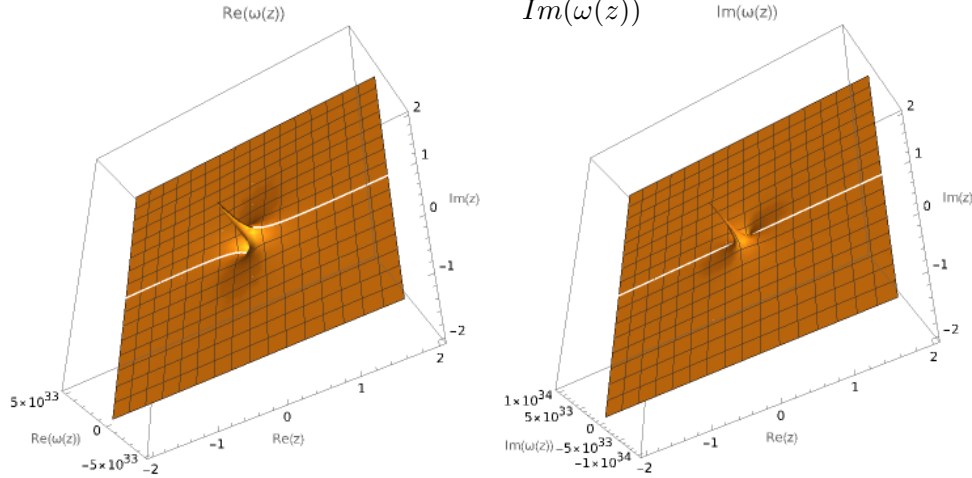


Figure 2.2: Behavior of $Re(\omega(z))$ and $Im(\omega(z))$



The following four graphs correspond to the imaginary and real parts of the radial displacement component " u " and the tangential displacement component " v ", obtained previously. The axis corresponding to χ represents the radial distance from the center of the circular hole, and the axis corresponding to θ represents the angular coordinates around the hole.

Figure 2.3: Behavior of $Re(u)$ and $Im(u)$

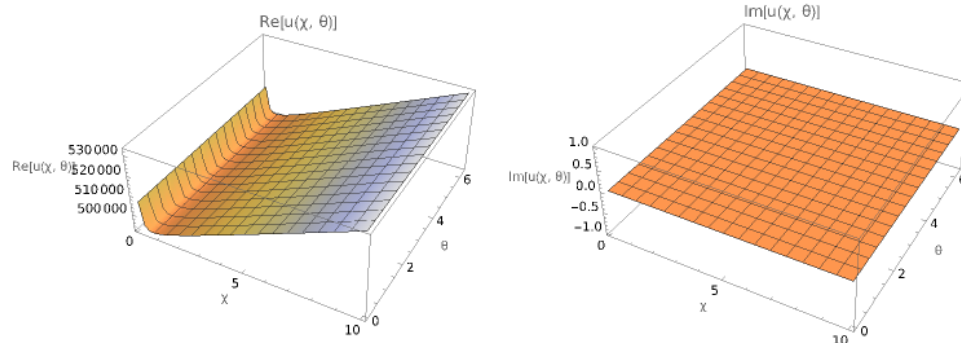
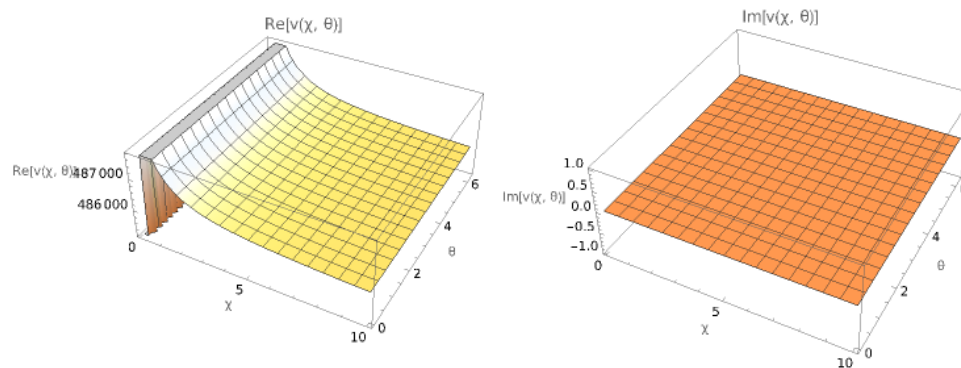


Figure 2.4: Behavior of $Re(v)$ and $Im(v)$



Capitol 3

Anisotropic micropolar media

In this section, the notations presented in the table below are used.

Notations	Physical Interpretation
a, b, c, d, h	coefficients constants specific heat
η	entropy specific per unit mass
I_{ij}	components of the microinertia tensor
t_{mn}	components of the stress tensor
τ_{mn}	components of the stress torque tensor
q_m	components of the thermal conduction vector
v_m	components of the displacement vector
φ_m	components of the torque vector
ϕ_m	components of the microrotation
f_m	mass force
g_m	mass torque
t_k	heat flux
τ_k	surface tractions
$n = (n_l)$	normal vector to boundary ∂D
T	conductive temperature
ϑ	thermodynamic temperature
ϑ_0	constant reference temperature
ρ	mass density
ϵ_{ijk}	Ricci symbol

Table 3.1: Notations

3.1 Uniqueness and instability in two-temperature thermoelasticity

3.1.1 Mixed boundary-value and initial-value problem

We consider the thermoelastic micropolar body occupying the three-dimensional domain Ω in the Euclidean space R^3 . The closure of Ω is denoted by $\bar{\Omega}$ and we have $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary of the domain Ω and is considered regular enough to allow the application of the divergence theorem. The unit vector of the exterior normal to $\partial\Omega$ has components denoted by n_i . The vector and tensor fields are denoted by bold letters. The notation v_i is used for the components

of a vector field $\mathbf{v}$, the notation u_{ij} is used for the components of a second-order tensor field $\mathbf{u}$, and so on. A fixed system of Cartesian axes Ox_i , $i = 1, 2, 3$ will be used to refer to the motion of the thermoelastic body.

Considering the geometric equations and the constitutive equations, which are introduced into the basic equations, we are led to the following system of partial differential equations:

$$\begin{aligned} A_{ijmn}(v_{n,mj} + \varepsilon_{kmn}\phi_{k,j}) + B_{ijmn}\phi_{n,mj} - \alpha_{ij}(\vartheta_{,j} + a\dot{\vartheta}_{,j}) &= \rho\ddot{v}_i, \\ B_{ijmn}(v_{n,mj} + \varepsilon_{kmn}\phi_{k,j}) + C_{ijmn}\phi_{n,mj} - \beta_{ij}(\vartheta_{,j} + a\dot{\vartheta}_{,j}) \\ + \varepsilon_{ijk}[A_{jkmn}(v_{n,m} + \varepsilon_{lmn}\phi_l) + B_{jkmn}\phi_{n,m} - \alpha_{ij}(\vartheta_{,j} + a\dot{\vartheta}_{,j})] &= I_{ij}\ddot{\phi}_i, \quad (3.1) \\ \kappa_{ij}T_{,ij} - \alpha_{ij}(\dot{v}_{j,i} + \varepsilon_{ijk}\dot{\phi}_k) - \beta_{ij}\dot{\vartheta}_{j,i} &= h\ddot{\vartheta} + d\dot{\vartheta}, \end{aligned}$$

which are verified for any $(x, t) \in \Omega \times (0, \infty)$.

By solving the mixed initial boundary value problem, in the theory of two-temperature thermoelasticity, of micropolar bodies in the cylinder $\Omega \times [0, \infty)$ we refer to an ordered set $(v_m, \phi_m, T, \vartheta)$ which verifies the above system of equations, the boundary conditions and the initial conditions.

3.1.2 Main result

We begin this section by specifying a law of conservation of energy, considering the variation of the conducting temperature, that is, considering the two-temperature relationship. This law has the following form:

$$E_1(t) = E_1(0), \quad t \in [0, \infty), \quad (3.2)$$

where

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_{\Omega} \left[\rho \dot{v}_m(t) \dot{v}_m(t) + I_{mn} \dot{\phi}_m(t) \dot{\phi}_n(t) \right. \\ &\quad + A_{mnkl} e_{mn}(t) e_{kl}(t) + 2B_{mnkl} e_{mn}(t) \sigma_{kl}(t) + C_{mnkl} \sigma_{mn}(t) \sigma_{kl}(t) \\ &\quad + c \kappa_{mn} T_{,m}(t) T_{,n}(t) + d \left(\vartheta(t) + \frac{h}{d} \dot{\vartheta}(t) \right)^2 + h \left(a - \frac{h}{d} \right) \dot{\vartheta}^2(t) \left. \right] dV \\ &\quad + \int_0^t \int_{\Omega} \left[\kappa_{mn} T_{,m}(s) T_{,n}(s) + c \left((\kappa_{mn} T_{,m}(s))_{,n} \right)^2 + (ad - h) \dot{\vartheta}^2(s) \right] dV ds. \end{aligned}$$

If we do not take into account the temperature variation of conductivity, the law of conservation of energy takes the following form:

$$E_2(t) = E_2(0), \quad t \in [0, \infty), \quad (3.3)$$

where

$$\begin{aligned}
E_2(t) = & \frac{1}{2} \int_{\Omega} \left[\rho \dot{v}_m(t) \dot{v}_m(t) + I_{mn} \dot{\phi}_m(t) \dot{\phi}_n(t) + A_{mnkl} e_{mn}(t) e_{kl}(t) \right. \\
& + 2B_{mnkl} e_{mn}(t) \sigma_{kl}(t) + C_{mnkl} \sigma_{mn}(t) \sigma_{kl}(t) \\
& + d \left(\vartheta(t) + \frac{h}{d} \dot{\vartheta}(t) \right)^2 + h \left(a - \frac{h}{d} \right) \dot{\vartheta}^2(t) \Big] dV \\
& + \int_0^t \int_{\Omega} \left[\kappa_{mn} T_{,m}(s) T_{,n}(s) + c \left((\kappa_{mn} T_{,m}(s))_{,n} \right)^2 + (ad - h) \dot{\vartheta}^2(s) \right] dV ds.
\end{aligned}$$

We denote by $\mathcal{P}$ the problem consisting of the system of equations (3.1), the boundary conditions, and the initial conditions.

Teoremă 7. *In the case of null initial data, the mixed problem $\mathcal{P}$ admits only the null identical solution.*

To obtain the second main result, concerning exponential instability for solving the mixed problem $\mathcal{P}$, we will have to assume that an additional condition is fulfilled. Namely, we must assume that the energy of the system, in its initial state, is not strictly positive, i.e. $E_2(0) \geq 0$.

We will start with some useful auxiliary considerations.

Let us consider a boundary value problem of the following form:

$$\begin{aligned}
(\kappa_{mn} u_{,m}(x))_{,n} &= d\vartheta^1 + h\vartheta^0 - (\alpha_{mn} e_{mn}^0 + \beta_{mn} \sigma_{mn}^0), \quad x \in \Omega, \\
\nu(x) &= 0, \quad x \in \partial\Omega,
\end{aligned} \tag{3.4}$$

where $u = u(x)$ is the unknown function, and the constants ϑ^0 , ϑ^1 , e_{mn}^0 and σ_{mn}^0 are initial data.

Based on the usual properties of boundary value problems, defined in the context of elliptic equations, we can deduce that the boundary value problem (3.4) admits a solution $u(x)$, defined on the domain Ω .

With the help of the function

$$\zeta(x, t) = \int_0^t T(x, s) ds,$$

from (3.4) we can see that the function $u(x)$ is a solution of the equation

$$\begin{aligned}
d\vartheta(x) + h\dot{\vartheta}(x) - [\kappa_{mn} (u_{,m}(x) + \zeta_{,m}(x))]_{,n} &= \\
&= \alpha_{mn} e_{mn}(x) + \beta_{mn} \sigma_{mn}(x).
\end{aligned}$$

Teoremă 8. *We assume positive coefficients and tensors.*

If in the mixed problem $\mathcal{P}$ the boundary values are zero, then any of its solutions, for which the condition $E_2(0) \geq 0$ holds, is exponentially unstable

3.2 Uniqueness, reciprocity and the variational principle

3.2.1 Problem formulation

We will consider an inhomogeneous, anisotropic and Cosserat body which, at the initial time $t = 0$, occupies the regular domain D of the three-dimensional Euclidean space R^3 . The domain D is bounded by the smooth closed surface ∂D . We will use both scalar functions and vector and tensor functions, and these depend on the points D , $x = (x_m)$ and the time variable $t \in [0, \infty)$.

If we consider the constitutive equations, then from the equations of motion and the energy equation we obtain the following system of partial differential equations:

$$\begin{aligned} \rho \ddot{v}_m &= A_{klmn} e_{kl,n} + B_{klmn} \sigma_{kl,n} + \alpha_{mn} (\vartheta_{,n} + a \dot{\vartheta}_{,n}), \\ I_{mn} \ddot{\phi}_n &= B_{mnkl} e_{kl,n} + C_{klmn} \sigma_{kl,n} + \beta_{mn} (\vartheta_{,n} + a \dot{\vartheta}_{,n}) \\ &\quad + \epsilon_{mjk} (C_{jkln} \sigma_{ln} + B_{jkln} \varepsilon_{ln} + \alpha_{jk} (\vartheta + a \dot{\vartheta})), \\ h \ddot{\vartheta} &= -d \dot{\vartheta} + \alpha_{mn} \dot{e}_{mn} + \beta_{mn} \dot{\varepsilon}_{mn} + \kappa_{mn} \vartheta_{,mn}, \end{aligned} \quad (3.5)$$

which takes place for any (t, x) in $[0, \infty) \times D$.

We will denote by $\mathcal{P}$ the mixed initial and boundary value problem in the context of Cosserat body thermodynamics, consisting of the equations (3.5), the initial conditions, and the boundary relations.

The qualitative results that we will address in the following relate to the solutions of the problem $\mathcal{P}$.

3.2.2 Reciprocity results

Our first result is a reciprocity result. For this we need a convolution product for two continuous functions. So, if φ and ψ are two scalar functions, defined on $[0, \infty) \times D$ and continuous in time, then their convolution product, denoted by " $\ast$ ", is defined by:

$$(\varphi \ast \psi)(t, x) = \int_0^t \varphi(t - \tau, x) \psi(\tau, x) d\tau.$$

Now we introduce the functions $p(t)$ and $r(t)$, useful in what follows, defined by:

$$p(t) = 1, \quad r(t) = (p \ast p)(t) = t, \quad \forall t \in [0, \infty), \quad (3.6)$$

and we will consider the following writing convention:

$$\hat{\varphi}(t, x) = \int_0^t \varphi(\tau, x) d\tau = (p \ast \varphi)(t, x). \quad (3.7)$$

To obtain a more accessible form of the energy equation, we consider the function ω defined on $[0, \infty) \times D$, by the relation:

$$\omega = \hat{S} + \vartheta_0(\eta_0 - \alpha), \quad (3.8)$$

where $\hat{S}$ is defined as in (3.7). Also, for a function u of the class $C^{0,1}$ on $[0, \infty) \times D$ we define the functions β and α by:

$$\beta u = u + \alpha u, \quad \gamma u = p * u + \alpha u. \quad (3.9)$$

In the following Proposition, we formulate the energy equation of in a different manner.

Propoziție 1.

If the functions $q_m \in C^{1,0}$ and $\eta \in C^{0,1}$ satisfy the energy equation and the initial condition $\eta(0, x) = \eta^0(x), x \in D$, then they satisfy the equation:

$$\hat{q}_{m,m} + \omega = \vartheta_0(\eta - \alpha), \quad \forall (t, x) \in [0, \infty) \times D. \quad (3.10)$$

The reciprocal statement is also true.

A reciprocal relationship refers to the connection between two external data systems:

$$\begin{aligned} \mathcal{D}^{(\nu)} = \{ & f_m^{(\nu)}, g_m^{(\nu)}, S^{(\nu)}, \bar{v}_m^{(\nu)}, \bar{\phi}_m^{(\nu)}, \bar{t}_m^{(\nu)}, \bar{\tau}_k^{(\nu)}, \bar{\vartheta}^{(\nu)}, \bar{q}^{(\nu)}, v_m^{0,(\nu)}, v_m^{1,(\nu)}, \\ & \phi_m^{0,(\nu)}, \phi_m^{1,(\nu)}, \eta^{0,(\nu)}, \vartheta^{0,(\nu)} \}, \end{aligned}$$

and the corresponding solutions to these data systems, more precisely:

$$\mathcal{S}^{(\nu)} = \left\{ v_m^{(\nu)}, \phi_m^{(\nu)}, \vartheta^{(\nu)}, \tau_{mn}^{(\nu)}, \sigma_{mn}^{(\nu)}, \eta^{(\nu)}, q_m^{(\nu)} \right\},$$

in both of the above systems we have $\nu = 1, 2$.

To simplify the writing of reciprocity relations, we need the notation:

$$\begin{aligned} \Gamma_{\nu\mu}(s, r) = & \int_{\partial D} \left[t_m^{(\nu)}(s, x) v_m^{(\mu)}(r, x) + \tau_k^{(\nu)}(s, x) \phi_m^{(\mu)}(r, x) - \frac{1}{\vartheta_0} b \bar{q}_m^{(\nu)}(s, x) \vartheta^{(\mu)}(r, x) \right] dA \\ & + \int_D \left[f_m^{(\nu)}(s, x) v_m^{(\mu)}(r, x) + g_m^{(\nu)}(s, x) \phi_m^{(\mu)}(r, x) - \frac{1}{\vartheta_0} b \omega^{(\nu)}(s, x) \vartheta^{(\mu)}(r, x) \right] dV \\ & + \int_D \left[\varrho \ddot{v}_m^{(\nu)}(s, x) v_m^{(\mu)}(r, x) + I_{mn} \ddot{\phi}_m^{(\nu)}(s, x) \phi_n^{(\mu)}(r, x) \right. \\ & \quad \left. - h \dot{\vartheta}^{(\nu)}(s, x) \vartheta^{(\mu)}(r, x) - a d \vartheta^{(\nu)}(s, x) \dot{\vartheta}^{(\mu)}(r, x) \right] dV \\ & + \frac{1}{\vartheta_0} \int_D b \bar{q}_m^{(\nu)}(s, x) \vartheta^{(\mu)}(r, x) dV, \quad \nu, \mu = 1, 2, \end{aligned} \quad (3.11)$$

where I used the following convention:

$$\begin{aligned} t_m^{(\nu)} &= t_{mk}^{(\nu)} n_k, \quad \tau_k^{(\nu)} = \tau_{kl}^{(\nu)} n_l, \\ \omega^{(\nu)} &= \bar{S}^{(\nu)} + \vartheta_0 \left(\eta^{0,(\nu)} - a \right), \quad q^{(\nu)} = q_k^{(\nu)} n_k. \end{aligned} \quad (3.12)$$

The following notations are also introduced:

$$\begin{aligned} J_{\nu\mu}(s, r) &= t_{mn}^{(\nu)}(s)v_{m,n}^{(\mu)}(r) + \tau_{mn}^{(\nu)}(s)\phi_{m,n}^{(\mu)}(r) \\ &\quad - b \left[\eta^{(\nu)}(s) - \alpha \right] \vartheta^{(\mu)}(r), \quad \nu, \mu = 1, 2, \end{aligned} \quad (3.13)$$

and

$$I_{\nu\mu}(s, r) = J_{\nu\mu}(s, r) + h\dot{\vartheta}^{(\nu)}(s)\vartheta^{(\mu)}(r) + ad\vartheta^{(\nu)}(s)\dot{\vartheta}^{(\mu)}(r), \quad \nu, \mu = 1, 2. \quad (3.14)$$

In (3.13) and (3.14) we avoided writing the dependence on the variable x . Now we formulate and prove the first reciprocity result.

Teoremă 9. *If the symmetry relations are satisfied, then for any $s, r \in [0, \infty)$ the following equality holds:*

$$\Gamma_{\nu\mu}(s, r) = \Gamma_{\mu\nu}(s, r), \quad \nu, \mu = 1, 2. \quad (3.15)$$

To obtain another reciprocity result, we will consider the notations:

$$\begin{aligned} F_m^{(\nu)} &= r * \left[f_m^{(\nu)}(s) + g_m^{(\nu)}(s) \right] + \varrho \left[t\dot{v}_m^{1,(\nu)} + v_m^{0,(\nu)} \right], \\ G_m^{(\nu)} &= r * \left[g_m^{(\nu)}(s) + f_m^{(\nu)}(s) \right] + \varrho \left[t\dot{v}_m^{1,(\nu)} + v_m^{0,(\nu)} \right], \\ R^{(\nu)} &= -t\vartheta^{0,(\nu)}, \quad \nu = 1, 2. \end{aligned} \quad (3.16)$$

Using Theorem 9 and the notations (3.16) we obtain a new reciprocity result.

Teoremă 10. *If the symmetry relations are satisfied and $\mathcal{S}^{(\nu)}$ is the solution corresponding to the external data system $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$, then the following equality holds:*

$$\begin{aligned} &\int_{\partial D} r * \left[t_m^{(1)}(s) * v_m^{(2)} + \tau_k^{(1)} * \phi_k^{(2)} - \frac{1}{\vartheta_0} q^{(1)} * c\vartheta^{(2)} \right] dA \\ &+ \int_D \left[F_m^{(1)} * v_m^{(2)} + G_m^{(1)} * \phi_m^{(2)} - \frac{1}{\vartheta_0} p * \vartheta^{(2)} * c\omega^{(1)} \right] dV \\ &\quad - \frac{a}{\vartheta_0} \int_{\partial D} p * q^{(1)} * R^{(2)} dA - \frac{a}{\vartheta_0} \int_D \omega^{(1)} * R^{(2)} dV \\ &\quad + \int_D \left[(h - ad)R^{(1)} * \vartheta^{(2)} + ap * \kappa_{mn}\vartheta_{,n}^{(1)} * R_{,m}^{(2)} \right] dV \\ &= \int_{\partial D} r * \left[t_m^{(2)}(s) * v_m^{(1)} + \tau_k^{(2)} * \phi_k^{(1)} - \frac{1}{\vartheta_0} q^{(2)} * c\vartheta^{(1)} \right] dA \\ &+ \int_D \left[F_m^{(2)} * v_m^{(1)} + G_m^{(2)} * \phi_m^{(1)} - \frac{1}{\vartheta_0} p * \vartheta^{(1)} * c\omega^{(2)} \right] dV \\ &\quad - \frac{a}{\vartheta_0} \int_{\partial D} p * q^{(2)} * R^{(1)} dA - \frac{a}{\vartheta_0} \int_D \omega^{(2)} * R^{(1)} dV \\ &\quad + \int_D \left[(h - ad)R^{(2)} * \vartheta^{(1)} + ap * \kappa_{mn}\vartheta_{,n}^{(2)} * R_{,m}^{(1)} \right] dV. \end{aligned} \quad (3.17)$$

To simplify the following relation, we will use the notation:

$$G(s, r) = - \int_D \left[\frac{1}{\vartheta_0} \omega(s) b \vartheta(r) - f_m(s) v_m(r) - g_m(s) \phi_m(r) \right] dV \\ - \int_{\partial D} \left[\frac{1}{\vartheta_0} \bar{q}(s) b \vartheta(r) - t_m(s) v_m(r) - \tau_k(s) \phi_m(r) \right] dA, \quad \forall s, r \in [0, \infty). \quad (3.18)$$

The reciprocity relation (3.15) underlies the result of the following theorem.

Teoremă 11.

If the symmetry relations are satisfied and $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$, is the solution corresponding to the external data system, then the following equality holds:

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_D [\varrho v_m v_m + I_{mn} \phi_m \phi_n + a \kappa_{mn} \bar{\vartheta}_{,m} \bar{\vartheta}_{,n}] dV \right\} \\ & + \frac{d}{dt} \left\{ \int_0^t \int_D [(ad - h) \vartheta^2 + \kappa_{mn} \bar{\vartheta}_{,n} \bar{\vartheta}_{,m}] dV ds \right\} = \\ & = \int_0^t [G(t - \tau, t + \tau) - G(t + \tau, t - \tau)] d\tau \quad (3.19) \\ & + \int_D [\varrho (\dot{v}_m(2t) v_m(0) + \dot{v}_m(0) v_m(2t)) + I_{mn} [\dot{\phi}_m(2t) \phi_m(0) + \dot{\phi}_m(0) \phi_m(2t)]] dV \\ & + \int_D [(ad - h) \vartheta(0) \vartheta(2t) + a \kappa_{mn} \bar{\vartheta}_{,n}(2t) \bar{\vartheta}_{,m}(0)] dV. \end{aligned}$$

The theorems presented above are essential to demonstrate that the solutions of the equations of a physical system respect certain principles of symmetry and conservation, which facilitates both the theoretical analysis and the practical application of these solutions. Now we can address the problem of the uniqueness of the solution of the problem $\mathcal{P}$.

Teoremă 12.. *We assume that:*

- the symmetry relations are satisfied;
 - ϱ and $ad - h$ are strictly positive;
 - the tensor κ_{mn} is positive semidefinite; - $a \geq 0$.
- Then the mixed problem $\mathcal{P}$ admits at most one solution.*

For the second result we introduce the functional $\mathcal{F}$, defined on H , by:

$$\begin{aligned}
\mathcal{F}(t, \mathcal{A}) = & \int_D p * [A_{klmn} e_{kl} * e_{mn} + B_{mnkl} \sigma_{kl} * e_{mn} + C_{klmn} \sigma_{kl} * e_{mn} \\
& + \varrho v_m * v_m + I_{mn} \phi_m * \phi_n - \frac{1}{\vartheta_0} r * \kappa_{mn} \vartheta_{,m} * \vartheta_{,n} - r * q_m * \vartheta_{,m} \\
& - [\varrho v_m - r * t_{mn,n} - F_m] * v_m + (I_{mn} \phi_m - p * \tau_{mn,n} - \epsilon_{mjk} t_{jk} - G_m) * \phi_m \\
& - \frac{\vartheta_0}{\alpha} r * (S - \alpha_{mn} e_{mn} - \beta_{mn} \varepsilon_{mn}) * (S - \alpha_{kl} e_{kl} - \beta_{kl} \varepsilon_{kl}) \\
& - r * (t_{mn} * e_{mn} + \tau_{mn} * \varepsilon_{mn}) - (p * S + r * q_{m,m} - R) * \vartheta] dV \\
& + \int_{\Sigma_1} r * t_m * \tilde{v}_m dA + \int_{\Sigma_1^c} r * (t_m - \tilde{t}_m) * v_m dA \\
& + \int_{\Sigma_2} r * \tau_k * \tilde{\phi}_k dA + \int_{\Sigma_2^c} r * (\tau_k - \tilde{\tau}_k) * \phi_k dA \\
& + \int_{\Sigma_3} r * q * \tilde{\vartheta} dA + \int_{\Sigma_3^c} r * (q - \tilde{q}) * \vartheta dA, \quad t \in [0, \infty),
\end{aligned} \tag{3.20}$$

for anything $\mathcal{A} = (v_m, \phi_m, \vartheta, e_{mn}, \sigma_{mn}, t_{mn}, \tau_{mn}, q_m, S) \in H$.

Theoremă 13.. *Suppose that the symmetry relations are satisfied, $\alpha \neq 0$ in the domain D and the thermoelastic state $\mathcal{A}$ is a solution of the mixed problem $\mathcal{P}$. Then the variation of the functional $\mathcal{A}$ is zero, more precisely*

$$\delta \mathcal{F}(t, \mathcal{A}) = 0, \quad t \in [0, \infty). \tag{3.21}$$

Remark.

It is not difficult to show that the statement in Theorem 13 is also valid reciprocally (see Gurtin [34]). In other words, if the identity (3.21) is true, then the state $\mathcal{A}$ for which this identity is true is the unique solution of our problem. The idea of the proof, which is also suggested by Gurtin in [34], is based on a particular choice of the thermoelastic state $\tilde{\mathcal{A}}$. In our case, the thermoelastic state proposed by Lebon in [21] can be used successfully.

Capitol 4

Dipolar Media

In this chapter, the notations presented in the table below will be used.

Notations	Physical Interpretation	
a, b, c, d, h	coefficients constants specific heat	
η	entropy specific per unit mass	
I_{ij}	components of the microinertia tensor	
t_{ij}	components of the tension tensor	
τ_{ij}	components of the tension torque tensor	
σ_{ijk}	components of the stress moment	
q_i	components of the thermal conduction vector	
v_i	components of the displacement vector	
ϕ_i	components of the microrotation	
τ_k	surface tractions	
g_{mn}	dipole force of the body f_m	mass force
$n = (nl)$	normal vector to the boundary ∂D	
T	conductive temperature	
φ	volume fraction change	
ϑ	thermodynamic temperature	
ϑ_0	constant reference temperature	
κ	microinertia	
l	extrinsic force of the body	
ρ	reference mass density	
ϵ_{ijk}	Ricci symbol	

Table 4.1: Notations

4.1 Uniqueness and instability

4.1.1 Mixed problem with initial and boundary data

To determine the uniqueness and instability of dipolar bodies with two temperatures, the same steps are used as in the case of micropolar media, the difference being represented by the complexity of the calculations given by the transition from vectors to tensors. Therefore, it is considered that a thermoelastic body with a dipolar structure occupies the three-dimensional domain Ω of the Euclidean space R^3 . The closure of Ω is denoted by $\bar{\Omega}$ and we have $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$

is the boundary of the domain Ω and is considered regular enough to allow the application of the divergence theorem. We relate the motion of the thermoelastic body to a fixed system of orthogonal axes Ox_i , $i = 1, 2, 3$. In order to characterize the evolution of our body, we consider the set of variables $(v_i, \phi_{ij}, T, \vartheta)$. Considering the geometric equations and the constitutive equations, which are introduced into the basic equations, we are led to the following system of partial differential equations:

$$\begin{aligned}
& (A_{ijmn} + E_{ijmn}) v_{n,mj} + (E_{mnij} + B_{ijmn}) (v_{n,mj} - \phi_{mn,j}) \\
& + (F_{ijklm} + D_{ijklm}) \phi_{lm,kj} - (\alpha_{ij} + \beta_{ij}) (\vartheta_{,j} + a\dot{\vartheta}_{,j}) = \rho \ddot{v}_i, \\
& F_{jklmn} v_{n,mj} + D_{mnjkl} (v_{n,mj} - \phi_{mn,j}) + C_{kljmnr} \phi_{nr,mj} - \delta_{klj} (\vartheta_{,j} + a\dot{\vartheta}_{,j}) \\
& + E_{klmn} v_{m,n} + B_{klmn} (v_{n,m} - \phi_{mn}) + D_{klmnr} \phi_{nr,m} - \beta_{kl} (\vartheta + a\dot{\vartheta}) = I_{kr} \ddot{\phi}_{lr}, \\
& \kappa_{ij} T_{,ij} - \alpha_{ij} \dot{v}_{i,j} - \beta_{ij} (\dot{v}_{i,j} - \dot{\phi}_{ij}) - \delta_{ijk} \dot{\phi}_{ij,k} = h\ddot{\vartheta} + d\dot{\vartheta},
\end{aligned} \tag{4.1}$$

which are satisfied for any $(x) \in \Omega \times (0, \infty)$.

By a solution we mean an ordered set $(v_i, \phi_{ij}, T, \vartheta)$ which satisfies the system of equations (4.1), the boundary conditions, and the initial conditions.

4.1.2 Main Results

We introduce the law of conservation of energy

$$W_1(t) = W_1(0), \quad t \in [0, \infty), \tag{4.2}$$

corresponding to the case where we consider the equation with two temperatures and where

$$\begin{aligned}
W_1(t) = & \frac{1}{2} \int_{\Omega} \left[\rho \dot{v}_i(t) \dot{v}_i(t) + I_{jk} \dot{\phi}_{ij}(t) \dot{\phi}_{ik}(t) + A_{ijkl} e_{ij}(t) e_{kl}(t) \right. \\
& + 2E_{ijkl} e_{ij}(t) \varepsilon_{kl}(t) + 2F_{ijklm} e_{ij}(t) \gamma_{klm}(t) + B_{ijkl} \varepsilon_{ij}(t) \varepsilon_{kl}(t) \\
& + 2G_{ijklm} \varepsilon_{ij}(t) \gamma_{klm}(t) + C_{ijklmn} \gamma_{ijk}(t) \gamma_{lmn}(t) \\
& \left. + c \kappa_{ij} T_{,i}(t) T_{,j}(t) + d \left(\vartheta(t) + \frac{h}{d} \dot{\vartheta}(t) \right)^2 + h \left(a - \frac{h}{d} \right) \dot{\vartheta}^2(t) \right] dV \\
& + \int_0^t \int_{\Omega} \left[\kappa_{ij} T_{,i}(s) T_{,j}(s) + c \left((\kappa_{ij} T_{,i}(s))_{,j} \right)^2 + (ad - h) \dot{\vartheta}^2(s) \right] dV ds.
\end{aligned}$$

If we consider the relation (??), that is, we do not take into account the temperature variation of the conductivity, the law of conservation of energy takes the form

$$W_2(t) = W_2(0), \quad t \in [0, \infty), \tag{4.3}$$

where

$$\begin{aligned}
W_2(t) = & \frac{1}{2} \int_{\Omega} \left[\rho \dot{v}_i(t) \dot{v}_i(t) + I_{jk} \dot{\phi}_{ij}(t) \dot{\phi}_{ik}(t) + A_{ijkl} e_{ij}(t) e_{kl}(t) \right. \\
& + 2E_{ijkl} e_{ij}(t) \varepsilon_{kl}(t) + 2F_{ijklm} e_{ij}(t) \gamma_{klm}(t) + B_{ijkl} \varepsilon_{ij}(t) \varepsilon_{kl}(t) \\
& + 2G_{ijklm} \varepsilon_{ij}(t) \gamma_{klm}(t) + C_{ijklmn} \gamma_{ijk}(t) \gamma_{lmn}(t) \\
& \left. + d \left(\vartheta(t) + \frac{h}{d} \dot{\vartheta}(t) \right)^2 + h \left(a - \frac{h}{d} \right) \dot{\vartheta}^2(t) \right] dV \\
& + \int_0^t \int_{\Omega} \left[\kappa_{ij} T_{,i}(s) T_{,j}(s) + c \left((\kappa_{ij} T_{,i}(s))_{,j} \right)^2 + (ad - h) \dot{\vartheta}^2(s) \right] dV ds.
\end{aligned}$$

The system of equations (4.1), the boundary conditions and the initial conditions together form the mixed problem that we will denote by $\mathcal{P}$.

Next, we will obtain two main results, a uniqueness and an instability, for solving the mixed problem $\mathcal{P}$. For uniqueness we use the usual procedure: we will show that the problem $\mathcal{P}$ admits only the null solution, if it is considered that the initial data are null.

Teoremă 14. *The mixed problem with initial and boundary conditions $\mathcal{P}$, in the case of null initial data, admits only the null solution.*

In the second main result, we want to prove that the solution to the mixed problem $\mathcal{P}$ is exponentially unstable, if certain conditions are met.

Specifically, we will assume that the initial energy of the system is not strictly positive.

First, we introduce a useful auxiliary result.

Let us denote by $\nu(x)$ the function that satisfies the following bounding problem.

$$\begin{aligned}
(\kappa_{ij} \nu_{,i}(x))_{,j} &= d\vartheta^1 + h\vartheta^0 - (\alpha_{ij} e_{ij}^0 + \beta_{ij} \epsilon_{ij}^0 + \delta_{ijk} \gamma_{ijk}^0), \quad x \in \Omega, \\
\nu(x) &= 0, \quad x \in \partial\Omega.
\end{aligned} \tag{4.4}$$

The fact that the boundary value problem (4.4) has a solution can be deduced from the usual properties of boundary value problems attached to elliptic equations.

From (4.4) we deduce that the function ν satisfies the equation:

$$\begin{aligned}
d\vartheta(x) + h\dot{\vartheta}(x) - [\kappa_{ij} (\nu_{,i}(x) + \xi_{,i}(x))]_{,j} &= \\
&= \alpha_{ij} e_{ij}(x) + \beta_{ij} \epsilon_{ij}(x) + \delta_{ijk} \gamma_{ijk}(x),
\end{aligned}$$

where the function $\xi(x)$ is defined above.

Now, consider the above problem $\mathcal{P}$ in the case where the initial data are inhomogeneous and the boundary data are homogeneous.

Teoremă 15. *Let us assume positive coefficients and tensors.*

If the mixed initial and boundary value problem $\mathcal{P}$, in the case of null boundary data, admits a solution for which $W_2(0) \geq 0$, then this solution is exponentially unstable.

4.2 The effect of voids and internal state variables on the elasticity of media with dipolar structure

4.2.1 Basic equations and conditions

We will consider Ω an open domain of the Euclidean space R^3 which is occupied, at the initial time $t = 0$, by an elastic medium with internal state variables and dipolar structure. The surface $\partial\Omega$ is the boundary of the domain D and is a closed and bounded set that allows the application of the divergence theorem. A point in Ω is represented as (x_i) or (x) . For the time variable t we assume that $t \in [0, t_0]$.

To characterize the evolution of an elastic dipolar medium with voids, the following kinematic variables are used:

$$v_m = v_m(x, t), \phi_{jk} = \phi_{jk}(x, t), \varsigma = \varsigma(x, t), (x, t) \in \Omega \times [0, t_0],$$

where ς represents the change in volume fraction.

In the following considerations we will use φ as the volume distribution function, where $\varphi = \varsigma - \varsigma_0$, where ς_0 is the value of ς in the initial state.

Using the procedure of Green and Rivlin, we can consider another deformation that differs from the given deformation only by a superposition of a rigid motion consisting of a rotation with constant angular velocity. We must assume that for this motion, all other properties of the media remain unaffected by this superposition. Consequently, we obtain the following kinematic relations, which provide expressions for the deformation measures, namely e_{ij} , ε_{ij} and γ_{ijk} , with respect to the motion variables:

$$\begin{aligned} e_{ij} &= \frac{1}{2}(u_{j,i} + u_{i,j}), \quad \varepsilon_{ij} = u_{j,i} - \phi_{ij}, \\ \gamma_{ijk} &= \phi_{jk,i}. \end{aligned} \tag{4.5}$$

We will denote the internal state variables by ξ_ν , $\nu = 1, 2, \dots, n$ and we will use the notation ξ_ν^0 for the values of the variables in the initial state of the body. Being in the context of a linear theory it is natural to use as internal variables the difference below, ω_ν , that is:

$$\omega_\nu = \xi_\nu - \xi_\nu^0. \tag{4.6}$$

We will consider only the particular case where solids have a point that is the center of symmetry. We also consider that for the body, which in its initial state is unstressed, the internal energy density is a quadratic form with respect to its independent constitutive variables. So, using the principle of conservation of energy, we obtain the following expression for the internal energy density:

$$\begin{aligned} \Psi &= \frac{1}{2}A_{ijmn}e_{ij}e_{mn} + G_{ijmn}e_{ij}\varepsilon_{mn} + F_{ijmnr}e_{ij}\gamma_{mnr} + \frac{1}{2}B_{ijmn}\varepsilon_{ij}\varepsilon_{mn} \\ &+ D_{ijmnr}\varepsilon_{ij}\gamma_{mnr} + \frac{1}{2}C_{ijkmnr}\gamma_{ijk}\gamma_{mnr} + a_{ijk}e_{ij}\varphi_{,k} + b_{ijk}\varepsilon_{ij}\varphi_{,k} + c_{ijkm}\gamma_{ijk}\varphi_{,k} \\ &+ \frac{1}{2}p_{mn}\varphi_{,m}\varphi_{,n} + \alpha_{ij\nu}e_{ij}\omega_\nu + \beta_{ij\nu}\varepsilon_{ij}\omega_\nu + \delta_{ijk\nu}\gamma_{ijk}\omega_\nu + f_{i\nu}\varphi_{,i}\omega_\nu. \end{aligned} \tag{4.7}$$

The above coefficients A_{ijmn} , B_{ijmn} , ..., a_{ijk} , ..., $f_{i\nu}$ characterize the elastic properties of a body with pores and an internal state variable and are called constitutive coefficients. In general, these coefficients depend on the point x , and in the particular case when the body is homogeneous, they have a constant value. With the help of this internal energy density, the following constitutive relations are obtained, which provide expressions for the stress measures, t_{ij} , τ_{ij} , m_{ijk} , h_i , as functions depending on the deformation tensors:

$$\begin{aligned} t_{ij} &= \frac{\partial \Psi}{\partial e_{ij}} = A_{ijmn} e_{mn} + G_{mni j} \varepsilon_{mn} + F_{mnri j} \gamma_{mnr} + a_{ijk} \varphi_{,k} + \alpha_{ij\nu} \omega_\nu, \\ \tau_{ij} &= \frac{\partial \Psi}{\partial \varepsilon_{ij}} = G_{ijmn} e_{mn} + B_{ijmn} \varepsilon_{mn} + D_{ijmnr} \gamma_{mnr} + b_{ijk} \varphi_{,k} + \beta_{ij\nu} \omega_\nu, \\ \sigma_{ijk} &= \frac{\partial \Psi}{\partial \gamma_{ijk}} = F_{ijkmn} e_{mn} + D_{mni jk} \varepsilon_{mn} + C_{mnri jk} \gamma_{mnr} + c_{ijkm} \varphi_{,m} + \delta_{ijk\nu} \omega_\nu, \\ h_i &= \frac{\partial \Psi}{\partial \varphi_{,i}} = a_{ijk} e_{jk} + b_{ijk} \varepsilon_{jk} + c_{ijsm} \gamma_{jsm} + f_{i\nu} \omega_\nu. \end{aligned} \quad (4.8)$$

The procedure used by Green and Rivlin in the case of classical elasticity can also be used to obtain the equilibrium laws in the context of the elasticity of porous media with dipolar structure, namely:

- the equations of motion

$$\begin{aligned} (t_{mn} + \tau_{mn})_{,n} + \rho f_m &= \rho \ddot{v}_m, \\ \sigma_{ijk,i} + \tau_{jk} + \rho g_{jk} &= I_{kr} \ddot{\phi}_{jr}; \end{aligned} \quad (4.9)$$

- the equation of equilibrium forces:

$$h_{i,i} + \rho l = \rho k \ddot{\varphi}. \quad (4.10)$$

In the context of a linear approximation we can use a suggestion from [72] so that the entropy production inequality leads to the following equation:

$$\dot{\omega}_\nu = f_\nu, \quad (4.11)$$

where

$$f_\nu = g_{ij\nu} e_{ij} + h_{ij\nu} \varepsilon_{ij} + l_{ijk\nu} \gamma_{ijk} + q_{\nu\beta} \omega_\beta. \quad (4.12)$$

The notations used are found in Table 2.1.

Since the tensor e_{ij} is symmetric, we can deduce the following symmetry relations:

$$\begin{aligned} A_{ijmn} &= A_{mni j} = A_{ijnm}, \quad B_{ijmn} = B_{mni j}, \quad G_{ijmn} = G_{ijnm}, \\ C_{ijk mnr} &= C_{mnri jk}, \quad p_{mn} = p_{nm}. \end{aligned}$$

To construct the mixed problem in the present context, we will add, in addition to the basic equations above, the following initial conditions:

$$\begin{aligned} v_m(x, 0) &= v_{0m}(x), \quad \dot{v}_m(x, 0) = v_{1m}(x), \\ \phi_{jk}(x, 0) &= \phi_{0jk}(x), \quad \dot{\phi}_{jk}(x, 0) = \phi_{1jk}(x), \\ \varphi(x, 0) &= \varphi_0(x), \quad \omega_\nu(x, 0) = \omega_{0\nu}(x), \quad x \in \Omega, \end{aligned} \quad (4.13)$$

and the boundary conditions given by:

$$\begin{aligned} v_m &= \tilde{v}_m, \text{ on } \bar{\Sigma}_1 \times [0, t_0], \quad t_j \equiv (t_{jk} + \tau_{jk}) n_k = \tilde{t}_j, \text{ pe } \Sigma_1^c \times [0, t_0], \\ \phi_{ij} &= \tilde{\phi}_{ij}, \text{ on } \bar{\Sigma}_2 \times [0, t_0], \quad \sigma_{jk} \equiv \sigma_{ijk} n_i = \tilde{\sigma}_{jk}, \text{ on } \Sigma_2^c \times [0, t_0], \\ \varphi &= \tilde{\varphi}, \text{ on } \bar{\Sigma}_3 \times [0, t_0], \quad h \equiv h_i n_i = \tilde{h}, \text{ on } \Sigma_3^c \times [0, t_0]. \end{aligned} \quad (4.14)$$

In (4.14) the surfaces $\bar{\Sigma}_1$, $\bar{\Sigma}_2$ and $\bar{\Sigma}_3$ together with their complements Σ_1^c , Σ_2^c and Σ_3^c are subsets of the boundary $\partial\Omega$ and satisfy the following two conditions:

$$\begin{aligned} \bar{\Sigma}_1 \cup \Sigma_1^c &= \bar{\Sigma}_2 \cup \Sigma_2^c = \bar{\Sigma}_3 \cup \Sigma_3^c = \partial\Omega, \\ \Sigma_1 \cap \Sigma_1^c &= \Sigma_2 \cap \Sigma_2^c = \Sigma_3 \cap \Sigma_3^c = \emptyset. \end{aligned}$$

The functions v_{0m} , v_{1m} , ϕ_{0jk} , ϕ_{1jk} , φ_0 , $\omega_{0\nu}$, $\tilde{v}_m$, $\tilde{t}_m$, $\tilde{\varphi}_{jk}$, $\tilde{\sigma}_{jk}$, $\tilde{h}$ and $\tilde{\varphi}$, from the above conditions (4.13) and (4.14), are prescribed and satisfy sufficient regularity conditions in their domain of definition.

We use the notation $\mathcal{P}$ for the mixed problem in the context of the theory of elasticity of media with pores and internal state variables and a dipolar structure. This includes equations (4.9)-(4.11), initial conditions (4.13) and boundary conditions (4.14).

A deformation state $(v_m, \phi_{jk}, \varphi, \omega_\nu)$ is called a solution to the mixed problem $\mathcal{P}$ if equations (4.9)-(4.11) and conditions (4.13) and (4.14) are verified for this deformation.

4.2.2 Basic results

In this section we will address the issue of the influence that holes and internal state variables can have on the behavior of media with dipolar structure. To obtain these results, we first introduce some useful estimates, contained in Theorems 16, 17 and 18.

The first auxiliary estimate is proved in Theorem 16.

Teoremă 16. *If $(v_m, \phi_{jk}, \varphi, \omega_\nu)$ is an arbitrary solution to the problem $\mathcal{P}_0$, then the following equality holds:*

$$\begin{aligned} \int_{\Omega} & (A_{ijmn} e_{ij} e_{mn} + 2G_{ijmn} e_{ij} \varepsilon_{mn} + 2F_{mnrij} e_{ij} \gamma_{mnr} \\ & + B_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + C_{ijsmnr} \gamma_{ijs} \gamma_{mnr} + 2D_{ijmnr} \varepsilon_{ij} \gamma_{mnr} + 2\alpha_{ij\nu} e_{ij} \omega_\nu \\ & + 2\beta_{ij\nu} \varepsilon_{ij} \omega_\nu + 2\delta_{ijr\nu} \gamma_{ijr} \omega_\nu + \rho \dot{v}_m \dot{v}_m + I_{kr} \dot{\phi}_{jr} \dot{\phi}_{jk}) dV \\ & = 2 \int_0^t \int_{\Omega} (\alpha_{ij\nu} e_{ij} + \beta_{ij\nu} \varepsilon_{ij} + \delta_{ijr\nu} \gamma_{ijr}) \dot{\omega}_\nu dV ds. \end{aligned} \quad (4.15)$$

In the next theorem we will prove another auxiliary estimate.

Teoremă 17. *Let $(v_m, \phi_{jk}, \varphi, \omega_\nu)$ be a solution to the problem P_0 . Then we can find the positive constant m_1 such that the following inequality is satisfied:*

$$\begin{aligned} & \int_{\Omega} (\alpha_{ij\nu} e_{ij} + \beta_{ij\nu} \varepsilon_{ij} + \delta_{ijs\nu} \gamma_{ijs}) \dot{\omega}_\nu dV \leq \\ & m_1 \int_{\Omega} (e_{ij} e_{ij} + \varepsilon_{ij} \varepsilon_{ij} + \gamma_{ijs} \gamma_{ijs} + \varphi_{,m} \varphi_{,m} + \omega_\nu \omega_\nu) dV. \end{aligned} \quad (4.16)$$

The last auxiliary estimate will be proven in the next theorem.

Teoremă 18. *Let us assume that the above assumptions are met and consider a solution $(v_m, \phi_{jk}, \varphi, \omega_\nu)$ for the problem $\mathcal{P}_0$. Then a constant $m_2 > 0$ can be determined such that the following inequality holds:*

$$\begin{aligned} & \int_{\Omega} \left(\dot{v}_m \dot{v}_m + \dot{\phi}_{kj} \dot{\phi}_{jk} + e_{ij} e_{ij} + \varepsilon_{ij} \varepsilon_{ij} + \gamma_{ijk} \gamma_{ijk} + \varphi^2 + \omega_\nu \omega_\nu \right) dV \leq \\ & m_2 \int_0^t \int_{\Omega} \left(\dot{v}_m \dot{v}_m + \dot{\phi}_{jk} \dot{\phi}_{jk} + e_{ij} e_{ij} + \varepsilon_{ij} \varepsilon_{ij} + \gamma_{ijk} \gamma_{ijk} + \varphi^2 + \omega_\nu \omega_\nu \right) dV ds, \end{aligned} \quad (4.17)$$

which takes place for any $t \in [0, t_0]$.

Our main result will be obtained by considering the estimates given by theorem 16, theorem 17 and theorem 18. So, we will prove the uniqueness of the mixed problem with initial and boundary data $\mathcal{P}$.

Teoremă 19. *We assume that the constitutive tensors and the mass density are positive definite. Then the above problem $\mathcal{P}$ defined by equations (4.9)-(4.10) with the initial data (4.13) and the boundary data (4.14) cannot admit more than one solution.*

Capitol 5

Final conclusions. Dissemination of research results. Future research directions

5.0.1 Final conclusions

In the present work, the behavior of isotropic and anisotropic micropolar media, as well as dipolar media, was analyzed in depth.

In the first part, the variational principles, continuous dependence and fundamental equations of isotropic micropolar media were studied, thus forming a solid basis for the subsequent analysis. We continued with the study of plane deformation and wave propagation with two delay times, results that provide a detailed perspective on the influence of heat sources and pores on the mechanical behavior.

In the second part, solutions based on complex potentials were obtained, applicable to the analysis of stresses and deformations around circular hole, which helps in the modeling and numerical simulation of these phenomena.

Regarding the study of anisotropic micropolar media, the uniqueness and instability in thermoelasticity were discussed, through variational formulations of problems with boundary and initial conditions. The reciprocity and variational principles were also addressed, highlighting their role in determining solutions.

Finally, in the analysis of dipolar media, the effects of gaps and internal state variables on their elasticity were obtained, results that contribute to the understanding of how the dipolar structure influences the mechanical behavior.

Therefore, the present work offers a broad perspective on the theories and applications of micropolar and dipolar media, using analytical and numerical methods to investigate complex phenomena. The results obtained have both theoretical and practical relevance, and can be applied in various fields of materials mechanics and structural engineering.

5.0.2 Dissemination of original results

Articles included in the doctoral thesis

- Marin, M., Vlase, S., **Fudulu, I.M.**, Precup, G.: Effect of Voids and Internal State Variables in Elasticity of Porous Bodies with Dipolar Structure, *Mathematics*, **9**(21), Art. No. 2741 (2021).
<https://doi.org/10.3390/math9212741>
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- Marin, M., **Fudulu, I.M.**, Vlase, S. :On some qualitative results in thermodynamics of Cosserat bodies, *Boundary Value Problem* 2022, **69** (2022).
<https://doi.org/10.1186/s13661-022-01652-8>
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- Marin, M., Vlase, S., **Fudulu, I. M.**: On instability in the theory of dipolar bodies with two-temperatures, *Carpathian Journal of Mathematics*, **38**, 459-468 (2022).
<https://doi.org/10.37193/CJM.2022.02.15>
 -wos:000761969400001
- **Fudulu, I.M.**:Plane strain of isotropic micropolar bodies with pores, *Bulletin of the Transilvania University of Braşov, Series III: Mathematics and Computer Science*, Vol. **3**(65), No. 1, 81-92 (2023).
<https://doi.org/10.31926/but.mif.2023.3.65.1.7>
- Neagu, D.M., **Fudulu, I.M.** , Marin, M. :Complex potentials solutions for isotropic Cosserat bodies with voids, *Boundary Value Problem* 2024, **129** (2024).
<https://doi.org/10.1186/s13661-024-01938-z>
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- Neagu, D.M., **Fudulu, I.M.**, Marin, M. et al. :Wave propagation with two delay times in an isotropic porous micropolar thermoelastic material, *Continuum Mech. Thermodyn.* **36**, 639–655 (2024).
<https://doi.org/10.1007/s00161-024-01287-3>
 -wos:001330082400001
- **Fudulu, I.M.**, Marin, M.: On a variational principle and continuous dependence result for a micropolar isotropic body, *Changes and Innovations in Social Systems, Studies in Systems, Decision and Control*, Springer, **505** (2025).
<https://doi.org/10.1007/978-3-031-43506-5-6>

Alte articole elaborate în timpul studiilor

- Bhatti, M.M., Marin, M., Ellahi, R., **Fudulu, I.M.** :Insight into the dynamics of EMHD hybrid nanofluid (ZnO/CuO-SA) flow through a pipe for geothermal energy applications, *Journal of Thermal Analysis and Calorimetry* **48**, 14261–14273 (2023).
<https://doi.org/10.1007/s10973-023-12565-8>

-wos: 001082625700001

Presentation of research results

• 4th Edition - MACOS 2022, *International Conference on Mathematics and Computer Science*, September 15-17, 2022, Braşov, Romania.

- **Fudulu, I.M.**, Marin, M.: *On a variational principle and continuous dependence result for a micropolar isotropic body*, *Changes and Innovations in Social Systems. Studies in Systems, Decision and Control*, Springer, **505** (2025). <https://doi.org/10.1007/978-3-031-43506-5-6>

• 5th *International Conference on Mathematics and its Applications in Science and Engineering*, September 16-18, 2024, Coimbra, Portugalia.

- Neagu, D.M., **Fudulu, I.M.** , Marin, M. :*Complex potentials solutions for isotropic Cosserat bodies with voids*, *Bound Value Probl* 2024, **129** (2024). <https://doi.org/10.1186/s13661-024-01938-z>.

Presentation of research results

5.0.3 Future research directions

The present work provides a detailed analysis of the behavior of isotropic and anisotropic micropolar media, as well as of dipolar media, highlighting both the theoretical foundations and advanced solution methods. The complexity of these fields allows for a wide range of future research directions, which would deepen and extend the obtained results.

A first possible direction could be the extension of the models to inhomogeneous and nonlinear media. In many practical applications, real materials exhibit variations of mechanical properties in space or nonlinear behaviors following extreme stresses. Including these aspects in theoretical models could allow a more detailed description of the behavior of complex structures and would contribute to the development of more precise analysis methods.

A promising direction is also the application of theoretical results to advanced materials. Memory materials, metamaterials and nanomaterials are just a few examples of structures that exhibit unusual mechanical properties, such as self-healing or negative stiffness. Investigating these materials from the perspective of micropolar and dipolar theories could lead to new design models in fields such as biomedicine, aeronautical engineering and robotics.

In addition, the development of numerical methods for three-dimensional cases represents an important technical challenge. The use of advanced numerical methods, such as the finite element method (FEM) or methods based on artificial intelligence, could allow a more detailed simulation of the phenomena and facilitate the optimization of the design of engineering structures.

Another essential aspect for model validation is experimental investigation. Although theoretical and numerical models are extremely useful, comparing them with experimental data can confirm or adjust the hypotheses used. Experiments

on real materials could contribute to improving predictions in the field of micropolar and dipolar elasticity.

Another important research direction is the study of wave propagation in complex structures. This topic is relevant for various applications, from acoustics and seismology to nondestructive analysis of materials. A deeper understanding of the propagation mechanisms could lead to improved defect detection techniques and to the development of structures capable of attenuating or controlling wave propagation.

In conclusion, the study of micropolar and dipolar media is a dynamic field, with numerous practical and theoretical applications. Future research directions can significantly contribute to the improvement of existing models, the development of new technologies, and a deeper understanding of the behavior of materials under complex conditions. Thus, this field remains of major interest both for researchers and engineers who aim to solve increasingly sophisticated problems in the mechanics of materials.

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