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TRANSILVANIA UNIVERSITY OF BRAŞOV
INTERDISCIPLINARY DOCTORAL SCHOOL

# Stability and dichotomy for stochastic skew-evolution semiflows

DOCTORAL THESIS
-Extended Abstract-

*PhD Supervisor:*
Prof. Univ. Dr. Ioan Lucian Popa

*PhD Student:*
Tímea Melinda Személy Fülöp (Moraru)

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# Scientific activity carried out during the doctoral studies

### I. Papers published/accepted for publication in journals/proceedings indexed in Web of Science

1. **T. M. Személy Fülöp**, *Integral characterizations of uniform h-dichotomy in mean for discrete-time stochastic skew-evolution semiflows*, Miskolc Math. Notes, (2025).
2. **T. M. Személy Fülöp**, *Datko type characterizations for uniform dichotomy in mean with growth rates for reversible stochastic skew-evolution semiflows in Banach spaces*, Glas.Mat.Ser.III, (2025), 60(1), 167–182, DOI:10.3336/gm.60.1.10.
3. **T. M. Személy Fülöp**, D.I. Borlea, *Integral characterizations of uniform h-dichotomy in mean for stochastic skew-evolution semiflows.* 2024 IEEE 18th International Symposium on Applied Computational Intelligence and Informatics (SACI), (2024), 000025–000030, DOI: 10.1109/SACI60582.2024.10619723.
4. **T. M. Személy Fülöp**, *Uniform h-dichotomy in mean for stochastic skew-evolution semiflows in Banach spaces*, Proc. 21th Int. Conf. Numer. Anal. Appl. Math., (ICNAAM), (2023), 3315(1), 280008, DOI: 10.1063/5.0286098.
5. **T. M. Személy Fülöp**, *Uniform stability in mean for stochastic skew-evoultion semiflows in Banach spaces*, Proc. 21th Int. Conf. Numer. Anal. Appl. Math. (ICNAAM), (2023), 3315(1), 400043, DOI: 10.1063/5.0291168
6. **T. M. Személy Fülöp**, M. Megan, D. I. Borlea (Pătraşcu), *On uniform stability with growth rates of stochastic skew-evolution semiflows in Banach Spaces*, Axioms, (2021), 10(3), 182, DOI: 10.3390/axioms10030182.

### II. Papers published/accepted for publication in specialty journals indexed in international databases (BDI)

1. **T. M. Személy Fülöp**, *On uniform dichotomy in mean of stochastic skew-evolution semiflows in Banach spaces*, An. Univ. Vest Timi, Ser. Mat.-Inform. 59 (1) (2023), 92-104, DOI: 10.2478/awutm-2023-0008

### III. Participation in international conferences

1. **T. M. Személy Fülöp**, *Integral characterizations for uniform dichotomy in mean with growth rates for reversible stochastic skew-evolution semiflows in Banach spaces* , XGEN International Conference on Science Communications, May 20-24, 2024, Technical University of Cluj-Napoca, Baia Mare, Romania, https://events.universitas.ro/event/4/

2. **T. M. Személy Fülöp**, D.I. Borlea, *Integral characterizations of uniform h-dichotomy in mean for stochastic skew-evolution semiflows.* 2024 IEEE 18th International Symposium on Applied Computational Intelligence and Informatics (SACI), May 21-25, 2024, Timişoara, România, https://conf.uni-obuda.hu/saci2024/

3. **T. M. Személy Fülöp**, *Uniform h-dichotomy in mean for stochastic skew-evolution semiflows in Banach spaces*, 21th International conference of numerical analysis and applied mathematics, September 11-17, 2023, Heraklion, Crete, Greece, https://history.icnaam.org

4. **T. M. Személy Fülöp**, *Uniform Stability In Mean For Stochastic Skew-Evoultion Semiflows in Banach Spaces*, 20th International conference of numerical analysis and applied mathematics, September 19-25, 2022, Heraklion, Crete, Greece, https://history.icnaam.org

5. A. Găină, M. Megan, **T. M. Személy Fülöp**, *Uniform stability concepts for stochastic skew-evolution cocycles in Banach spaces* , Le XVe Colloque Franco-Roumain de Mathmatiques Appliques, 29 août - 2 septembre 2022, Toulouse, France,
https://15colfrro.sciencesconf.org/resource/listeparticipants?lang=fr

6. A. Găină, M. Megan, **T. M. Személy Fülöp** , *Uniform Stability Concepts for Discrete-Time Skew-Evolution Cocycles*, ISIM & ISWIM, June 26-29, 2022, Bucharest, Romania,
https://2022.isimconference.eu/images/2022

7. A. Găină, C.F. Popa, **T. M. Személy Fülöp**, *On stability of evolution cocycles in Banach spaces* , The 28th International Conference in Operator Theory, June 27 - July 1, 2022, Timişoara, Romania,
https://sites.google.com/site/ot28conference/all-abstracts

## IV. Participation in national conferences

1. **T. M. Személy Fülöp**, *Uniform h-Dichotomy in mean for stochastic skew-evolution semiflows in Banach spaces*, Sixth Romanian Itinerant Seminar on Mathematical Analysis and its Applications, May 30-31, 2024, Cluj-Napoca, Romania,
www.cs.ubbcluj.ro/wp-content/uploads/ProgramRISMAA2024-v2.pdf

2. A. Găină, C.F. Popa, **T. M. Személy Fülöp**, *On uniform dichotomy of linear discrete time skew- evolution cocycles in Banach spaces*, Fourth Romanian Itinerant Seminar on Mathematical Analysis and its Applications, May 19-21, 2022, Brasov, Romania,
mateinfo.unitbv.ro/ro/admitere/admitere-masterat/520-rismaa-participants

3. **T. M. Személy Fülöp** , A. Găină, C.F. Popa, *On uniform stability of linear discrete-time systems in Banach spaces* , Third Romanian Itinerant Seminar on Mathematical Analysis and its Applications, 11 September 2021, Alba-Iulia, Romania, rismaa.uab.ro

## V. Citations

### V.1. Citations in peer-reviewed journals indexed in Web of Science

- **T. M. Személy Fülöp**, M. Megan, D. I. Borlea (Pătraşcu), *On uniform stability with growth rates of stochastic skew-evolution semiflows in Banach Spaces*, Axioms. (2021), 10(3), 182. **is cited in**:

  (*i*) T.Yue, *On uniform instability in mean of stochastic skew-evolution semi-flows.* Glas.Mat., (2024), 59(1), 107–123, https://doi.org/10.3336/gm.59.1.05

  (*ii*) T.Yue, K. Liu, J. Zhang, *Some results on the uniform polynomial behavior in average of cocycles over maps.* U.P.B. Sci. Bull., Series A, (2025), 87(4), 99–108.

# Table of Contents

# Introduction

The classical theory of asymptotic behaviors of exponential type, polynomial type, or with general growth rates such as stability, instability, dichotomy, trichotomy, and splitting for deterministic differential equations has undergone an impressive development over time and has been approached from different perspectives. Solutions of differential equations have been obtained by means of evolutionary processes or cocycles over flows or semiflows in Hilbert or Banach spaces. Moreover, the aforementioned asymptotic properties have been studied both in the uniform and in the nonuniform setting.

In the context of the development of the theory of stochastic differential equations in infinite-dimensional spaces, a series of fundamental works provide essential perspectives for the analysis of the asymptotic behavior and the stability of such systems.

Thus, Bensoussan and Flandoli [8] proved the existence of stochastic inertial manifolds for degenerate parabolic equations by employing the LyapunovPerron method, which opened the way to further studies on the geometric structure of stochastic solutions. On the other hand, in 1999, S. Bonaccorsi [11] extended the variation of constants formula to nonlinear stochastic equations in infinite-dimensional spaces.

The work of C. Chicone and Y. Latushkin [28] (1999) provides a unified approach to the study of evolution operators and semigroups, with applications in spectral stability, control theory, and the analysis of energy transfer in nonlinear dynamical systems. In parallel, W. Coppel [29] (1978) offered a fundamental treatment of dichotomy in stability theory, developing solid criteria for the characterization of stability and instability of differential systems.

The theory of infinite-dimensional linear systems was detailed in the classical work of R. Curtain and Pritchard [31] (1978), and more recently R. Curtain and H. Zwart [30] (2020) further advanced this direction through a state-space approach, also suitable for the control of nonlinear systems.

In the studies of G. Da Prato and Ichikawa [32] (1987), remarkable contributions emerged concerning Lyapunov equations for time-varying systems, while I. Gikhman and A. Skorokhod [49] (1972) laid the theoretical foundations of It calculus in infinite-dimensional spaces. In the same vein, the works of H. Kunita [58] on infinitesimal generators of random positive semigroups, together with the treatises of P. Protter [94] and J. Zabczyk [137] devoted to stochastic calculus and control theory, complete the conceptual framework necessary for modern approaches in the field.

The theoretical foundations used in this work rely on concepts from general topology and measure theory, for which classical references include the works of P. Halmos [57] and J. Munkres [86].

A general framework for the analysis of asymptotic behaviors of nonlinear dynamical systems, both in the deterministic and the stochastic context, is provided by the notion of evolution cocycle, first introduced in the literature by M. Megan and C. Stoica [76]. This notion generalizes the classical concepts of evolution operators, operator semigroups, and cocycles over semiflows, offering a flexible and efficient tool for modeling dynamical systems in Banach spaces.

The extension of this framework to more general forms, capable of including various asymptotic behaviors, has been achieved in numerous recent contributions. For instance, in [77] and [80], the authors investigated different notions of stability and dichotomy for evolution cocycles, both in the uniform and in the nonuniform setting. Moreover, integral approaches and characterizations based on families of invariant projectors have been developed in [82], [83], [100], [107], [108], and [109].

In parallel, in [46], A. Gin analyzed the notion of uniform $h$-dichotomy for evolution cocycles in Banach spaces, providing necessary and sufficient conditions. The work [47] extends these results to the nonuniform case, introducing general classes of growth rates and offering relevant characterizations for evolution cocycles. A distinct contribution is brought by the study of dichotomy with differentiable growth rates in [67].

Other important contributions can be found in [70], [66], and [78], where the authors employ both classical methods and modern tools of functional analysis in order to obtain criteria applicable in Banach spaces.

Trichotomic behavior, as well as the associated characterizations, is also well represented in the recent literature, through works such as [99], [101], and the monograph [103], which provide a unified framework for the study of uniform and nonuniform asymptotic behaviors of evolution cocycles.

Subsequently, attention has increasingly focused on stochastic differential equations (SDEs) in infinite-dimensional Banach or Hilbert spaces. These provide a natural framework for modeling phenomena affected by random noise, with applications in physics, biology, or economics. The problems concerning the existence of cocycles generated by stochastic differential equations have been intensively studied by authors such as A. M. Ateiwi [4], T. Caraballo [26], [24], and [25], A. Carverhill [27], G. Da Prato and J. Zabczyk [33], F. Flandoli [45], B. Schmalfu [98], S.-E. Mohammed [84], and Mohammed with his collaborators [85]. Within this framework, the concept of stochastic cocycle also emerged, introduced systematically by

L. Arnold [1] and later extended to various contexts [2], [3].

The main theme of this thesis is the study of mean asymptotic behaviors of stability and dichotomy type for stochastic evolution cocycles. The analysis is carried out in continuous time, within a uniform framework, in Banach spaces. The work aims to generalize the results already known in the deterministic case, by adapting classical concepts and tools to the stochastic context and by focusing on asymptotic behaviors with growth rates. These allow for a unified treatment of the exponential and polynomial cases, providing a more flexible framework for asymptotic analysis.

Since exponential or polynomial asymptotic behaviors are more restrictive, it was necessary to introduce a more general concept, that of growth rates.

R. Naulin and M. Pinto, in [87] and [89], introduced the concept of asymptotic behaviors with growth rates, which was later developed in works such as [89] and [65]. This approach enabled the generalization of existing results in the exponential and polynomial cases, opening new perspectives on Datko-, Lyapunov-, and Barbashin-type characterizations. Asymptotic behaviors of the $h$-stability or $h$-dichotomy type have proven to be useful in providing a uniform treatment of a wide class of systems, both in the deterministic and in the stochastic context.

In recent years, a series of scientific works authored by T. M. Személy Fülöp have made important contributions to the theory of asymptotic behaviors in mean of stochastic evolution cocycles in Banach spaces. Thus, in the paper [130], written in collaboration with M. Megan and D. I. Borlea, the concept of uniform stability in mean with growth rates is introduced and studied, with sufficient conditions being formulated and relevant examples provided that illustrate generalizations of exponential and polynomial mean stability. Continuing this direction, the paper [124] proposes a rigorous formalization of uniform dichotomy in mean, using invariant families of projectors for stochastic skew-evolution semiflows in continuous time. In [128], connections are established between the concepts of uniform $h$-dichotomy in mean and uniform exponential dichotomy in mean, as well as between uniform $h$-dichotomy in mean and uniform polynomial dichotomy in mean. Furthermore, in the paper [123], written together with D. I. Borlea, integral characterizations of uniform $h$-dichotomy in mean are provided, using invariant and strongly invariant families of projectors. In [129], uniform stability in mean of exponential and polynomial type is analyzed, while the paper [125] addresses the reversible case of stochastic skew-evolution semiflows, obtaining Datko-type characterizations for uniform $h$-dichotomy in mean. Also in the direction of stochastic evolution cocycles, C. Stoica introduced a unified perspective in [105], discussing stochastic formulations and interpretations of these dynamical structures. In the recent paper [106], several fundamental issues related to the study of evolution equations in a stochastic context are analyzed, providing useful insights for extending classical methods. In addition, the researcher T. Yue has addressed related topics concerning instability in mean and its characterizations. In the works [135], [136], and [134], Datko- and Barbashin-type theorems are formulated.

These works have been developed in the spirit of the earlier contributions of the authors P. V. Hai in [55], R. Boruga in [13], D. Dragičević in [38], D. Stoica in [111], [112], [113], [114], [115], [116], D. Stoica and M. Megan in [118], [119], [120], [121], D. Stoica and L. Lemle in [117], D. Stoica, M. Megan and L. Lemle in [122], C. Tudor in [131], and V. M. Ungureanu in [132].

Within the theory of stability of differential equations, several theorems have laid the foundations for the development of this field, among the most important being the characterization known as the Datko–Pazy theorem. According to it, a family of evolution operators $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ defined on a Banach space $X$ is exponentially stable if and only if there exists an exponent $p \in [1,\infty)$ such that

$$\sup_{s\geq 0} \int_s^\infty \|U(t,s)x\|^p \, dt < \infty, \quad \text{for all } x \in X.$$

Later, R. Datko proved this result for the particular case $p = 2$ for semigroups in Hilbert spaces in [36], and for evolution operators in [37].

The Datko theorem has constituted the foundation of important studies on the uniform exponential stability of evolution equations. Subsequently, numerous works have been devoted to this subject, mentioning only a few of them [53], [61], [134], [9], and [7]. At the same time, in the analysis of asymptotic behaviors of dynamical systems, a reference point is the monograph of A. M. Lyapunov [63], which laid the basis for the development of fundamental methods, in particular through the use of Lyapunov-type functions, a technique frequently encountered in the specialized literature. We also mention here other significant works [40], [60].

The classical result of E. A. Barbashin [5] represents one of the most important starting points in stability theory, since it provided integral characterizations, in which the integration parameter is the second one, for uniform exponential stability. This direction has also been continued by other authors, among them [39], [52], [134], and [133].

The concept of uniform stability, of exponential type, polynomial type, or with growth rates, occupies a central place in the theory of asymptotic behaviors of nonautonomous dynamical systems. The fundamental work of L. Barreira and C. Valls [6] provides an in-depth perspective on the stability of nonautonomous differential systems, constituting an important theoretical reference in the definition and study of these concepts.

In Banach spaces, stability criteria have been extended through Datko–Pazy type methods and modern operator approaches. In this regard, R. Boruga and his collaborators have made significant contributions: in [19], integral characterizations for uniform stability with growth rates are provided, while in [22], the analysis of uniform stability is detailed using functional conditions applicable within a generalized framework. The corresponding instability is discussed in [21], while in [12], majoration criteria are introduced.

A notable result is the formulation of a logarithmic criterion for polynomial stability [16], providing a new characterization of uniform polynomial stability for evolution operators in Banach spaces. Moreover, C. Buşe and collaborators [23] developed a strengthened version of Barbashin's theorem, useful in integral characterizations of stability in the nonautonomous context.

Uniform stability has been addressed in discrete or general contexts by H. Damak [35], C.-L. Mihiţ [81], P. V. Hai [54], [56], M. Megan, C. Stoica, L. Buliga [79], C. Stoica [102], as well as by Megan, Sasu and Sasu in the articles [69], [71], [73]. Also, in the work of C. Stoica and M. Megan [110], both exponential stability and instability for cocycles over semiflows are studied.

A remarkable recent contribution belongs to the same authors (Megan, A. L. Sasu and B. Sasu), who in [72] propose a Zabczyk–Rolewicz type criterion for the stability of nonlinear nonautonomous systems, bringing a new perspective on the functional analyses involved.

In addition, the article [41] links stability to the admissibility properties of discrete systems, while the analysis of C.-L. Mihiţ in [81] deepens the conditions of uniform $h$-stability for evolution operators.

All these research directions strengthen the importance of uniform stability with growth rates, providing essential theoretical foundations for the extension of this concept to the context of stochastic evolution cocycles, which constitutes one of the main objectives of the present thesis.

The generalization of the concept of stability is achieved through the introduction of the notion of dichotomy, which involves the existence of a family of projectors allowing the decomposition of the state space into a direct sum of two closed subspaces: one associated with the stable behavior and the other with the unstable behavior. The importance of exponential dichotomy for linear differential equations was established through two fundamental works, namely the classical monograph by J. L. Massera and J. J. Schäffer, published in 1966 [64], and the treatise by J. L. Daleckii and M. G. Krein, published in 1974 [34].

The concept of uniform exponential dichotomy has been significantly extended in the last decades, being addressed in numerous important works [59], [62], [74], [92], [96], [97], which have provided integral characterizations, necessary and sufficient conditions, or connections with the theory of evolution semigroups.

In addition, more general forms of this behavior have also been analyzed, such as nonuniform exponential dichotomy [93], [95], as well as intermediate or mixed forms of trichotomy, developed especially in the works of C. Stoica [101], [99] and of S. Elaydi and O. Hájek [43].

A recent research direction is represented by dichotomy with growth rates, or $h$-dichotomy, which generalizes the exponential and polynomial cases. In this regard, we mention the significant contributions of R. Boruga and M. Megan, who in the works [14], [17], [18], [20] have provided characterizations of the dichotomy concept as well as Datko-type criteria for this behavior, including in the nonuniform variant.

Moreover, in the works [9] and [10], the authors extended the theory of stable manifolds in the presence of nonuniform polynomial dichotomies, demonstrating their applicability to nonlinear nonautonomous equations.

In modeling real phenomena, arising from fields such as biology, economics, or environmental sciences, it is often observed that events do not evolve continuously, but rather occur at discrete moments of time. This aspect justifies the increasing interest devoted to the study of dynamical systems in discrete time. Thus, the theoretical framework of stochastic skew-evolution semiflows in discrete time has proved to be extremely valuable for the description and analysis of the asymptotic behavior of such systems.

In the author's work [126], characterizations of the property of uniform $h$-dichotomy in mean for stochastic skew-evolution semiflows in discrete time have been highlighted. More precisely, necessary and sufficient conditions are provided by using both invariant families of projectors and strongly invariant families of projectors for stochastic skew-evolution semiflows in discrete time. As a consequence, integral characterizations are obtained for the concept of uniform exponential dichotomy in mean. For recent contributions in this field we mention the works [15], [42], [44], [48], [50], [51], [68], [75], [90], [91], and [104].

The thesis is structured into three chapters, preceded by the present Introduction and followed by the Bibliography. The main objective consists in deepening and generalizing the deterministic results within the stochastic framework. The first chapter, entitled **Stochastic skew-evolution semiflows and stochastic families of projectors**, consists of five sections: *Stochastic skew-evolution semiflows, Growth concepts for stochastic skew-evolution semiflows, Stochastic invariant and strongly invariant families of projectors for stochastic skew-evolution semiflows, Growth concepts for pairs of the form* $(C, P)$*, Bibliographic comments.* The first section presents the introductory notions in stochastic analysis, the notions of stochastic evolution semiflow, stochastic evolution semiflows, stochastic skew-evolution semiflows, and several illustrative examples are provided in this regard. The second section introduces the concepts of uniform $h$-growth for stochastic skew-evolution semiflows and establishes the connections between mean growth concepts. The third section deals with the notions of invariant and strongly invariant families of projectors, as well as the relationships between them. The following section generalizes the concepts of $h$-growth to the general case of pairs of the form $(C, P)$ in the uniform setting. Moreover, the connections between the concepts are presented, along with characterizations involving invariant and strongly invariant families of projectors for the concept of uniform $h$-growth.

The second chapter, entitled **Stability concepts for stochastic skew-evolution semiflows**, addresses the concept of stability in mean with growth rates and consists of four sections: *Preliminaries, Connections between the concepts of uniform stability in mean, Characterizations of the concept of uniform stability in mean,* *Bibliographic comments.*

In the first section of this chapter, the notions of $h$-stability in mean as well as uniform $h$-stability in mean were introduced, mentioning also the particular cases of the concept of uniform stability, namely the concept of uniform exponential stability in mean and the concept of uniform polynomial stability in mean. Several illustrative examples were also provided.

The second section of this chapter addresses the connections between the concepts of uniform stability in mean. The third section is dedicated to the characterization of the concept of uniform stability in mean. The main results of this section are Theorems 2.3.1, 2.3.2, 2.3.3 and Corollaries 2.3.1, 2.3.2, 2.3.3, 2.3.4, 2.3.5, 2.3.6, which form the basis for the results to be proved throughout this chapter. Thus, Barbashin-, Datko-, and Lyapunov-type integral characterizations are obtained for the concept of uniform $h$-stability in mean.

The third chapter, entitled **Dichotomy concepts for stochastic skew-evolution semiflows**, contains the most general results of the thesis and consists of three sections: *Preliminaries, Connection between the concepts of uniform dichotomy in mean, Characterization of the concept of uniform dichotomy in mean, Bibliographic comments.*

In the first section of this chapter, the notions of uniform $h$-dichotomy in mean are introduced, recalling also the particular cases, namely the concept of uniform exponential dichotomy in mean and the concept of uniform polynomial dichotomy in mean. In Section 3.2, emphasis is placed on establishing the connections between the concepts of uniform dichotomy, and Theorems 3.2.1, 3.2.2, as well as Corollaries 3.2.1, 3.2.2, represent some of the most important results of this section and have been published in [128]. The property of uniform $h$-dichotomy in mean is addressed in Section 3.3 with invariant and strongly invariant families of projectors. In Theorems 3.3.9, 3.3.10, the connection between uniform $h$-dichotomy in mean in the discrete time and uniform $h$-dichotomy in mean in the continuous time is studied. In this sense, it is obtained that uniform $h$-dichotomy in mean in continuous time is equivalent to uniform $h$-dichotomy in mean in discrete time, under conditions of uniform $h$-growth. Another important result of this section is Theorem 3.3.2, which was published in [124]. Other important results have been obtained later and published in [123] and [125].

The original results presented in this thesis and highlighted in the Bibliographic comments at the end of each chapter have been published or accepted for publication, with the exception of Theorems 3.3.9 and 3.3.10, which have been submitted for publication in [127]. All these results belong partly to the author of the thesis as sole author, or have been obtained in collaboration with Professor Emeritus Mihail Megan, the scientific coordinator of the work, and with D. I. Borlea. The publications appeared in ISI-indexed journals [125], in specialized journals from Romania and abroad, with peer-review scientific committees [124], [130], as well as in the proceedings of international conferences [129], [128], [123]. In addition, a series of other original results, which are not directly included in the content of the thesis but are related to its topic and to the individual training plan, have been published in the ISI-indexed journal [126] and have been presented at the national conference “Sixth Romanian Itinerant Seminar on Mathematical Analysis and its Applications”, May 30–31, 2024, Cluj-Napoca, Romania.

The results included in this work have also been presented and discussed within the “Seminar on Mathematical Analysis and Control Theory” at the West University of Timişoara, a seminar founded and long coordinated by the late Professor Emeritus Mihail Megan, a prominent figure of the Romanian mathematical school. His scientific contributions, the support offered to young researchers, and the generous guidance provided to the author of this thesis have had an essential impact on the development of the research presented in this work.

The Bibliography brings together the articles published by the author on the topic of the thesis in specialized journals, the works presented at national and international conferences, as well as a selection of works consulted and cited by the author in the elaboration of the research.

This thesis is dedicated to the memory of Professor Emeritus Mihail Megan, my scientific mentor and a fundamental reference in my professional development. His rigorous thinking, constant support, and the trust he placed in me throughout my training were decisive for the development of this work.

I would like to express my sincere gratitude to Professor Ioan Lucian Popa, PhD Habil., for the support, guidance, and encouragement he offered during the difficult period that followed the loss of my scientific coordinator. I also wish to thank the members of the Advisory Committee: Professor Adina Luminiţa Sasu, PhD Habil., Lecturer Aurelian Crăciunescu, PhD, Lecturer Larisa Elena Biriş, PhD, and Professor Marin Marin, PhD Habil., for the valuable advice and guidance received in the preparation of this thesis.

Special thanks are addressed to Associate Professor Traian Ceauşu, PhD, for his valuable observations and the support provided during the stages of writing this thesis.

I also express my gratitude to my colleagues from the research group at the West University of Timişoara, whose suggestions, guidance, and encouragement have contributed significantly to the completion of this work.

I wholeheartedly thank my dear friends, who have stood by me throughout this journey and have offered me support, encouragement, and warm thoughts filled with affection and gratitude.

To my entire family I am deeply grateful for the love, support, patience, and trust they have unconditionally offered me. I thank them for their prayers, for their encouragement, for the peace they provided at home, and for all the sacrifices made in silence. Without their constant presence and the generosity with which they supported me, this journey would not have been possible.

# 1. Stochastic skew-evolution semiflows and stochastic families of projectors

## 1.1 Stochastic skew-evolution semiflows

**Definition 1.1.1.** [57] Let $\Omega$ be a nonempty set. A family of subsets $\mathcal{B} \subset \mathcal{P}(\Omega)$ is called a *$\sigma$-algebra on* $\Omega$ if the following conditions are satisfied:

1. The empty set belongs to $\mathcal{B}$:
$$\emptyset \in \mathcal{B}.$$
2. If $A \in \mathcal{B}$, then its complement with respect to $\Omega$ also belongs to $\mathcal{B}$:
$$A^c = \Omega \setminus A \in \mathcal{B}.$$
3. If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$, then their countable union also belongs to $\mathcal{B}$:
$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}.$$

The family $\mathcal{B}$ is also called a *$\sigma$-algebra of subsets of* $\Omega$.

**Definition 1.1.2.** [86, p. 76] Let $K$ be a nonempty set. A family of subsets $\mathcal{D} \subset \mathcal{P}(K)$ is called a *topology on* $K$ if the following conditions are satisfied:

1. The empty set and the whole set belong to $\mathcal{D}$:
$$\emptyset \in \mathcal{D}, \quad K \in \mathcal{D}.$$
2. The arbitrary (possibly infinite) union of sets from $\mathcal{D}$ belongs to $\mathcal{D}$, that is, for any family $(U_i)_{i\in I} \subset \mathcal{D}$ we have
$$\bigcup_{i\in I} U_i \in \mathcal{D}.$$

3. The finite intersection of sets from $\mathcal{D}$ belongs to $\mathcal{D}$, that is, for any $U_1, \ldots, U_n \in \mathcal{D}$ we have

$$\bigcap_{k=1}^{n} U_k \in \mathcal{D}.$$

The pair $(K, \mathcal{D})$ is called a *topological space*, and the elements of $\mathcal{D}$ are called *open sets*.

**Definition 1.1.3.** [57] Let $\Omega$ be a nonempty set and let $\mathcal{B}$ be a $\sigma$-algebra of subsets of $\Omega$. Then $(\Omega, \mathcal{B})$ is called a *measurable space*, and the elements of $\mathcal{B}$ are called measurable sets.

**Definition 1.1.4.** [86] Let $(K, \mathcal{D})$ be a topological space. Then the $\sigma$-algebra generated by the family $\mathcal{D}$ is called the *Borel $\sigma$-algebra* on $K$ and is denoted by $\mathcal{B}(K) = \sigma(\mathcal{D})$. Its elements are called *Borel sets*.

**Definition 1.1.5.** [57] Let $(\Omega, \mathcal{B})$ be a measurable space and $(K, \mathcal{D})$ a topological space. A function $f : \Omega \to K$ is said to be *measurable* if for every $D \in \mathcal{D}$ we have $f^{-1}(D) \in \mathcal{B}$ (that is, the inverse image of every open set under the function $f$ is a Borel set).

**Definition 1.1.6.** [57] A function $f : \Omega \to K$, measurable with respect to the measurable spaces $(\Omega, \mathcal{B})$ and $(K, \mathcal{D})$, is called a *random variable*. We say that a random variable is *simple* if it takes only a finite number of values.

**Definition 1.1.7.** [57] Let $(\Omega, \mathcal{B})$ be a measurable space. A $\sigma$-additive function $\mu : \mathcal{B} \to [0, 1]$ with $\mu(\Omega) = 1$ is called a *probability measure*. The triple $(\Omega, \mathcal{B}, \mu)$ is called a *probability space*.

**Lemma 1.1.1.** If $X$ is a separable Banach space and $f$ is a random variable defined on $(\Omega, \mathcal{B})$ with values in $(X, \|\cdot\|)$, then the real-valued function $\|f(\cdot)\|$ is measurable.

For the proof see [33, p. 19].

Let $f : (\Omega, \mathcal{B}) \to (X, \|\cdot\|)$.

**Definition 1.1.8.** [33] The random variable $f : (\Omega, \mathcal{B}) \to (X, \|\cdot\|)$ is called *Bochner integrable* (or *simply integrable*) with respect to the probability measure $\mu$ if

$$\int_{\Omega} \|f(\omega)\| \, d\mu(\omega) < \infty.$$

If $f$ is an integrable random variable, we define

$$\int_{\Omega} f(\omega) \, d\mu(\omega) = \lim_{n \to \infty} \int_{\Omega} f_n(\omega) \, d\mu(\omega),$$

where $f_n$ is a sequence of simple random variables such that the sequence $\|f(\omega) - f_n(\omega)\|$ decreases monotonically to zero. This integral is well defined because the above limit exists [33] and is independent of the sequence of simple random variables $(f_n)_{n\in\mathbb{N}}$ chosen for the approximation.

We also mention that this integral is called the *mean* or the *Bochner integral* of the random variable $f$. It is denoted by $E(f)$, that is,

$$E(f) = \int_{\Omega} f(\omega)\, d\mu(\omega).$$

The necessary conditions for a linear operator to commute with the integral defined above are given by the following proposition:

**Proposition 1.1.1.** Suppose that $X$ and $Y$ are two Banach spaces and $A : D(A) \subseteq X \to Y$ is a closed operator whose domain $D(A)$ is a Borel subset of $X$, where $D(A)$ is endowed with the graph norm of $A$. If $f : \Omega \to X$ is a random variable such that $f(\omega) \in D(A)$ with probability 1, then $A(f)$ is a random variable with values in $Y$. Moreover, if

$$\int_{\Omega} \|Af(\omega)\|\, d\mu(\omega) < \infty,$$

then

$$A \int_{\Omega} f(\omega)\, d\mu(\omega) = \int_{\Omega} (Af)(\omega)\, d\mu(\omega). \tag{1.1}$$

*Proof.* For the proof see [33, p. 21]. □

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Let $X$ be a real or complex Banach space. Denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on $X$, and the norms on $X$, respectively on $\mathcal{B}(X)$, will be denoted by $\|\cdot\|$. We also define the sets

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\},$$

$$T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\},\ t_0 \in [0, \infty)\}$$

**Definition 1.1.9.** [76] A measurable random field $\varphi : \Delta \times \Omega \to \Omega$ is said to be a *stochastic evolution semiflow* on $\Omega$ if the following properties hold:

- $(es_1)$ $\varphi(t, t, \omega) = \omega$, for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$,
- $(es_2)$ $\varphi(t, s, \varphi(s, t_0, \omega)) = \varphi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.

**Example 1.1.1.** [76] Let $\varphi : \Delta \times \Omega \to \Omega$, $\varphi(t, s, \omega) = \omega$. It is easy to verify that $\varphi$ is an evolution semiflow on $\Omega$.

**Example 1.1.2.** [76] Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a decreasing function on $[0, \infty)$ such that $\lim_{t\to\infty} f(t) = l$. Let $\Omega$ be the closure in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, with respect to the topology of uniform convergence, of the set $\{f_t \mid t \in \mathbb{R}_+\}$, where $f_t(\tau) = f(t + \tau)$ for every $\tau \in \mathbb{R}_+$.

The mapping $\varphi : \Delta \times \Omega \to \Omega$, $\varphi(t, s, \omega) = \omega_{t-s}$, where $\omega_{t-s}(\tau) = \omega(t - s + \tau)$, is a stochastic evolution semiflow on $\Omega$.

Indeed,

$$\varphi(t, s, \varphi(s, t_0, \omega)) = \varphi(t, s, \omega(s - t_0 + \tau)) = \omega_{t-t_0}(\tau) = \varphi(t, t_0, \omega).$$

**Example 1.1.3.** [2] Let $X$ be a Banach space and let $\Omega$ be the space of all paths $\omega : \mathbb{R}_+ \to X$ such that $\omega(0) = 0$, endowed with the compact-open topology. Let $\mathcal{F}_t$, $t \geq 0$, be the $\sigma$-algebra generated by the set $\{\omega(u) - \omega(v) \in X \mid u, v \leq t\}$, and let $\mathcal{B}$ be the Borel $\sigma$-algebra associated with $\Omega$. Then, the mapping defined by $\varphi_0(t, \omega)(s) = \omega(t + s) - \omega(t)$ is a stochastic semiflow on $\Omega$, for all $(t, s, \omega) \in \Delta \times \Omega$.

**Definition 1.1.10.** [76] Let $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ be a measurable map. We say that $\Phi$ is a *stochastic evolution cocycle* associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ if the following conditions hold:

$(ce_1)$ $\Phi(s, s, \omega) = I$ (the identity operator on $X$), for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$;

$(ce_2)$ $\Phi(t, s, \varphi(s, t_0, \omega))\Phi(s, t_0, \omega) = \Phi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$;

$(ce_3)$ $(s, t_0, \omega) \mapsto \Phi(s, t_0, \omega)$ is continuous for all $\omega \in \Omega$.

If $\varphi : \Delta \times \Omega \to \Omega$ is a stochastic evolution semiflow on $\Omega$ and $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is a stochastic evolution cocycle over $\varphi$, then the pair $C = (\Phi, \varphi)$ is referred to as a *stochastic skew-evolution semiflow* on $\Delta \times \Omega$.

Moreover, an evolution operator can be regarded as a stochastic evolution cocycle. For this, we recall that a mapping $U : \Delta \to \mathcal{B}(X)$ is called an *evolution operator* if the following conditions are satisfied:

$(u_1)$ $U(t, t) = I$ (the identity operator), for all $t \in \mathbb{R}_+$;

$(u_2)$ $U(t, s)U(s, t_0) = U(t, t_0)$, for all $(t, s, t_0) \in T$.

**Definition 1.1.11.** [125] The stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is said to be *reversible* if for all $(t, s, \omega) \in \Delta \times \Omega$, the map $\Phi(t, s, \omega)$ is bijective.

**Example 1.1.4.** [55] Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator. Then, for any stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$, the mapping $\Phi_U : \Delta \times \Omega \to \mathcal{B}(X)$ defined by

$$\Phi_U(t, s, \omega) = U(t, s), \quad \text{for every} \quad (t, s, \omega) \in \Delta \times \Omega$$

is a stochastic evolution cocycle. Consequently, for any stochastic evolution semiflow $\varphi$, the pair $C = (\Phi_U, \varphi)$ is a stochastic skew-evolution semiflow.

**Example 1.1.5.** [2] Let us consider the stochastic evolution semiflow $\varphi_0$ defined in Example 1.1.3. We denote

$$\varphi(t,s,\omega)=\varphi_0(t-s,\omega)$$

and

$$\Phi(t,s,\omega)=\Phi_0(t-s,\omega)$$

for all $(t,s,\omega)\in\Delta\times\Omega$. Then $C(\Phi,\varphi)$ is a stochastic skew-evolution semiflow on $\Delta\times\Omega$.
It can be observed that

$$\varphi(s,s,\omega)=\omega$$

and

$$\begin{aligned}\varphi(t,s,\varphi(s,t_0,\omega))&=\varphi(t+s,t_0,\omega)-\varphi(t,t_0,\omega)=\\&=\omega(t+s+t_0)-\omega(t+s)-\omega(t+t_0)+\omega(t)=\\&=\varphi(t+t_0,s,\omega)-\varphi(t,s,\omega)=\\&=\varphi(t,t_0,\omega).\end{aligned}$$

Consequently, the stochastic skew-evolution semiflow generalizes the concept of the classical stochastic cocycle introduced by L. Arnold in [2].

**Definition 1.1.12.** [89] An nondecreasing function $h:\mathbb{R}_+\to[1,\infty)$ with $\lim\limits_{t\to\infty}h(t)=\infty$ is called a *growth rate*.

**Example 1.1.6.** [130] Let $C=(\Phi,\varphi)$ be a stochastic skew-evolution semiflow on $\Delta\times\Omega$. For an invertible growth rate $h:\mathbb{R}_+\to[1,\infty)$, we consider

$$h_e:\mathbb{R}_+\to\mathbb{R}_+,\quad h_e(t)=h^{-1}(e^t)$$

and

$$\varphi_e:\Delta\times\Omega\to\Omega,\quad \varphi_e(t,s,\omega)=\varphi(h_e(t),h_e(s),\omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_e:\Delta\times\Omega\to\mathcal{B}(X),\quad \Phi_e(t,s,\omega)=\Phi(h_e(t),h_e(s),\omega)$$

a stochastic evolution cocycle associated with $\varphi_e$.
Then $C=(\Phi_e,\varphi_e)$ is a stochastic skew-evolution semiflow on $\Delta\times\Omega$.

**Example 1.1.7.** [130] Let $C=(\Phi,\varphi)$ be a stochastic skew-evolution semiflow on $\Delta\times\Omega$. For an invertible growth rate $h:\mathbf{R}_+\to[1,\infty)$, we consider

$$h_p:\mathbb{R}_+\to\mathbb{R}_+,\quad h_p(t)=h^{-1}(t+1)$$

and

$$\varphi_p:\Delta\times\Omega\to\Omega,\quad \varphi_p(t,s,\omega)=\varphi(h_p(t),h_p(s),\omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_p : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_p(t, s, \omega) = \Phi(h_p(t), h_p(s), \omega)$$

a stochastic evolution cocycle associated with $\varphi_p$.
Then $C = (\Phi_p, \varphi_p)$ is a stochastic skew-evolution semiflow on $\Delta \times \Omega$.

The following example is obtained from Example 1.1.6 for $h(t) = e^t$.

**Example 1.1.8.** [130] Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$. We denote

$$\varphi_p^1 : \Delta \times \Omega \to \Omega, \quad \varphi_p^1(t, s, \omega) = \varphi(e^t - 1, e^s - 1, \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_p^1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_p^1(t, s, \omega) = \Phi(e^t - 1, e^s - 1, \omega)$$

a stochastic evolution cocycle associated with $\varphi_p^1$.

Then $C = (\Phi_p^1, \varphi_p^1)$ is a stochastic skew-evolution semiflow on $\Delta \times \Omega$.

The following example is a particular case of Example 1.1.7 for the growth rate $h(t) = t + 1$.

**Example 1.1.9.** [130] Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$. We denote

$$\varphi_e^1 : \Delta \times \Omega \to \Omega, \quad \varphi_e^1(t, s, \omega) = \varphi(\ln(t + 1), \ln(s + 1), \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_e^1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_e^1(t, s, \omega) = \Phi(\ln(t + 1), \ln(s + 1), \omega)$$

a stochastic evolution cocycle associated with $\varphi_e^1$.

Then $C = (\Phi_e^1, \varphi_e^1)$ is a stochastic skew-evolution semiflow on $\Delta \times \Omega$.

We denote by $L(\Omega, X, \mu)$ the Banach space of all Bochner-measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$.

In the following, particular classes of stochastic evolution cocycles are defined. These classes are used in the integral characterization of exponential stability, polynomial stability and $h$-stability, respectively of exponential dichotomy, polynomial dichotomy and $h$-dichotomy in mean.

**Definition 1.1.13.** [66] The stochastic evolution cocycle $C = (\Phi, \varphi)$ is called *strongly measurable* if, for every $(t_0, x) \in \mathbb{R}_+ \times L(\Omega, X, \mu)$, the mapping

$$s \longmapsto \int_\Omega \|\Phi(s, t_0, \omega)x(\omega)\| \, d\mu(\omega)$$

is measurable on $[t_0, \infty)$.

**Definition 1.1.14.** [66] The stochastic evolution cocycle $C = (\Phi, \varphi)$ is called $*$-*strongly measurable* if, for every $(t, t_0, x^*) \in T \times L(\Omega, X^*, \mu)$, the mapping

$$s \longmapsto \int_\Omega \|\Phi(t, s, \varphi(s, t_0, \omega))^* x^*(\omega)\| \, d\mu(\omega)$$

is measurable on $[t_0, t]$.

## 1.2 Growth concepts for stochastic skew-evolution semiflows

**Definition 1.2.1.** The stochastic skew-evolution semiflow $(\Phi, \varphi)$ has *uniform h-growth in mean (h.g.m.)* if there are $M > 1$, $\varepsilon \geq 0$ and $\alpha > 0$ with

$$h(s)^\alpha \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| d\mu(\omega) \leq M h(s)^\varepsilon h(t)^\alpha \int_\Omega \|x(\omega)\| d\mu(\omega);$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and all $x \in L(\Omega, X, \mu)$.

**Definition 1.2.2.** [130] The stochastic skew-evolution semiflow $(\Phi, \varphi)$ has *uniform h- growth in mean (u.h.g.m.)* if there are $M > 1$ and $\alpha > 0$ with

$$h(s)^\alpha \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| d\mu(\omega) \leq M h(t)^\alpha \int_\Omega \|x(\omega)\| d\mu(\omega);$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and all $x \in L(\Omega, X, \mu)$.

**Remark 1.2.1.** [130] We say that the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform h- growth in mean if and only if there are $M > 1$ and $\alpha > 0$ with

$$h(s)^\alpha \int_\Omega \|\Phi(t, t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq M h(t)^\alpha \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| d\mu(\omega);$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and all $x_0 \in L(\Omega, X, \mu)$.

As particular cases, we have

- (i) for $h(t) = e^t$ we obtain the concept of uniform exponential growth in mean $(u.e.g.m.)$.

- (ii) for $h(t) = t + 1$ it results the concept of uniform polynomial growth in mean (*u.p.g.m.*).

In the following, two majorization criteria for the concept of uniform exponential growth are presented:

**Proposition 1.2.1.** Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow. The following statements are equivalent:

$(i)$ $C = (\Phi, \varphi)$ has uniform exponential growth in mean;

$(ii)$ there exists a nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to\infty} g(t) = \infty$ such that

$$\int_\Omega \|\Phi(t, s, \omega)x(\omega)\| \, d\mu(\omega) \leq g(t - s) \int_\Omega \|x(\omega)\| \, d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and all $x \in L(\Omega, X, \mu)$;

$(iii)$ there exists a nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to\infty} g(t) = \infty$ such that

$$\int_\Omega \|\Phi(t, t_0, \omega)x_0(\omega)\| \, d\mu(\omega) \leq g(t - s) \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| \, d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and all $x_0 \in L(\Omega, X, \mu)$.

**Proposition 1.2.2.** The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform polynomial growth in mean if and only if the stochastic skew-evolution semiflow $C_p^1 = (\Phi_p^1, \varphi_p^1)$ defined in Example 1.1.8 has uniform exponential growth in mean.

**Proposition 1.2.3.** The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform exponential growth in mean if and only if the stochastic skew-evolution semiflow $C_e^1 = (\Phi_e^1, \varphi_e^1)$ defined in Example 1.1.9 has uniform polynomial growth in mean.

**Proposition 1.2.4.** The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform $h$-growth in mean if and only if the stochastic skew-evolution semiflow $C_e = (\Phi_e, \varphi_e)$ defined in Example 1.1.6 has uniform exponential growth in mean.

**Proposition 1.2.5.** The pair $C = (\Phi, \varphi)$ has uniform $h$-growth in mean if and only if the stochastic skew-evolution semiflow $C_e = (\Phi_e, \varphi_e)$ defined in Example 1.1.7 has uniform polynomial growth in mean.

In what follows, we emphasize the interconnections among the growth concepts discussed in this section.

**Remark 1.2.2.** The relationship between the growth concepts defined above is given by the following diagrams:

$$\begin{array}{ccc} u.p.g.m. & \Rightarrow & p.g.m. \\ \Downarrow & & \Downarrow \\ u.e.g.m. & \Rightarrow & e.g.m. \end{array}$$

and

$$u.h.g.m. \ \Rightarrow \ h.g.m.$$

## 1.3 Stochastic invariant and strongly invariant families of projectors for stochastic skew-evolution semiflows

**Definition 1.3.1.** [124] We say that a continuous mapping $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a *family of projectors* on $X$ if

$$P^2(s,\omega) = P(s,\omega), \quad \text{for all } (s,\omega) \in \mathbb{R}_+ \times \Omega.$$

**Remark 1.3.1.** [124] If $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a family of projectors for a stochastic skew-evolution semiflow $C = (\Phi, \varphi)$, then $Q : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ defined by $Q(s,\omega) = I - P(s,\omega)$ is also a family of projectors for $C$, called the *family of projectors complementary to* $P$.

**Definition 1.3.2.** [124] We say that the family of projectors $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is *invariant* for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ if the relation

$$\Phi(t,s,\omega)P(s,\omega) = P(t,\varphi(t,s,\omega))\Phi(t,s,\omega)$$

holds for every $(t,s,\omega) \in \Delta \times \Omega$.

**Remark 1.3.2.** [124] If the family of projectors $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is invariant for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$, then the complementary family of projectors $Q : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is also invariant for $C = (\Phi, \varphi)$.

**Example 1.3.1.** [124] If $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a family of projectors on $X$, then

$$P_h : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_h(s,\omega) = P(h^{-1}(e^s),\omega)$$

is also a family of projectors on $X$. It is easy to observe that

$$\begin{aligned} P_h(s,\omega) \cdot P_h(s,\omega) &= P(h^{-1}(e^s),\omega) \cdot P(h^{-1}(e^s),\omega) \\ &= P(h^{-1}(e^s),\omega) \\ &= P_h(s,\omega). \end{aligned}$$

**Example 1.3.2.** [124] If $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a family of projectors on $X$, then

$$P_h : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_h(s,\omega) = P(h^{-1}(s+1),\omega)$$

is also a family of projectors on $X$. Indeed,

$$\begin{aligned} P_h(s,\omega) \cdot P_h(s,\omega) &= P(h^{-1}(s+1),\omega) \cdot P(h^{-1}(s+1),\omega) \\ &= P(h^{-1}(s+1),\omega) \\ &= P_h(s,\omega). \end{aligned}$$

In what follows, if $P$ is an invariant family of projectors for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$, then we denote by $\Phi_P : \Delta \times \Omega \to \mathcal{B}(X)$ the mapping defined by

$$\Phi_P(t,s,\omega) = \Phi(t,s,\omega)P(s,\omega). \tag{1.2}$$

**Proposition 1.3.1.** [123] The mapping $\Phi_P$ has the following properties:

*(i)* $\Phi_P(t,s,\omega) = P(t,\varphi(t,s,\omega))\Phi_P(t,s,\omega)$, for all $(t,s,\omega) \in \Delta \times \Omega$;

*(ii)* $\Phi_P(t,t,\omega) = P(t,\omega)$, for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$;

*(iii)* $\Phi_P(t,t_0,\omega) = \Phi_P(t,s,\varphi(s,t_0,\omega))\Phi_P(s,t_0,\omega)$, for all $(t,s,t_0,\omega) \in T \times \Omega$.

**Remark 1.3.3.** [125] If the stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is reversible and the family of projectors $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is invariant for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$, then

$$P(s,\omega)\Phi^{-1}(t,s,\omega) = \Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega)),$$

for all $(t,s,\omega) \in \Delta \times \Omega$.

**Proposition 1.3.2.** [125] If $\Phi_P(t,s,\omega) : \Delta \times \Omega \to \mathcal{B}(X)$ and $\Phi_P^{-1}(t,s,\omega)$ is its inverse, then:

*(i)* $\Phi(t,s,\omega)\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega)) = P(t,\varphi(t,s,\omega))$, for all $(t,s,\omega) \in \Delta \times \Omega$;

*(ii)* $\Phi^{-1}(t,s,\omega)\Phi(t,s,\omega)P(s,\omega) = P(s,\omega)$, for all $(t,s,\omega) \in \Delta \times \Omega$;

*(iii)* $\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega)) = P(s,\omega)\Phi^{-1}(t,s,\omega)P(t,\varphi(t,s,\omega))$, for all $(t,s,\omega) \in \Delta \times \Omega$.

If $Q$ is an invariant family of projectors for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$, then we denote by $\Phi_Q : \Delta \times \Omega \to \mathcal{B}(X)$ the mapping defined by

$$\Phi_Q(t,s,\omega) = Q(t,\varphi(t,s,\omega))\Phi(t,s,\omega).$$

**Proposition 1.3.3.** [123] The mapping $\Phi_Q$ has the following properties:

*(i)* $\Phi_Q(t,s,\omega)=Q(t,\varphi(t,s,\omega))\Phi_Q(t,s,\omega)$, for all $(t,s,\omega)\in\Delta\times\Omega$;

*(ii)* $\Phi_Q(t,t,\omega)=Q(t,\omega)$, for all $(t,\omega)\in\mathbb{R}_+\times\Omega$;

*(iii)* $\Phi_Q(t,t_0,\omega)=\Phi_Q(t,s,\varphi(s,t_0,\omega))\Phi_Q(s,t_0,\omega)$, for all $(t,s,t_0,\omega)\in T\times\Omega$.

**Definition 1.3.3.** [123] The family of projectors $P:\mathbb{R}_+\times\Omega\to\mathcal{B}(X)$ is said to be *strongly invariant* to $C=(\Phi,\varphi)$ if it is invariant to $C$ and for all $(t,s,\omega)\in\Delta\times\Omega$, the map $\Phi(t,s,\omega)$ is an isomorphism from *Range* $Q(s,\omega)$ to *Range* $Q(t,\varphi(t,s,\omega))$.

**Remark 1.3.4.** [123] If the family of projectors $P:\mathbb{R}_+\times\Omega\to\mathcal{B}(X)$ is strongly invariant to the stochastic skew-evolution semiflow $C=(\Phi,\varphi)$, then there exists $\Psi:\Delta\times\Omega\to\mathcal{B}(X)$ such that for all $(t,s,\omega)\in\Delta\times\Omega$, $\Psi$ is an isomorphism from *Range* $Q(t,\varphi(t,s,\omega))$ to *Range* $Q(s,\omega)$.

We will use the following notation:

$$\Psi_Q(t,s,\omega)=\Psi(t,s,\omega)Q(t,\varphi(t,s,\omega)).$$

**Proposition 1.3.4.** [123] If the family of projectors $P:\mathbb{R}_+\times\Omega\to\mathcal{B}(X)$ is strongly invariant to the stochastic skew-evolution semiflow $C=(\Phi,\varphi)$ then the map $\Psi_Q$ has the following properties:

*(i)* $\Phi_Q(t,s,\omega)\Psi_Q(t,s,\omega)=Q(t,\varphi(t,s,\omega))$, for all $(t,s,\omega)\in\Delta\times\Omega$;

*(ii)* $\Psi_Q(t,s,\omega)\Phi_Q(t,s,\omega)=Q(s,\omega)$, for all $(t,s,\omega)\in\Delta\times\Omega$;

*(iii)* $\Psi_Q(t,s,\omega)=Q(s,\omega)\Psi_Q(t,s,\omega)$, for all $(t,s,\omega)\in\Delta\times\Omega$.

*(iv)* $\Psi_Q(t,t,\omega)=Q(t,\omega)$, for all $(t,\omega)\in\mathbb{R}_+\times\Omega$;

*(v)* $\Psi_Q(t,t_0,\omega)=\Psi_Q(s,t_0,\omega)\Psi_Q(t,s,\varphi(s,t_0,\omega))$, for all $(t,s,t_0,\omega)\in T\times\Omega$.

**Definition 1.3.4.** The pair $(C,P)$ is called a *dichotomic pair* if:

*(i)* the family of projectors $P$ is invariant for the stochastic skew-evolution semiflow $C=(\Phi,\varphi)$;

*(ii)* for all $(t,s,\omega)\in\Delta\times\Omega$, the stochastic evolution cocycle $\Phi(t,s,\omega)$ is an isomorphism from *Range* $P(s,\omega)$ to *Range* $P(t,\varphi(t,s,\omega))$;

*(iii)* for all $(t,s,\omega)\in\Delta\times\Omega$, the stochastic evolution cocycle $\Phi(t,s,\omega)$ is an isomorphism from *Range* $Q(s,\omega)$ to *Range* $Q(t,\varphi(t,s,\omega))$,

for all $(t,s,\omega)\in\Delta\times\Omega$.

**Remark 1.3.5.** The pair $(C,P)$ is called a dichotomic pair if and only if the following conditions are satisfied:

*(i)* for all $(t, s, \omega) \in \Delta \times \Omega$ there exists $\Psi : \Delta \times \Omega \to \mathcal{B}(X)$ such that $\Psi_P(t, s, \omega)$ is an isomorphism from Range $P(t, \varphi(t, s, \omega))$ to Range $P(s, \omega)$ and

$$\Phi(t, s, \omega)\Psi_P(t, s, \omega)P(t, \varphi(t, s, \omega)) = P(t, \varphi(t, s, \omega))$$

and

$$\Psi_P(t, s, \omega)\Phi(t, s, \omega)P(s, \omega) = P(s, \omega);$$

*(ii)* for all $(t, s, \omega) \in \Delta \times \Omega$ there exists $\Psi : \Delta \times \Omega \to \mathcal{B}(X)$ such that $\Psi_Q(t, s, \omega)$ is an isomorphism from Range $Q(t, \varphi(t, s, \omega))$ to Range $Q(s, \omega)$ and

$$\Phi(t, s, \omega)\Psi_Q(t, s, \omega)Q(t, \varphi(t, s, \omega)) = Q(t, \varphi(t, s, \omega))$$

and

$$\Psi_Q(t, s, \omega)\Phi(t, s, \omega)Q(s, \omega) = Q(s, \omega).$$

## 1.4 Growth concepts for pairs of the form $(C, P)$

**Definition 1.4.1.** [124] The pair $(C, P)$ is said to be with *uniform h-growth in mean (u.h.g.m.)* if there exist constants $M \geq 1$ and $\alpha > 0$ such that:

*$(uhg_1m)$* $$h(s)^\alpha \int_\Omega \|\Phi(t, t_0, \omega)P(t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq$$
$$\leq M h(t)^\alpha \int_\Omega \|\Phi(s, t_0, \omega)P(t_0, \omega)x_0(\omega)\| d\mu(\omega);$$

*$(uhg_2m)$* $$h(s)^\alpha \int_\Omega \|\Phi(s, t_0, \omega))Q(t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq$$
$$\leq M h(t)^\alpha \int_\Omega \|\Phi(t, t_0, \omega))Q(t_0, \omega)x_0(\omega)\| d\mu(\omega),$$
for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

**Definition 1.4.2.** [124] We say that the pair $(C, P)$ has *h-growth in mean* (hg.m.) if there exist constants $M \geq 1$, $\varepsilon \geq 0$ and $\alpha > 0$ such that:

*$(hg_1m)$* $$h(s)^\alpha \int_\Omega \|\Phi_P(t, s, \omega)x(\omega)\| d\mu(\omega) \leq M h(s)^\varepsilon h(t)^\alpha \int_\Omega \|P(s, \omega)x(\omega)\| d\mu(\omega);$$

*$(hg_2m)$* $$h(s)^\alpha \int_\Omega \|Q(s, \omega)x(\omega)\| d\mu(\omega) \leq M h(t)^\varepsilon h(t)^\alpha \int_\Omega \|\Phi_Q(t, s, \omega)x(\omega)\| d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

As specific cases we note that when the growth rate is $e^t$, this establishes the concept of *uniform exponential growth in mean* and if the growth rate is $t + 1$, then we arrive at the concept of *uniform polynomial growth in mean* respectively.

**Proposition 1.4.1.** [123] The following conditions hold true equivalently:

*(i)* The pair $(C, P)$ has u.h.g.m;

*(ii)* There are $M \geq 1, \alpha > 0$ with:

$$(uhg_1'm)\ \ h(s)^\alpha \int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq Mh(t)^\alpha \int_\Omega \|\Phi_P(s,t_0,\omega)x_0(\omega)\|d\mu(\omega);$$

$$(uhg_2'm)\ \ h(s)^\alpha \int_\Omega \|\Phi_Q(s,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq Mh(t)^\alpha \int_\Omega \|\Phi_Q(t,t_0,\omega)x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

*(iii)* There are $M \geq 1, \alpha > 0$ with:

$$(uhg_1''m)\ \ h(s)^\alpha \int_\Omega \|\Phi_P(t,s,\omega)x(\omega)\|d\mu(\omega) \leq Mh(t)^\alpha \int_\Omega \|P(s,\omega)x(\omega)\|d\mu(\omega);$$

$$(uhg_2''m)\ \ h(s)^\alpha \int_\Omega \|\Psi_Q(t,s,\omega)x(\omega)x(\omega)\|d\mu(\omega) \leq Mh(t)^\alpha \int_\Omega \|Q(t,\varphi(t,s,\omega))x(\omega)\|d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$;

*(iv)* There are $M \geq 1, \alpha > 0$ such that:

$$(uhg_1'''m)\ \ h(s)^\alpha \int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq Mh(t)^\alpha \int_\Omega \|\Phi_P(s,t_0,\omega)x_0(\omega)\|d\mu(\omega);$$

$$(uhg_2'''m)\ \ h(t_0)^\alpha \int_\Omega \|\Psi_Q(t,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(s)^\alpha \int_\Omega \|\Psi_Q(t,s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Remark 1.4.1.** [123] By taking $h(t) = e^t$ and, respectively, $h(t) = t + 1$ in Proposition 1.4.1, we obtain a characterization of uniform growth in mean for pairs of the form $(C, P)$ in the exponential case and, respectively, in the polynomial case.

The following theorem provides a characterization of the concept of uniform $h$-growth in mean in the case where $\Phi$ is a reversible stochastic evolution cocycle.

**Theorem 1.4.1.** [125] The pair $(C, P)$ is uniformly $h$-dichotomic in mean with $\Phi$ reversible stochastic evolution cocycle if and only if there exist $M > 1$ and $\alpha > 0$ with:

$$(uhg_1'''m)\qquad h(s)^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq M \cdot h(t)^\alpha \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega);$$

$$(uhg_2'''m) \qquad h(s)^\alpha \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq M \cdot h(t)^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$
for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

In the following, we present majorization criteria for the concepts of uniform growth in mean, using stochastic skew-evolution semiflows defined in Examples 1.1.6 and 1.1.7, as well as the associated families of projectors.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors for $C$. Let $h : \mathbb{R}_+ \to [1, \infty)$ be an invertible growth rate. We define

$$h_e : \mathbb{R}_+ \to \mathbb{R}_+, \quad h_e(t) = h^{-1}(e^t)$$

and

$$\varphi_e : \Delta \times \Omega \to \Omega, \quad \varphi_e(t,s,\omega) = \varphi(h_e(t), h_e(s), \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_e : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_e(t,s,\omega) = \Phi(h_e(t), h_e(s), \omega)$$

a stochastic evolution cocycle associated to $\varphi_e$, and

$$P_e : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_e(s,\omega) = P(h_e(s), \omega)$$

an invariant family of projectors for the stochastic skew-evolution semiflow $C_e$, for all $(t,s,\omega) \in \Delta \times \Omega$.

**Proposition 1.4.2.** The pair $(C, P)$ has uniform $h$-growth in mean if and only if the pair $(C_e, P_e)$ has uniform exponential growth in mean.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors for $C$. For an invertible growth rate $h : \mathbb{R}_+ \to [1, \infty)$, we consider

$$h_p : \mathbb{R}_+ \to \mathbb{R}_+, \quad h_p(t) = h^{-1}(t+1)$$

and

$$\varphi_p : \Delta \times \Omega \to \Omega, \quad \varphi_p(t,s,\omega) = \varphi(h_p(t), h_p(s), \omega)$$

a stochastic skew-evolution semiflow on $\Omega$,

$$\Phi_p : \Delta \times \Omega \to \Omega, \quad \Phi_p(t,s,\omega) = \Phi(h_p(t), h_p(s), \omega)$$

a stochastic evolution cocycle associated to $\varphi_p$,

$$P_p : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_p(s,\omega) = P(h_p(s), \omega)$$

an invariant family of projectors for the stochastic skew-evolution semiflow $C_p$, for all $(t,s,\omega) \in \Delta \times \Omega$.

**Proposition 1.4.3.** The pair $(C, P)$ has uniform $h$-growth in mean if and only if the dichotomic pair $(C_p, P_p)$ has uniform polynomial growth in mean.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors for $C$. We denote

$$\varphi_p^1 : \Delta \times \Omega \to \Omega, \quad \varphi_p^1(t, s, \omega) = \varphi(e^t - 1, e^s - 1, \omega)$$

a stochastic skew-evolution semiflow on $\Omega$,

$$\Phi_p^1 : \Delta \times \Omega \to \Omega, \quad \Phi_p^1(t, s, \omega) = \Phi(e^t - 1, e^s - 1, \omega)$$

a stochastic evolution cocycle associated to $\varphi_p^1$,

$$P_p^1 : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_p^1(s, \omega) = P(e^s - 1, \omega)$$

an invariant family of projectors for $C_p^1$, for all $(t, s, \omega) \in \Delta \times \Omega$.

**Proposition 1.4.4.** The pair $(C, P)$ has uniform polynomial growth in mean if and only if the pair $(C_p^1, P_p^1)$ has uniform exponential growth in mean.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors for $C$. We denote

$$\varphi_e^1 : \Delta \times \Omega \to \Omega, \quad \varphi_e^1(t, s, \omega) = \varphi(\ln(t + 1), \ln(s + 1), \omega)$$

a stochastic skew-evolution semiflow on $\Omega$,

$$\Phi_e^1 : \Delta \times \Omega \to \Omega, \quad \Phi_e^1(t, s, \omega) = \Phi(\ln(t + 1), \ln(s + 1), \omega)$$

a stochastic evolution cocycle associated to $\varphi_e^1$,

$$P_e^1 : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_e^1(s, \omega) = P(\ln(s + 1), \omega)$$

an invariant family of projectors for $C_e^1$, for all $(t, s, \omega) \in \Delta \times \Omega$.

**Proposition 1.4.5.** The pair $(C, P)$ has uniform exponential growth in mean if and only if the dichotomic pair $(C_e^1, P_e^1)$ has uniform polynomial growth in mean.

## *1.5 Bibliographic comments*

In the last decades, the research activity regarding the asymptotic behavior of stochastic evolution equations in infinite-dimensional spaces has known a significant intensification. Many results can be obtained not only for differential equations and evolution operators or evolution cocycles but also for stochastic skew-evolution semiflows.

In this chapter, we present the concepts of $h$-growth for stochastic skew-evolution semiflows $C = (\Phi, \varphi)$, as well as the growth concepts for pairs of the form $(C, P)$ in the uniform case. These notions are encountered throughout the thesis in the statements of theorems characterizing the uniform stable or dichotomic behavior of stochastic skew-evolution semiflows in Banach spaces.

In the first section of this chapter (*Stochastic skew-evolution semiflows*) we introduced the notions of stochastic evolution semiflow and stochastic skew-evolution semiflow, which represent generalizations of the classical concepts defined by L. Arnold in [2] and by G. Da Prato and J. Zabczyk in [33]. In the deterministic case, the notion of evolution cocycle was introduced by M. Megan and C. Stoica in [76]. The examples presented highlight the generalized character of the notion of stochastic skew-evolution semiflow compared to the classical notions of evolution cocycle or evolution operator. The original results of this section are represented by Examples 1.1.6, 1.1.7, 1.1.8, 1.1.9, which were published in [130].

Section 1.2 (*Uniform growth concepts for stochastic skew-evolution semiflows*) introduces the concepts of $h$-growth for stochastic skew-evolution semiflows $C = (\Phi, \varphi)$ in the uniform case. In the particular cases $h(t) = e^t$, respectively $h(t) = t + 1$, one obtains the concepts of exponential growth and polynomial growth for stochastic skew-evolution semiflows. As original results, we mention Definition 1.2.2 and Remark 1.2.1, published in [130], as well as Propositions 1.2.1, 1.2.2, 1.2.3, 1.2.4, 1.2.5, which were presented at the conference *Le XVe Colloque Franco-Roumain de Mathmatiques Appliques, 29 août – 2 septembre 2022, Toulouse, France* and constitute material for a scientific paper currently under preparation.

In the next section (*Invariant and strongly invariant stochastic families of projectors*), we present the concepts related to families of projectors (invariant, strongly invariant), as well as several results that will be used throughout the paper. Definition 1.3.4 introduces the concept of a dichotomic pair $(C, P)$, where $C$ is a stochastic skew-evolution semiflow and $P$ is a family of projectors invariant for $C$. Definition 1.3.4 and Remark 1.3.5 are original and represent results of ongoing research. The original results of this section include Definitions 1.3.1, 1.3.2, Remarks 1.3.1, 1.3.2, and Examples 1.3.1, 1.3.2, published in [124], as well as Remark 1.3.3 and Proposition 1.3.2, published in [125]. In [123], Definition 1.3.3, Remark 1.3.4, and Propositions 1.3.3, 1.3.4 were introduced, representing generalizations of results from [83].

Section 1.4 (*Uniform growth concepts for pairs of the form* $(C, P)$) presents the concepts of exponential growth, polynomial growth, and growth with respect to growth rates for pairs of the form $(C, P)$.

The original results are given by Definitions 1.4.1 and 1.4.2, which were published in [124]. Proposition 1.4.1 and Remark 1.4.1 provide characterizations of the concept of uniform $h$-growth in mean, as well as characterizations of uniform exponential growth in mean and uniform polynomial growth in mean, and were published in

[123].

Propositions 1.4.2, 1.4.3, 1.4.4, and 1.4.5 establish majorization criteria in the case of exponential and polynomial growth, based on the connection between uniform $h$-growth in mean and uniform exponential growth in mean, respectively between uniform $h$-growth in mean and uniform polynomial growth in mean. These propositions are original and are part of an ongoing study.

Another original result is Theorem 1.4.1, published in [125].

# 2. Stability concepts for stochastic skew-evolution semiflows

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, $X$ a real or complex Banach space. Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on $X$, $\varphi : \Delta \times \Omega \to \Omega$ a stochastic evolution semiflow on $\Omega$, $C = (\Phi, \varphi)$ a stochastic skew-evolution semiflow on $\Delta \times \Omega$, and let $h : \mathbb{R}_+ \to [1, \infty)$ be a growth rate, i.e., $h$ is a bijective and increasing function. Denote by $L(\Omega, X, \mu)$ the Banach space of all Bochner measurable functions.

## *2.1 Preliminaries*

**Definition 2.1.1.** [113] We say that the pair $C = (\Phi, \varphi)$ is *uniformly stable in mean (u.s.m.)* if there exists $N \geq 1$ such that

$$\int_\Omega \|\Phi(t, t_0, \omega)x_0(\omega)\| \, d\mu(\omega) \leq N \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| \, d\mu(\omega)$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Definition 2.1.2.** [130] The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is called *uniformly h-stable in mean (u.h.s.m.)* if there exist $N > 1$ and $\nu > 0$ such that

$$h(t)^\nu \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| \, d\mu(\omega) \leq N h(s)^\nu \int_\Omega \|x(\omega)\| \, d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Remark 2.1.1.** [130] The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exist $N > 1$ and $\nu > 0$ such that

$$h(t)^\nu \int_\Omega \|\Phi(t, t_0, \omega)x(\omega)\| \, d\mu(\omega) \leq N h(s)^\nu \int_\Omega \|\Phi(s, t_0, \omega)x(\omega)\| \, d\mu(\omega)$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x \in L(\Omega, X, \mu)$.

We note the following particular cases:

(i) for $h(t) = e^t$, one obtains the concept of *uniform exponential stability in mean (u.e.s.m.)*;

(ii) for $h(t) = t + 1$, one obtains the concept of *uniform polynomial stability in mean (u.p.s.m.)*.

**Remark 2.1.2.** [129] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean, then $C = (\Phi, \varphi)$ is uniformly stable in mean.

The converse implication does not hold, as illustrated by the following example.

**Example 2.1.1.** [129] Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator defined by

$$U(t, s)x = \frac{h(s)}{h(t)}x.$$

The mapping $\Phi_U : \Delta \times \Omega \to \mathcal{B}(X)$ defined by

$$\Phi_U(t, s, \omega) = U(t, s)$$

is a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ defined by

$$\varphi(t, s, \omega) = \frac{h(s)}{h(t)}x.$$

Then the pair $C = (\Phi_U, \varphi)$ is a stochastic skew-evolution semiflow which is uniformly stable in mean, but not uniformly $h$-stable in mean.

**Remark 2.1.3.** [129] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly stable in mean, then it has uniform $h$-growth in mean.

Indeed, from the fact that $C = (\Phi, \varphi)$ is uniformly stable in mean, it follows that there exists $M \geq 1$ such that

$$\int_\Omega \|\Phi(t, s, \omega)x(\omega)\| \, d\mu(\omega) \leq M \int_\Omega \|x(\omega)\| \, d\mu(\omega) \leq M \left(\frac{h(t)}{h(s)}\right)^\alpha \int_\Omega \|x(\omega)\| \, d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Thus, $C = (\Phi, \varphi)$ has uniform $h$-growth in mean.

The converse implication does not hold, as illustrated by the following example:

**Example 2.1.2.** [129] Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator defined by

$$U(t, s)x = \frac{h(t)}{h(s)}x.$$

Then the mapping $\Phi_U : \Delta \times \Omega \to \mathcal{B}(X)$ defined by

$$\Phi_U(t, s, \omega) = U(t, s)$$

is a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ defined by

$$\varphi(t, s, \omega) = \frac{h(t)}{h(s)} x.$$

Hence, the pair $C = (\Phi_U, \varphi)$ is a stochastic skew-evolution semiflow which has uniform $h$-growth in mean, but is not uniformly stable in mean.

**Proposition 2.1.1.** The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is

*(i) uniformly stable in mean* if and only if there exists $N \geq 1$ such that

$$\int_\Omega \|\Phi(t, t_0, \omega)x_0(\omega)\| \, d\mu(\omega) \leq N \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| \, d\mu(\omega)$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

*(ii) uniformly exponentially stable in mean* if and only if there exist constants $N \geq 1$ and $\nu > 0$ such that

$$\int_\Omega \|\Phi(t, t_0, \omega)x_0(\omega)\| \, d\mu(\omega) \leq N e^{-\nu(t-s)} \int_\Omega \|\Phi(s, t_0, \omega)x_0(\omega)\| \, d\mu(\omega)$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Remark 2.1.4.** [129] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly exponentially stable in mean, then it is uniformly stable in mean.

Indeed, from the fact that $C = (\Phi, \varphi)$ is uniformly exponentially stable (u.e.s.m.), it follows that there exist $N \geq 1$, $\nu > 0$ such that

$$\int_\Omega \|\Phi(t, s, \omega)x(\omega)\| \, d\mu(\omega) \leq N e^{-\nu(t-s)} \int_\Omega \|x(\omega)\| \, d\mu(\omega) \leq N \int_\Omega \|x(\omega)\| \, d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

In the following example, it can be observed that the converse implication is not true.

**Example 2.1.3.** [100] Let $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ be an increasing and bounded function. The mapping $\varphi : \Delta \times \Omega \to \Omega$, defined by

$$\varphi(t, s, \omega) = t - s + \omega,$$

is a stochastic evolution semiflow on $\mathbb{R}_+$ for all $(t, s, \omega) \in \Delta \times \Omega$, and the mapping $\Phi : \Delta \times \Omega \to \mathcal{B}(\mathbb{R})$, defined by

$$\Phi(t, s, \omega) = \frac{f(\omega)}{f(t - s + \omega)},$$

is a stochastic evolution cocycle on $\mathbb{R}$ for all $(t,s,\omega)\in\Delta\times\Omega$.

It is easy to verify that $C=(\Phi,\varphi)$ is uniformly stable with $N=1$. On the other hand, assume by contradiction that there exist constants $N\geq 1$ and $\nu>0$ such that

$$\frac{f(\omega)}{f(t-s+\omega)}\leq Ne^{-\nu(t-s)}$$

for all $(t,s,\omega)\in\Delta\times\Omega$.

Taking $s=0$, we obtain

$$\frac{e^{\nu t}}{f(t+\omega)}\leq\frac{N}{f(\omega)},$$

which leads to a contradiction as $t\to\infty$. Therefore, $C=(\Phi,\varphi)$ is not uniformly exponentially stable in mean.

## 2.2 Connections between the concepts of uniform stability in mean

In the next theorem we present the connection between the concept of uniform $h$-stability in mean and the concept of uniform exponential stability in mean.

Let $C=(\Phi,\varphi)$ be a stochastic skew-evolution semiflow on $\Delta\times\Omega$ and let $C_e=(\Phi_e,\varphi_e)$ be the stochastic skew-evolution semiflow defined in Example 1.1.6.

**Theorem 2.2.1.** [130] The stochastic skew-evolution semiflow $C=(\Phi,\varphi)$ is uniformly $h$-stable in mean if and only if the stochastic skew-evolution semiflow $C_e=(\Phi_e,\varphi_e)$ is uniformly exponentially stable in mean.

The connection between uniform polynomial stability in mean and uniform exponential stability in mean is given by the following corollary.

Let $C=(\Phi,\varphi)$ be a stochastic skew-evolution semiflow on $\Delta\times\Omega$ and let $C_p^1=(\Phi_p^1,\varphi_p^1)$ be the stochastic skew-evolution semiflow defined in Example 1.1.8.

**Corollary 2.2.1.** [130] The stochastic skew-evolution semiflow $C=(\Phi,\varphi)$ is uniformly polynomially stable in mean if and only if the stochastic skew-evolution semiflow $C_p^1=(\Phi_p^1,\varphi_p^1)$ is uniformly exponentially stable in mean.

The next theorem presents the connection between uniform $h$-stability in mean and uniform polynomial stability in mean.

Let $C=(\Phi,\varphi)$ be a stochastic skew-evolution semiflow on $\Delta\times\Omega$ and let $C_p=(\Phi_p,\varphi_p)$ be the stochastic skew-evolution semiflow defined in Example 1.1.7.

**Theorem 2.2.2.** [130] The stochastic skew-evolution semiflow $C=(\Phi,\varphi)$ is uniformly $h$-stable in mean if and only if the stochastic skew-evolution semiflow $C_p=(\Phi_p,\varphi_p)$ is uniformly polynomially stable in mean.

A particular case of the previous theorem is the connection between uniform exponential stability in mean and uniform polynomial stability in mean.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $C_e^1 = (\Phi_e^1, \varphi_e^1)$ be the stochastic skew-evolution semiflow defined in Example 1.1.9.

**Corollary 2.2.2.** [130] The stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly exponentially stable in mean if and only if the stochastic skew-evolution semiflow $C_e^1 = (\Phi_e^1, \varphi_e^1)$ is uniformly polynomially stable in mean.

**Remark 2.2.1.** The connection between stability concepts and growth concepts is illustrated in the following diagrams:

$$u.h.s.m. \;\Rightarrow\; u.h.g.m.$$

and

$$\begin{array}{ccc} u.e.s.m. & \Rightarrow & u.p.s.m. \\ \Downarrow & & \Downarrow \\ u.e.g.m. & \Leftarrow & u.p.g.m. \end{array}$$

## 2.3 Characterizations of the concept of uniform stability in mean

An initial characterization of the concept of uniform stability in mean for stochastic skew-evolution semiflows is given by

**Theorem 2.3.1.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform exponential growth in mean, then $C = (\Phi, \varphi)$ is uniformly exponentially stable in mean if and only if there exist $r > 1$ and $c \in (0, 1)$ such that

$$\int_\Omega \|\Phi(r+s, s, \omega)x(\omega)\| d\mu(\omega) \leq c \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.1.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform polynomial growth in mean, then $C = (\Phi, \varphi)$ is uniformly polynomially stable in mean if and only if there exist constants $r > e - 1$ and $c \in (0, 1)$ such that

$$\int_\Omega \|\Phi(rs+r+s, s, \omega)x(\omega)\| d\mu(\omega) \leq c \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.2.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform $h$-growth in mean, then $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exist constants $r > e$ and $c \in (0, 1)$ such that

$$\int_\Omega \|\Phi(h^{-1}(rs), h^{-1}(s), \omega)x(\omega)\| d\mu(\omega) \leq c \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Another characterization of uniform stability in mean is given by

**Theorem 2.3.2.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform polynomial growth in mean, then $C = (\Phi, \varphi)$ is uniformly polynomially stable in mean if and only if there exists $L > 1$ such that

$$\int_\Omega \|\Phi(t,s,\omega)x(\omega)\| d\mu(\omega) \ \ln \frac{t+1}{s+1} \leq L \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.3.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform $h$-growth in mean, then $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exists $L > 1$ such that

$$\int_\Omega \|\Phi(t,s,\omega)x(\omega)\| d\mu(\omega) \ln \frac{h(t)}{h(s)} \leq L \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.4.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform exponential growth in mean, then $C = (\Phi, \varphi)$ is uniformly exponentially stable in mean if and only if there exists $L > 1$ such that

$$(t-s) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| d\mu(\omega) \leq L \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Theorem 2.3.3.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform exponential growth in mean, then $C = (\Phi, \varphi)$ is uniformly exponentially stable in mean if and only if there exist $M > 1$ and an increasing function $g : [1, \infty) \to \mathbf{R}_+$ with $\lim\limits_{t\to\infty} g(t) = \infty$ such that

$$g(t-s) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| d\mu(\omega) \leq M \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.5.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform polynomial growth in mean, then $C = (\Phi, \varphi)$ is uniformly polynomially stable in mean if and only if there exist $M > 1$ and an increasing function $g : [1, \infty) \to \mathbf{R}_+$ with $\lim\limits_{t\to\infty} g(t) = \infty$ such that

$$g\left(\frac{t+1}{s+1}\right) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| d\mu(\omega) \leq M \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Corollary 2.3.6.** [130] If the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ has uniform $h$-growth in mean, then $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exist $M > 1$ and an increasing function $g : [1, \infty) \to \mathbf{R}_+$ with $\lim_{t\to\infty} g(t) = \infty$ such that

$$g\left(\frac{h(t)}{h(s)}\right) \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| d\mu(\omega) \leq M \int_\Omega \|x(\omega)\| d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

For Datko-type characterizations, we will use the following class of functions introduced in [123].

Let $\mathcal{H}$ be the set of growth rates $h : \mathbb{R}_+ \to [1, \infty)$ which satisfy the following conditions:

*($h_1$)* there exists $H > 1$ such that $h(t + 1) \leq Hh(t)$, for all $t \geq 0$;

*($h_2$)* for every $\beta < 0$ there exists $H_1 > 1$ such that

$$\int_s^\infty h(t)^\beta dt \leq H_1\, h(s)^\beta, \quad \forall\, s \geq 0;$$

*($h_3$)* for every $\beta > 0$ there exists $H_2 > 1$ such that

$$\int_0^t h(s)^\beta ds \leq H_2\, h(t)^\beta, \quad \forall\, t \geq 0.$$

**Remark 2.3.1.** [123] If $h(t) = e^t$, then $h \in \mathcal{H}$.

The following theorems are of Datko type for the concept of uniform $h$-stability in mean for stochastic skew-evolution semiflows.

**Theorem 2.3.4.** Let $h \in \mathcal{H}$ and $C = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow with uniform $h$-growth. Then $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exist constants $D > 1$ and $d \in (0, 1)$ such that

$$\int_s^\infty h(t)^{d-1} \left( \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| d\mu(\omega) \right) dt \leq Dh(s)^d \int_\Omega \|x(\omega)\| d\mu(\omega), \qquad (2.1)$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Theorem 2.3.5.** Let $h \in \mathcal{H}$ and $C = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow with uniform $h$-growth in mean. Then $C = (\Phi, \varphi)$ is uniformly $h$-stable in mean if and only if there exist $D > 1$ and $d \in (0, 1)$ such that

$$\int_s^\infty h(t)^d \left( \int_\Omega \|\Phi(t, s, \omega)x(\omega)\| d\mu(\omega) \right) dt \leq D \cdot h(s)^d \int_\Omega \|x(\omega)\| d\mu(\omega), \tag{2.2}$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Theorem 2.3.6.** Let $h \in \mathcal{H}$ and $C = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow with uniform $h$-growth. Then $C = (\Phi, \varphi)$ is uniformly $h$-stable if and only if there exist $D > 1$ and $d \in (0, 1)$ such that

$$\int_{t_0}^t (h(s)^d)^{-1} \left( \int_\Omega \|\Phi(s, t_0, \omega)x(\omega)\| d\mu(\omega) \right)^{-1} ds \leq D(h(t)^d)^{-1} \left( \int_\Omega \|\Phi(t, t_0, \omega)x(\omega)\| d\mu(\omega) \right)^{-1}, \tag{2.3}$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x \in L(\Omega, X, \mu)$.

We now introduce the notion of a first type $h$-Lyapunov function:

**Definition 2.3.1.** A function $L_1 : \Delta \times \Omega \times L(\Omega, X, \mu) \to \mathbb{R}_+$ is called a *Lyapunov h-function of first type* for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ if there exists a constant $D > 1$ such that the following conditions are satisfied:

*($hL_1$)* $L_1(s, t_0, \omega, x(\omega)) \leq D \int_\Omega \|\Phi(s, t_0, \omega)x(\omega)\| d\mu(\omega)$,
for all $(s, t_0, \omega) \in \Delta \times \Omega$ and all $x \in L(\Omega, X, \mu)$;

*($hL_2$)* $L_1(t, t_0, \omega, x(\omega)) + \int_s^t \frac{h(\tau)^d}{h(s)^d} \left( \int_\Omega \|\Phi(\tau, t_0, \omega)x(\omega)\| d\mu(\omega) \right) d\tau = L_1(s, t_0, \omega, x(\omega))$,
for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x \in L(\Omega, X, \mu)$.

The following result provides a characterization of uniform $h$-stability in mean using a Lyapunov $h$-function of first type:

**Theorem 2.3.7.** A strongly measurable stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ with uniform $h$-growth is uniformly $h$-stable in mean if and only if there exists a first type $h$-Lyapunov function for $C = (\Phi, \varphi)$.

**Definition 2.3.2.** A function $L_2 : \Delta \times \Omega \times L(\Omega, X, P) \to \mathbb{R}_+$ is called a *Lyapunov h-function of second type* for the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ if there exists a constant $D > 1$ such that the following conditions are satisfied:

($hL_1'$) $L_2(t,t_0,\omega,x(\omega)) \leq D\left(\int_\Omega \|\Phi(t,t_0,\omega)x(\omega)\|d\mu(\omega)\right)^{-1}$, for all $(t,t_0,\omega)\in\Delta\times\Omega$ and $x\in L(\Omega,X,\mu)$;

($hL_2'$) $L_2(s,t_0,\omega,x(\omega))+\int\limits_s^t \left(\frac{h(t)}{h(\tau)}\right)^d \left(\int_\Omega \|\Phi(\tau,t_0,\omega)x(\omega)\|d\mu(\omega)\right)^{-1} d\tau = L_2(t,t_0,\omega,x(\omega))$, for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x\in L(\Omega,X,\mu)$.

The following result provides a characterization of uniform $h$-stability in mean in terms of a Lyapunov $h$-function of second type.

**Theorem 2.3.8.** A strongly measurable stochastic skew-evolution semiflow $C = (\Phi,\varphi)$ with uniform $h$-growth in mean is uniformly $h$-stable in mean if and only if there exists a second type $h$-Lyapunov function for $C=(\Phi,\varphi)$.

For Barbashin-type characterizations, we use the following classes of functions introduced in [19]:

(i) $\mathcal{H}$ is the set of all functions $h:\mathbb{R}_+\to[1,\infty)$ with the property that there exists $H>1$ such that $h(t+1)\leq Hh(t)$, for all $t\geq 0$.

(ii) $\mathcal{H}_1$ is the set of all functions $h:\mathbb{R}_+\to[1,\infty)$ with the property that there exists $H_1>1$ such that $h(h(t))\leq H_1h(t)$, for all $t\geq 0$.

(iii) $\mathcal{H}_B^1$ is the set of all functions $h:\mathbb{R}_+\to[1,\infty)$ with the property that for all $\alpha\in(0,1)$, there exists $H_1>1$ such that $\int_0^t h(s)^{\alpha-1}\,ds \leq H_1h(t)^\alpha$, for all $t\geq 0$.

(iv) $\mathcal{H}_B^2$ is the set of all functions $h:\mathbb{R}_+\to[1,\infty)$ with the property that for all $\alpha\in(0,1)$, there exists $H_2>1$ such that $\int_0^t h(s)^\alpha\,ds\leq H_2h(t)^\alpha$, for all $t\geq 0$.

**Remark 2.3.2.** The classes of functions $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_B^1$, $\mathcal{H}_B^2$ introduced above represent a generalization of exponential and polynomial growth rates. Moreover, they also allow a characterization of $*$-strongly measurable cocycles with growth rates $h\in\mathcal{H}_1\cap\mathcal{H}_B^1$, which will be presented in the following theorem.

**Remark 2.3.3.** [19] If $e$ is an exponential function, then

$$e\in\mathcal{H}\cap\mathcal{H}_B^2\subset\mathcal{H}\cap\mathcal{H}_B^1.$$

**Remark 2.3.4.** [19] If $p$ is a polynomial function, then

$$p\in(\mathcal{H}\cap\mathcal{H}_1)\cup\left(\mathcal{H}_B^1\setminus\mathcal{H}_B^2\right).$$

For a detailed discussion regarding the existence of these integrals and the properties of the considered classes of functions, see Section 3 in [19].

**Theorem 2.3.9.** [129] Let $h \in \mathcal{H}_1 \cap \mathcal{H}_B^1$ and let $C = (\Phi, \varphi)$ be a $*$-strongly measurable skew-evolution semiflow with uniform $h$-growth. Then $C = (\Phi, \varphi)$ is uniformly $h$-stable if and only if there exist $B > 1$ and $b \in (0, 1)$ such that

$$\int_0^t h(s)^{-(b+1)} \left( \int_\Omega \|\Phi(t, s, \varphi(s, t_0, \omega))^* x^*(\omega)\| d\mu(\omega) \right) ds \leq h(t)^b B \int_\Omega \|x^*(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, x^*) \in T \times \Omega$ and $x^* \in L(\Omega, X, \mu)$.

**Remark 2.3.5.** The statement in Remark 2.3.2 also holds for growth rates $h \in \mathcal{H} \cap \mathcal{H}_B^2$.

**Theorem 2.3.10.** [129] Let $h \in \mathcal{H} \cap \mathcal{H}_B^2$ and let $C = (\Phi, \varphi)$ be a $*-$strongly measurable stochastic skew-evolution semiflow with uniform $h$-growth. Then $C = (\Phi, \varphi)$ is uniformly $h$-stable if and only if there exist $B > 1$ and $b \in (0, 1)$ such that

$$\int_0^t h(s)^{-(b+1)} \left( \int_\Omega \|\Phi(t, s, \varphi(s, t_0, \omega))^* x^*(\omega)\| d\mu(\omega) \right) ds \leq h(t)^b B \int_\Omega \|x^*(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, x^*) \in T \times \Omega$ and $x^* \in L(\Omega, X, \mu)$.

## 2.4 Bibliographical comments

The study of differential equations, both in finite-dimensional and infinite-dimensional spaces, has generated over time a rich literature concerning the concept of uniform exponential stability. This concept was introduced in 1930 by O. Perron [88], in the context of the stability analysis of the system

$$\dot{x}(t) = A(t)x(t) + f(t, x)$$

in an infinite-dimensional framework. Later, in 1972, R. Datko [37] provided an integral characterization of uniform exponential stability for evolution operators, under the assumption of uniform exponential growth. This contribution proved to be an essential starting point for a large series of subsequent investigations in the field.

In the first section of this chapter (*Preliminaries*), emphasis is placed on the definition of uniform $h$-stability in mean, and several significant examples are presented in this context. Some of these results were published in [130] and represent the original contributions of this section: Definition 2.1.2 and Remarks 2.1.1.

Other original results include Remarks 2.1.2, 2.1.3, 2.1.4, Proposition 2.1.1, and Examples 2.1.1, 2.1.2, which were presented at the conference *The 20th International Conference on Numerical Analysis and Applied Mathematics, September 19–25, 2022, Heraklion, Crete, Greece* and published in [129] in the context of exponential stability in mean.

In the next paragraph (*Connections between the concepts of uniform stability in mean*), the connections are established between the concepts of uniform $h$-stability in mean and uniform exponential stability in mean, Theorem 2.2.1, and between uniform $h$-stability in mean and uniform polynomial stability in mean, Theorem 2.2.2. Corollaries 2.2.1 and 2.2.2 represent particular cases of the above theorems. These theorems and corollaries represent the original results of this paragraph and were published in [130].

The last paragraph (*Characterizations of the concept of uniform stability in mean*) is dedicated to the characterization of uniform stability in mean. The study of uniform stability in mean is carried out with growth rates and represents a generalization of the concepts of uniform exponential stability in mean and uniform polynomial stability in mean. The function $h : \mathbb{R}_+ \to [1, \infty)$ is called a growth rate ($h$ is a bijective and increasing function). This concept was mentioned for the first time in the paper of M. Pinto [89]. The characterization of uniform $h$-stability in mean is carried out by logarithmic-type theorems, majorization theorems, Hai-type results, Datko-type theorems, Lyapunov-type theorems, and Barbashin-type theorems. These are generalizations of Datko-, Lyapunov-, and Barbashin-type theorems published in the works of R. Borug, M. Megan, and D.M.M. Toth [19], and C.L. Mihi [81].

The original results of this section are given by Theorems 2.3.1, 2.3.2, 2.3.3 and Corollaries 2.3.1, 2.3.2, 2.3.3, 2.3.4, 2.3.5, 2.3.6, published in [130].

Theorems 2.3.4, 2.3.5 and 2.3.6 were presented at the conference *Le XVe Colloque Franco-Roumain de Mathmatiques Appliques, 29 août – 2 septembre 2022, Toulouse, France* and will be the subject of a scientific paper currently under preparation.

Furthermore, Definitions 2.3.1, 2.3.2 and Theorems 2.3.7, 2.3.8 constitute original results, presented in the context of the study of stability of evolution cocycles at the conference *The 28th International Conference in Operator Theory, June 27 – July 1, 2022, Timioara, Romania.*

The particular cases of Theorems 2.3.9 and 2.3.10 were published in [129].

# 3. Dichotomy concepts for stochastic skew-evolution semiflows

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $X$ be a real or complex Banach space.

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on $X$, $\varphi : \Delta \times \Omega \to \Omega$ an evolution semiflow on $\Omega$, $C = (\Phi, \varphi)$ a stochastic skew-evolution semiflow on $\Delta \times \Omega$, and let $h : \mathbb{R}_+ \to [1, \infty)$ be a growth rate, that is, $h$ is a bijective and increasing function. Denote by $L(\Omega, X, \mu)$ the Banach space of all Bochner measurable functions.

We consider the pair $(C, P)$, where $C = (\Phi, \varphi)$ is a stochastic skew-evolution semiflow and $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a family of projectors invariant for $C$.

## *3.1 Preliminaries*

**Definition 3.1.1.** [125] We say that the pair $(C, P)$ is *uniformly dichotomic in mean (u.d.m.)* if there exists $N \geq 1$ such that

*($ud_1m$)* $$\int_\Omega \|\Phi_P(t,s,\omega)x(\omega)\| d\mu(\omega) \leq N \int_\Omega \|P(s,\omega)x(\omega)\| d\mu(\omega);$$

*($ud_2m$)* $$\int_\Omega \|Q(s,\omega)x(\omega)\| d\mu(\omega) \leq N \int_\Omega \|\Phi_Q(t,s,\omega)x(\omega)\| d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Definition 3.1.2.** [124] We say that the pair $(C, P)$ is *uniformly h-dichotomic in mean (u.h.d.m.)* if there exist $N \geq 1$ and $\nu > 0$ such that

*($uhd_1m$)* $$h(t)^\nu \int_\Omega \|\Phi_P(t,s,\omega)x(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|P(s,\omega)x(\omega)\| d\mu(\omega);$$

*($uhd_2m$)* $$h(t)^\nu \int_\Omega \|Q(s,\omega)x(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi_Q(t,s,\omega)x(\omega)\| d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

In particular,

(i) for $h(t) = e^t$ we obtain the concept of uniform exponential dichotomy in mean (*u.e.d.m.*);

(ii) for $h(t) = t + 1$ we obtain the concept of uniform polynomial dichotomy in mean (*u.p.d.m.*).

**Remark 3.1.1.** The pair $(C, P)$ is *uniformly h-dichotomic in mean* if and only if there exist $N \geq 1$ and $\nu > 0$ such that

$$(uhd_1'm)\ \ h(t)^\nu \int_\Omega \|\Phi_P(t, t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi_P(s, t_0, \omega)x_0(\omega)\| d\mu(\omega);$$

$$(uhd_2'm)\ \ h(t)^\nu \int_\Omega \|\Phi_Q(s, t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi_Q(t, t_0, \omega)x(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in \Delta \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

## 3.2 Connection between the concepts of uniform dichotomy in mean

The connection between uniform $h$-dichotomy in mean and uniform exponential dichotomy in mean is established below:

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$, and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors to $C$. For an invertible growth rate $h : \mathbb{R}_+ \to [1, \infty)$, we consider

$$h_e : \mathbb{R}_+ \to \mathbb{R}_+, \quad h_e(t) = h^{-1}(e^t)$$

and

$$\varphi_e : \Delta \times \Omega \to \Omega, \quad \varphi_e(t, s, \omega) = \varphi(h_e(t), h_e(s), \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_e : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_e(t, s, \omega) = \Phi(h_e(t), h_e(s), \omega)$$

a stochastic evolution cocycle associated to $\varphi_e$,

$$P_e : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_e(s, \omega) = P(h_e(s), \omega)$$

an invariant family of projectors to $C_e$,

for all $(t, s, \omega) \in \Delta \times \Omega$.

**Theorem 3.2.1.** [128] The pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if the pair $(C_e, P_e)$ is uniformly exponentially dichotomic in mean.

The connection between uniform $h$-dichotomy in mean and uniform polynomial dichotomy in mean is presented below:

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors to $C$. For an invertible growth rate $h : \mathbf{R}_+ \to [1, \infty)$, we consider

$$h_p : \mathbb{R}_+ \to \mathbb{R}_+, \quad h_p(t) = h^{-1}(t+1)$$

and define

$$\varphi_p : \Delta \times \Omega \to \Omega, \quad \varphi_p(t,s,\omega) = \varphi(h_p(t), h_p(s), \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_p : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_p(t,s,\omega) = \Phi(h_p(t), h_p(s), \omega)$$

a stochastic evolution cocycle associated to $\varphi_p$,

$$P_p : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_p(s,\omega) = P(h_p(s), \omega)$$

an invariant family of projectors to $C_p$,
for all $(t,s,\omega) \in \Delta \times \Omega$.

**Theorem 3.2.2.** [128] The pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if the pair $(C_p, P_p)$ is uniformly polynomially dichotomic in mean.

The connection between uniform polynomial dichotomy in mean and uniform exponential dichotomy in mean is given below.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors to $C$. We denote

$$\varphi_p^1 : \Delta \times \Omega \to \Omega, \quad \varphi_p^1(t,s,\omega) = \varphi(e^t - 1, e^s - 1, \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_p^1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_p^1(t,s,\omega) = \Phi(e^t - 1, e^s - 1, \omega)$$

a stochastic evolution cocycle associated to $\varphi_p^1$,

$$P_p^1 : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_p^1(s,\omega) = P(e^s - 1, \omega)$$

an invariant family of projectors to $C_p^1$,
for all $(t,s,\omega) \in \Delta \times \Omega$.

**Corollary 3.2.1.** The pair $(C, P)$ is uniformly polynomially dichotomic in mean if and only if the pair $(C_p^1, P_p^1)$ is uniformly exponentially dichotomic in mean.

The connection between uniform exponential dichotomy in mean and uniform polynomial dichotomy in mean is presented below.

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and let $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ be an invariant family of projectors to $C$. We denote

$$\varphi_e^1 : \Delta \times \Omega \to \Omega, \quad \varphi_e^1(t, s, \omega) = \varphi(\ln(t+1), \ln(s+1), \omega)$$

a stochastic evolution semiflow on $\Omega$,

$$\Phi_e^1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_e^1(t, s, \omega) = \Phi(\ln(t+1), \ln(s+1), \omega)$$

a stochastic evolution cocycle associated to $\varphi_e^1$,

$$P_e^1 : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X), \quad P_e^1(s, \omega) = P(\ln(s+1), \omega)$$

an invariant family of projectors to $C_e^1$,
for all $(t, s, \omega) \in \Delta \times \Omega$.

**Corollary 3.2.2.** The pair $(C, P)$ is uniformly exponentially dichotomic in mean if and only if the pair $(C_e^1, P_e^1)$ is uniformly polynomially dichotomic in mean.

## *3.3 Characterization of the concept of uniform dichotomy in mean*

The following proposition provides characterizations of the concept of uniform dichotomy in mean with invariant families of projectors and with strongly invariant families of projectors.

**Proposition 3.3.1.** [123] The following statements are equivalent:

*(i)* The pair $(C, P)$ is u.h.d.m.;

*(ii)* There exist $N \geq 1,\ \nu > 0$ such that:

$$(uhd_1'm)\ \ h(t)^\nu \int_\Omega \|\Phi_P(t, t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq N h(s)^\nu \int_\Omega \|\Phi_P(s, t_0, \omega)x_0(\omega)\| d\mu(\omega);$$

$$(uhd_2'm)\ \ h(t)^\nu \int_\Omega \|\Phi_Q(s, t_0, \omega)x_0(\omega)\| d\mu(\omega) \leq N h(s)^\nu \int_\Omega \|\Phi_Q(t, t_0, \omega)x_0(\omega)\| d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

*(iii)* There exist $N \geq 1,\ \nu > 0$ such that:

$$(uhd_1''m)\ \ h(t)^\nu \int_\Omega \|\Phi_P(t, s, \omega)x(\omega)\| d\mu(\omega) \leq N h(s)^\nu \int_\Omega \|P(s, \omega)x(\omega)\| d\mu(\omega);$$

$(uhd_2''m)$ $h(t)^\nu \int_\Omega \|\Psi_Q(t,s,\omega)x(\omega)\|d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|Q(t,\varphi(t,s,\omega))x(\omega)\|d\mu(\omega),$

for all $(t,s,\omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$;

*(iv)* There exist $N \geq 1,\ \nu > 0$ such that:

$(uhd_1'''m)$ $h(t)^\nu \int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq Nh(s)^\nu \int_\Omega \|\Phi_P(s,t_0,\omega)x_0(\omega)\|d\mu(\omega);$

$(uhd_2'''m)$ $h(s)^\nu \int_\Omega \|\Psi_Q(t,t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq Nh(t_0)^\nu \int_\Omega \|\Psi_Q(t,s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Remark 3.3.1.** [123] The particular cases result by taking $h(t) = e^t$ and $h(t) = t+1$ in Proposition 3.3.1. We obtain a characterization of uniform dichotomy in mean for pairs of the form $(C, P)$ for the exponential case and for the polynomial case.

The following theorem provides a characterization for the concept of uniform $h$-dichotomy in mean in the case when $\Phi$ is a reversible stochastic evolution cocycle.

**Theorem 3.3.1.** [125] The pair $(C, P)$ is $h$-uniformly dichotomic in mean with $\Phi$ a reversible stochastic evolution cocycle if and only if there exist $N > 1$ and $\nu > 0$ such that:

$$(uhd_1^r m)\ \ h(t)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Nh(s)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

$$(uhd_2^r m)\ \ h(t)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Nh(s)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

The following theorem provides a characterization for the concept of uniform $h$-dichotomy in mean with a logarithmic criterion, a majorization criterion and a Hai-type criterion.

**Theorem 3.3.2.** [124] If the pair $(C, P)$ has uniform $h$-growth in mean, then the following statements are equivalent:

*(1)* $(C, P)$ is uniformly $h$-dichotomic in mean;

*(2)* there exists a constant $L > 1$ such that

$$(uhl_1m)\ \ \int_\Omega \|\Phi_P(t,s,\omega)x(\omega)\|d\mu(\omega) \ln \frac{h(t)}{h(s)} \leq L \int_\Omega \|P(s,\omega)x(\omega)\|d\mu(\omega)$$

$$(uhl_2m)\ \int_\Omega \|Q(s,\omega)x(\omega)\|d\mu(\omega)\ln\frac{h(t)}{h(s)} \leq L\int_\Omega \|\Phi_Q(t,s,\omega)x(\omega)\|d\mu(\omega),$$
for all $(t,s,\omega)\in\Delta\times\Omega$ and $x\in L(\Omega,X,\mu)$.

*(3)* there exists a strictly increasing bijective function $g:[1,\infty)\to\mathbf{R}_+$ with $\lim_{t\to\infty} g(t)=\infty$, $g(e)=1$, and a constant $L>1$ such that:

$$(uhM_1m)\ g\left(\frac{h(t)}{h(s)}\right)\int_\Omega \|\Phi_P(t,s,\omega)x(\omega)\|d\mu(\omega) \leq L\int_\Omega \|P(s,\omega)x(\omega)\|d\mu(\omega)$$

$$(uhM_2m)\ g\left(\frac{h(t)}{h(s)}\right)\int_\Omega \|Q(s,\omega)x(\omega)\|d\mu(\omega) \leq L\int_\Omega \|\Phi_Q(t,s,\omega)x(\omega)\|d\mu(\omega),$$
for all $(t,s,\omega)\in\Delta\times\Omega$ and $x\in L(\Omega,X,\mu)$.

*(4)* there exist $r>e$ and $c\in(0,1)$ such that

$$(uhH_1m)\ \int_\Omega \|\Phi_P(rs,s,\omega)x(\omega)\|d\mu(\omega) \leq c\int_\Omega \|P(s,\omega)x(\omega)\|d\mu(\omega)$$

$$(uhH_2m)\ \int_\Omega \|Q(s,\omega)x(\omega)\|d\mu(\omega) \leq c\int_\Omega \|\Phi_Q(rs,s,\omega)x(\omega)\|d\mu(\omega),$$
for all $(s,\omega)\in\mathbb{R}_+\times\Omega$ and $x\in L(\Omega,X,\mu)$.

**Remark 3.3.2.** [124] For $h(t)=e^t$ we obtain the version of this theorem for the concept of uniform exponential dichotomy, which represents generalizations of results presented by C. Stoica in [100].

**Remark 3.3.3.** [124] For $h(t)=t+1$ we obtain the version of this theorem for the concept of uniform polynomial dichotomy, which was proved by R. Boruga and M. Megan in [17], and by R. Boruga in [12] for evolution operators in Banach spaces.

**Remark 3.3.4.** The connection between the concepts of dichotomy and growth is given in the following diagram:

$$u.h.d.m. \ \Rightarrow\ u.h.g.m.$$

and

$$\begin{array}{ccc} u.e.d.m & \Rightarrow & u.p.d.m \\ \Downarrow & & \Downarrow \\ u.e.g.m. & \Leftarrow & u.p.g.m. \end{array}$$

In what follows, let $h\in\mathcal{H}$, where the class $\mathcal{H}$ is defined in 2.3.

The following theorems provide integral characterizations in terms of families of invariant projectors and, respectively, strongly invariant families of projectors.

**Theorem 3.3.3.** [123] Assume that $C=(\Phi,\varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C,P)$ has uniform $h$-growth in mean and $h\in\mathcal{H}$. The pair $(C,P)$ is uniformly $h$-dichotomic in mean if and only if there exist constants $D\geq 1$ and $d\in(0,1)$ such that

$(uhD_1^1m_c)$ $\displaystyle\int_s^\infty h(t)^d \left(\int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\| d\mu(\omega)\right) dt \leq$

$\leq D\ h(s)^d \left(\int_\Omega \|\Phi_P(s,t_0,\omega)x_0(\omega)\| d\mu(\omega)\right);$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

$(uhD_1^2m_c)$ $\displaystyle\int_s^\infty \frac{h(t)^d}{\int_\Omega \|\Phi_Q(t,t_0,\omega)x_0(\omega)\| d\mu(\omega)} dt \leq \frac{D\ h(s)^d}{\int_\Omega \|\Phi_Q(s,t_0,\omega)x_0(\omega)\| d\mu(\omega)},$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$

with $\displaystyle\int_\Omega \|\Phi_Q(s,t_0,\omega)x_0(\omega)\| \neq 0$.

**Remark 3.3.5.** [123] In Theorem 3.3.3, if the growth rate is $e^t$, we obtain the version of this theorem for the concept of uniform exponential dichotomy in mean.

**Theorem 3.3.4.** [123] Assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C, P)$ has uniform $h$-growth in mean and $h \in \mathcal{H}$. The pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in (0, 1)$ such that

$(uhD_1^2m_c)$ $\displaystyle\int_{t_0}^t \frac{h(s)^{-d}}{\int_\Omega \|\Phi_P(s,t_0,\omega)x_0(\omega)\| d\mu(\omega)} ds \leq \frac{D\ h(t)^{-d}}{\int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\| d\mu(\omega)};$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$, with

$\displaystyle\int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\| \neq 0,$

$(uhD_2^2m_c)$ $\displaystyle\int_{t_0}^t \frac{\int_\Omega \|\Phi_Q(s,t_0,\omega)x_0(\omega)\| d\mu(\omega)}{h(s)^d} ds \leq \frac{D\ \int_\Omega \|\Phi_Q(t,t_0,\omega)x_0(\omega)\| d\mu(\omega)}{h(t)^d},$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Remark 3.3.6.** [123] Taking $h(t) = e^t$, we obtain the version of Theorem 3.3.4 for the concept of uniform exponential dichotomy in mean.

**Theorem 3.3.5.** [123] Assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C, P)$ has strongly uniform $h$-growth in mean and $h \in \mathcal{H}$. The pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in (0, 1)$ such that

$(uhD_1^3m_c)$ $\displaystyle\int_t^\infty h(\tau)^d \left(\int_\Omega \|\Phi_P(\tau,t_0,\omega)x_0(\omega)\| d\mu(\omega)\right) d\tau \leq$

$\leq D\ h(t)^d \left(\int_\Omega \|\Phi_P(t,t_0,\omega)x_0(\omega)\| d\mu(\omega)\right);$

for all $(t, t_0, \omega) \in \Delta \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

$$(uhD_2^3m_c) \int_{t_0}^{t} h(\tau)^{-d} \left( \int_\Omega \|\Psi_Q(t, \tau, \varphi(\tau, t_0, \omega))x_0(\omega)\| d\mu(\omega) \right) d\tau \leq$$
$$\leq Dh(t)^{-d} \left( \int_\Omega \|Q(t, \varphi(t, t_0, \omega))x_0(\omega)\| d\mu(\omega) \right),$$
for all $(t, t_0, \omega) \in \Delta \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$.

**Remark 3.3.7.** [123] Theorem 3.3.5 provides a version of the theorem for uniform exponential dichotomy in mean, in the case when $h(t) = e^t$.

**Theorem 3.3.6.** [123] Assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C, P)$ has strongly uniform $h$-growth in mean and $h \in \mathcal{H}$. The pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if there exist constants $D \geq 1$ and $d \in (0, 1)$ such that

$$(uhD_1^4m_c) \int_{t_0}^{t} \frac{h(s)^{-d}}{\int_\Omega \|\Phi_P(s, t_0, \omega)x_0(\omega)\| d\mu(\omega)} ds \leq \frac{D\ h(t)^{-d}}{\int_\Omega \|\Phi_P(t, t_0, \omega)x_0(\omega)\| d\mu(\omega)},$$
for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $\int_\Omega \|\Phi_P(t, t_0, \omega)x_0(\omega)\| \neq 0$.

$$(uhD_2^4m_c) \int_{t_0}^{t} \frac{h(s)^{d}}{\int_\Omega \|\Psi_Q(t, s, \varphi(s, t_0, \omega))x_0(\omega)\| d\mu(\omega)} ds \leq \frac{D\ h(t)^{d}}{\int_\Omega \|\Psi_Q(t, t_0, \omega)x_0(\omega)\| d\mu(\omega)},$$
for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $\int_\Omega \|\Psi_Q(t, t_0, \omega)x_0(\omega)\| \neq 0$.

**Remark 3.3.8.** [123] As an immediate consequence of Theorem 3.3.6, if $h(t) = e^t$, we obtain its version for the concept of uniform exponential dichotomy in mean.

The following two theorems provide integral characterizations for the concept of uniform $h$-dichotomy in mean in the case when $\Phi$ is a reversible stochastic evolution cocycle.

**Theorem 3.3.7.** [125] Assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C, P)$ has uniform $h$-growth in mean and $h \in \mathcal{H}$. The pair $(C, P)$ is uniformly $h$-dichotomic in mean with $\Phi$ a reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0, 1)$ such that

$$(uhD_1^1m) \quad \int\limits_s^\infty \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}\, dt \leq$$
$$\leq \frac{D\, h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}$$
for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$, with $P(s,\varphi(s,t_0,\omega))x_0(\omega) \neq 0$;

$$(uhD_2^1m) \quad \int\limits_s^\infty h(t)^d \left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)\right) dt \leq$$
$$\leq D\, h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega),$$
for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$.

**Corollary 3.3.1.** [125] Assume that $C = (\Phi,\varphi)$ is a strongly measurable stochastic skew-evolution semiflow and $(C,P)$ has uniform exponential growth in mean. The pair $(C,P)$ is uniformly exponential dichotomic in mean with $\Phi$ a reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0,1)$ such that

$$(ueD_1^1m) \quad \int\limits_s^\infty \frac{e^{dt}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}\, dt \leq$$
$$\leq \frac{D\, e^{ds}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)};$$
for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$ with $P(s,\varphi(s,t_0,\omega))x_0(\omega) \neq 0$;

$$(ueD_2^1m) \quad \int\limits_s^\infty e^{dt} \left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)\right) dt \leq$$
$$\leq D\, e^{ds} \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega),$$
for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$.

**Theorem 3.3.8.** [125] Assume that $C = (\Phi,\varphi)$ is a strongly measurable stochastic skew-evolution semiflow, $(C,P)$ has uniform $h$-growth in mean and $h \in \mathcal{H}$. The pair $(C,P)$ is uniformly $h$-dichotomic in mean with $\Phi$ a reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0,1)$ such that

$$(uhD_1^2m) \quad \int\limits_{t_0}^t \frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}{h(s)^d}\, ds \leq$$
$$\leq \frac{D \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}{h(t)^d};$$
for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$;

$$(uhD_2^2m)\quad \int_{t_0}^{t} \frac{h(s)^{-d}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}\, ds \leq$$
$$\leq \frac{D\, h(t)^{-d}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)},$$
for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $Q(t,\varphi(t,t_0,\omega))x_0(\omega) \neq 0$.

**Corollary 3.3.2.** [125] Assume that $C = (\Phi, \varphi)$ is a strongly measurable stochastic skew-evolution semiflow and $(C, P)$ has uniform exponential growth in mean. The pair $(C, P)$ is uniformly exponential dichotomic in mean with $\Phi$ a reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0, 1)$ such that

$$(ueD_1^2m)\quad \int_{t_0}^{t} \frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}{e^{ds}}\, ds \leq$$
$$\leq \frac{D\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}{e^{dt}};$$
for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

$$(ueD_2^2m)\quad \int_{t_0}^{t} \frac{e^{-ds}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)}\, ds \leq$$
$$\leq \frac{D\, e^{-dt}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|\, d\mu(\omega)},$$
for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $Q(t,\varphi(t,t_0,\omega))x_0(\omega) \neq 0$.

In the following we present a theorem of equivalence between the concept of uniform $h$-dichotomy in mean in discrete time and the concept of uniform $h$-dichotomy in mean in continuous time with invariant families of projectors.

We denote by $\mathcal{H}$ the set of growth rates $h : \mathbb{R}_+ \to [1, \infty)$ with the property that there exists $H > 1$ such that $h(t + 1) < Hh(t)$ for all $t \geq 0$. Moreover, we denote

$$\Delta_d = \{(m, n) \in \mathbf{N}^2 : m \geq n\}.$$

**Theorem 3.3.9.** [128] Let $h \in \mathcal{H}$. Then the pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if the pair $(C, P)$ has uniform $h$-growth in mean and there exist two constants $N_1 \geq 1$ and $\nu > 0$ such that

$$(uhd_1^e m)\quad h(m)^\nu \int_\Omega \|\Phi_P(m,n,\omega)x(\omega)\| d\mu(\omega) \leq N_1\, h(n)^\nu \int_\Omega \|P(n,\omega)x(\omega)\| d\mu(\omega)$$

$$(uhd_2^e m)\quad h(m)^\nu \int_\Omega \|Q(n,\omega)x(\omega)\| d\mu(\omega) \leq N_1\, h(n)^\nu \int_\Omega \|\Phi_Q(m,n,\omega)x(\omega)\| d\mu(\omega),$$
for all $(m,n,\omega) \in \Delta_d \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Remark 3.3.9.** [127] For $h(t) = e^t$, respectively $h(t) = t + 1$, we obtain the equivalence theorem between u.e.d.m. in discrete time and u.e.d.m. in continuous time, respectively the equivalence theorem between u.p.d.m. in discrete time and u.p.d.m. in continuous time.

The following theorem presents the equivalence between uniform $h$-dichotomy in mean in discrete time and uniform $h$-dichotomy in mean in continuous time with strongly invariant families of projectors.

**Theorem 3.3.10.** [127] Let $h \in \mathcal{H}$. Then the pair $(C, P)$ is uniformly $h$-dichotomic in mean if and only if the pair $(C, P)$ has uniform $h$-growth in mean and there exist two constants $N_1 \geq 1$ and $\nu > 0$ such that

$$(uhd_1^t m)\ \ h(m)^\nu \int_\Omega \|\Phi_P(m, n, \omega)x(\omega)\| d\mu(\omega) \leq N_1\ h(n)^\nu \int_\Omega \|P(n, \omega)x(\omega)\| d\mu(\omega);$$

$$(uhd_2^t m)\ \ h(m)^\nu \int_\Omega \|\Psi_Q(m, n, \omega)x(\omega)\| d\mu(\omega) \leq N_1\ h(n)^\nu \int_\Omega \|Q(m, \varphi(m, n, \omega))x(\omega)\| d\mu(\omega),$$

for all $(m, n, \omega) \in \Delta_d \times \Omega$ and $x \in L(\Omega, X, \mu)$.

**Remark 3.3.10.** [127] For $h(t) = e^t$, respectively $h(t) = t + 1$, we obtain the equivalence theorem between u.e.d.m. in discrete time and u.e.d.m. in continuous time, respectively the equivalence theorem between u.p.d.m. in discrete time and u.p.d.m. in continuous time with strongly invariant families of projectors.

## *3.4 Bibliographical Comments*

A significant interest in the theory of dynamical systems has been devoted to the phenomenon known as uniform exponential dichotomy.

The property of exponential dichotomy for linear differential equations began to gain considerable importance with the appearance of two important monographs by J. L. Massera & J. J. Schäffer [64] and J. L. Daleckii & M. G. Krein [34].

In the stochastic sense, the concept of exponential dichotomy has been studied by many authors, such as A. M. Ateiwi [4] or T. Caraballo et al. [24].

In this chapter, we address the general concept of dichotomy in mean with growth rates, or $h$-dichotomy in mean, where $h : \mathbb{R}_+ \to [1, \infty)$ is a growth rate (more precisely, $h$ is a bijective and increasing function with $\lim\limits_{t\to\infty} h(t) = \infty$) for stochastic skew-evolution semiflows in Banach spaces. Thus, we obtain various characterizations for stochastic skew-evolution semiflows in Banach spaces for this concept, as well as necessary and sufficient conditions for the particular cases of exponential and polynomial dichotomy in mean.

In the first paragraph of this chapter (*Preliminaries*), we state the definitions of uniform dichotomy in mean, which generalize the notion of uniform $h$-stability in mean described in the second chapter. Definition 3.1.2 and Remark 3.1.1 represent the original results of this paragraph.

In the second paragraph (*Connections between the concepts of uniform dichotomy in mean*), we present the connections between the concepts of uniform dichotomy in mean. The original results of this paragraph are Theorems 3.2.1, 3.2.2, Corollaries 3.2.1, 3.2.2, published in [128] and disseminated at the *21st International Conference on Numerical Analysis and Applied Mathematics, September 11–17, 2023, Heraklion, Crete, Greece.*

In the following paragraph (*Characterizations of the concept of uniform dichotomy in mean*), characterizations of the concept of uniform dichotomy in mean are obtained using families of invariant projectors, respectively strongly invariant families of projectors, in Proposition 3.3.1 and Remark 3.3.1, published in [123].

Moreover, a logarithmic-type criterion, a majorization-type criterion and a Hai-type criterion are obtained, published by T. M. Személy Fülöp in [124].

The integral characterizations are obtained in Theorems 3.3.3, 3.3.4, 3.3.5, 3.3.6 and Remarks 3.3.5, 3.3.6, 3.3.7, 3.3.8, and represent original results published in [123] and presented at the conference *2024 IEEE 18th International Symposium on Applied Computational Intelligence and Informatics (SACI), May 21–25, 2024, Timişoara, România.*

In [125], integral characterizations for the concept of uniform $h$-dichotomy in mean are presented in the case when the stochastic evolution cocycle $\Phi$ is reversible. These results are original and are stated in Theorems 3.3.1, 3.3.7, 3.3.8, respectively Corollaries 3.3.1, 3.3.2, being published in [125] and presented at the conference *XGEN International Conference on Science Communications, May 20–24, 2024, Technical University of Cluj-Napoca, Baia Mare, Romania.*

Theorem 3.3.9 represents an equivalence theorem between the concept of uniform $h$-dichotomy in mean in discrete time and the concept of uniform $h$-dichotomy in mean in continuous time, using families of invariant projectors. This result is original and was published in [128].

On the other hand, Theorem 3.3.10 presents the equivalence between the concept of uniform $h$-dichotomy in mean in discrete time and the concept of uniform $h$-dichotomy in mean in continuous time, using families of strongly invariant projectors. This result is original and has been submitted for publication in [127].

Both results were presented at the conference *Sixth Romanian Itinerant Seminar on Mathematical Analysis and its Applications, May 30–31, 2024, Cluj-Napoca, Romania.*

# 4. Conclusions and Perspectives

The thesis extends the classical deterministic results to the stochastic context, providing a unified framework for the study of stability and dichotomy with growth rates.

Future directions include:

- Investigation of $h$-dichotomy in mean for stochastic skew-evolution semiflows in discrete time.
- Applications to stochastic differential equations in physics and economics.
- Development of constructive criteria for the control of stochastic systems.

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